GENERALIZED FOKKER-PLANCK EQUATIONS FOR COLOURED, MULTIPLICATIVE GAUSSIAN NOISE

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(recibido 22 de mayo, 1984; aceptado 7 de agosto, 1984)

ABSTRACT

With the help of Novikov's theorem, it is possible to derive a master equation for a coloured, multiplicative, Gaussian random process;

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the coefficients of this master equation satisfy a complicated auxiliary integro-differential equation. For small values of the Kubo number, the master equation reduces to an approximate generalized Fokker-Planck equation. The diffusion coefficient is explicitly written in terms of correlation functions. Finally, a straightforward and elementary second-order perturbative treatment is proposed to derive the same approximate Fokker-Planck equation.

RESUMEN

Con la ayuda del teorema de Novikov es posible obtener una ecuación maestra para un proceso Gaussiano al azar, multiplicativo, coloreado; los coeficientes de esta ecuación maestra satisfacen una ecuación integro-diferencial auxiliar complicada. Para valores pequeños del número de Kubo, la ecuación maestra se reduce a una ecuación generalizada de Fokker-Planck aproximada. El coeficiente de difusión se presenta explícitamente en términos de funciones de correlación. Finalmente se propone un método directo y elemental de perturbaciones de segundo orden para obtener la misma ecuación Fokker-Planck aproximada.

1. INTRODUCTION

There has been in recent times a growth of interest in the derivation of master equations for stochastic processes with arbitrary spectra and correlation times. In the process of development of new methods to study this problem, a theorem established first by Furutsu, Donsker and Novikov[1] for Gaussian noises and afterwards generalized to arbitrary Gaussian and non-Gaussian functionals[2,3] has been of considerable value. Interesting examples of the use of these powerful theorems in the derivation of generalized Fokker-Planck equations for different physical systems may be found in the papers cited above, as well as in Refs. 4 and 5.

In this paper we approach the same general problem and use the method based on Novikov's theorem to perform an efficient derivation of the master equation for a coloured, multiplicative, Gaussian noise. Similar results are presented in Ref. 4, the fundamental difference being that here we are able to go a step further in obtaining a closed formula for the response function, (see Eq. (21) below without resorting to the operator method of Martin, Siggia and Rose, or to any other perturbative method (see Sections 2 and 3)).

As is well known, despite the fact that one is able to construct generalized Fokker-Planck equations for arbitrary noises, they often are of little value as closed expressions due to their formal nature. They become useful in practice only when they can be reduced to second-order equations, either due to particular properties of the system, or by means of approximations. Since the latter seems to be the most frequent case, we include in Section 5 a separate derivation of the generalized Fokker-Planck equation for coloured, multiplicative noise with a small Kubo number.[6] This derivation, which is based on a perturbative method, is unusually simple and unsophisticated, and perhaps more appropriate for pedagogical use than are most of the known derivations.

With the exception of Section 5, we limit our study to Gaussian noises, just for simplicity purposes, since most of the results take on a much more elaborate form in the general case. There exists no problem of principle, however, in extending the treatment to arbitrary distributions for the stochastic force, using, *e.g.*, the method developed by Bochkov, Dubkov, Malakhov and others, to which we refer the interested reader[2,3].

2. DERIVATION OF A MASTER EQUATION

Consider a dynamical system represented by the equation*

$$\dot{x}_\alpha = f_\alpha(\vec{x},t) + g_{\alpha K}(\vec{x},t)\xi_K(t) \qquad , \tag{1}$$

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*Instead of Eq. (1), we should consider in principle the more general equation of motion

$$\dot{x}_\alpha = f_\alpha(\vec{x},t) + K_\alpha(\vec{x},t;\ \xi(t)) \ .$$

In many of the results that follow, the substitution $g_{\alpha K}\xi_K \rightarrow K_\alpha$ can be directly performed. In some equations, however (see, *e.g.*, Eq. (23)), the factor $g_{\alpha K}$ occurs alone; in these instances the stochastic function $\partial K_\alpha(t)/\partial\xi_K(t)$ would appear instead of the sure function $g_{\alpha K}$. This complicates the expressions, but does not substantially modify the physics; hence for simplicity we have chosen the (linear) multiplicative noise problem defined by Eq. (1).

where $\xi_K(t)$ are arbitrary Gaussian, not necessarily stationary, random functions with zero mean value and with a correlation tensor

$$\langle\xi_i(t)\xi_j(s)\rangle = \varphi_{ij}(t,s) \quad , \tag{2}$$

$\vec{x}$ is a vector in (n-dimensional) phase-space, and a summation is implied over repeated indices, both Greek and Latin.

Let $R(\vec{x},t)$ represent the density of systems at point $\vec{x}$ and time t, and let $\vec{x}(t)$ represent the solution of Eq. (1) such that $\vec{x}(t=0) = \vec{x}_0$. We then know that the function

$$R(\vec{x},t) = \delta(\vec{x} - \vec{x}(t)) \tag{3}$$

satisfies the stochastic continuity equation[7,8]

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x_\alpha} \dot{x}_\alpha R = 0 \tag{4}$$

subject to the initial condition

$$R(\vec{x},0) = \delta(\vec{x} - \vec{x}_0) \quad . \tag{5}$$

We are assuming that each member of the ensemble starts from the same initial condition $\vec{x}_0$; otherwise one would have to write $R(\vec{x},0) = \delta(\vec{x} - \vec{x}_\delta(0))$ instead of Eq. (5), and average over the distribution of $\vec{x}_\delta(0)$.

By Eq. (1), Eq. (4) becomes

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x_\alpha} f_\alpha R = - \frac{\partial}{\partial x_\alpha} g_{\alpha K}\xi_K R \quad . \tag{6}$$

To obtain an equation of evolution for the average of the phase-space density over all realizations of $\xi(t)$, $W(\vec{x},t) \equiv \langle R\rangle$, we take the average of Eq. (6):

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} f_\alpha W = - \frac{\partial}{\partial x_\alpha} g_{\alpha K}\langle\xi_K R\rangle \quad . \tag{7}$$

As is clear from physical considerations, the function W is just the phase-space probability density; a proof of this may be found in Refs. 8 and 9.

In order to find a more suitable expression for the r.h.s. term of Eq. (7), we use Novikov's theorem[1], which in our notation reads

$$<\xi_K(t)R(t)> = \int dt_1 \psi_{Kn}(t,t_1) < \frac{\delta R(t)}{\delta \xi_n(t_1)} > \quad . \tag{8}$$

This formula is valid for any Gaussian random function $\xi(t)$ with zero average. The integral extends over the region in which the functions are defined. In the present case we realize from causality considerations that R(t) depends on $\xi_n(t_1)$ for $t_1 \leq t$ only (see Eq. (1)); hence since our initial time is $t = 0$, the integral runs from 0 to t.

Taking into account that R is a functional of ξ_n through $\vec{x}(t)$, we write Eq. (8) as

$$\begin{aligned} <\xi_K(t)R(t)> &= \int_0^t dt_1 \psi_{Kn}(t,t_1) < \frac{\partial R(t)}{\partial x_\beta(t)} \frac{\delta x_\beta(t)}{\delta \xi_n(t_1)} > \\ &= - \frac{\partial}{\partial x_\beta} \int_0^t dt_1 \psi_{Kn}(t,t_1) < R(t) \frac{\delta x_\beta(t)}{\delta \xi_n(t_1)} > \quad . \end{aligned} \tag{9}$$

In writing the second equality, use was made of Eq. (1) and of the fact that the remaining terms of the integrand do not depend on $\vec{x}$.

By introducing the function $\zeta(s,t)$, which may be interpreted as a short of weighted, non-averaged response function:

$$\zeta_{K\beta}(s,t) = \int_0^t dt_1 \psi_{Kn}(s,t_1) \frac{\delta x_\beta(t)}{\delta \xi_n(t_1)} \quad , \tag{10}$$

we can write Eq. (9) as

$$<\xi_K(t)R(t)> = - \frac{\partial}{\partial x_\beta} <\zeta_{K\beta}(t,t)R(t)> \tag{11}$$

and Eq. (7) transforms into

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} f_\alpha W = \frac{\partial}{\partial x_\alpha} g_{\alpha K} \frac{\partial}{\partial x_\beta} <\zeta_{K\beta} R> \quad , \tag{12}$$

with $\zeta_{K\beta} = \zeta_{K\beta}(t,t)$. To calculate ζ we perform a formal time integration of Eq. (1) ($\vec{x}'$ stands for $\vec{x}(t')$):

$$x_\beta(t) = \int_0^t dt' f_\beta(\vec{x}',t') + \int_0^t dt' g_{\beta K}(\vec{x}',t')\xi_K(t') + x_{o\beta} \tag{13}$$

and take the functional derivative with respect to $\xi_n(t_1)$:

$$\frac{\delta x_\beta(t)}{\delta \xi_n(t_1)} = \int_{t_1}^t dt' \frac{\delta f_\beta(\vec{x}',t')}{\delta \xi_n(t_1)} + \int_{t_1}^t dt' \frac{\delta g_{\beta K}(\vec{x}',t')}{\delta \xi_n(t_1)} \xi_K(t') + g_{\beta n}(\vec{x}(t_1),t_1). \tag{14}$$

The lower integration limit is once more due to causality. Since by Eq. (1) we have for $t_1 \leq t'$

$$\frac{\delta f_\beta(\vec{x}',t')}{\delta \xi_n(t_1)} + \left[\frac{\delta g_{\beta K}(x',t')}{\delta \xi_n(t_1)}\right]\xi_K(t') = \frac{\partial \dot{x}_\beta(\vec{x}',t')}{\partial x_\gamma(t')} \frac{\delta x_\gamma(t')}{\delta \xi_n(t_1)} \quad , \tag{15}$$

Eq. (14) can be written as

$$\frac{\delta x_\beta(t)}{\delta \xi_n(t_1)} = \int_{t_1}^t dt' \frac{\partial \dot{x}_\beta(\vec{x}',t')}{\partial x_\gamma(t')} \frac{\delta x_\gamma(t')}{\delta \xi_n(t_1)} + g_{\beta n}(\vec{x}(t_1),t_1) \quad .$$

By multiplying this result by $\psi_{Kn}(s,t_1)$ and integrating over t_1 we obtain, after changing the order of integration in the first r.h.s. term,

$$\zeta_{K\beta}(s,t) = \int_0^t dt' \frac{\partial \dot{x}_\beta(\vec{x}',t')}{\partial x_\gamma(t')} \zeta_{K\gamma}(s,t') + \int_0^t dt' \psi_{Kn}(s,t') g_{\beta n}(\vec{x}',t') \ . \tag{16}$$

We write the solution of Eq. (16) as

$$\zeta_{K\beta}(s,t) = \int_0^t dt' \ G_{\beta\gamma}(t,t') g_{\gamma n}(\vec{x}',t') \psi_{Kn}(s,t') \quad , \tag{17}$$

where the Green function satisfies the differential equation

$$\dot{G}_{\alpha\beta}(t,t') = \frac{\partial \dot{x}_\alpha(\vec{x},t)}{\partial x_\lambda(t)} G_{\lambda\beta}(t,t') \quad , \tag{18}$$

and is subject to the condition

$$G_{\alpha\beta}(t,t) = \delta_{\alpha\beta} \quad . \tag{19}$$

The solution of Eq. (18) is

$$G_{\alpha\beta}(t,t') = \frac{\partial x_\alpha(t)}{\partial x_\beta(t')} \tag{20}$$

and Eq. (17) gives therefore

$$\zeta_{\kappa\beta}(s,t) = \int_0^t dt' \psi_{Kn}(s,t') \frac{\partial x_\beta(t)}{\partial x_\lambda(t')} g_{\lambda n}(\vec{x}',t') \tag{21}$$

Notice that this is an exact solution for the response function and it can be calculated in some particular cases. We give explicit results for the linear general case and for the white noise problem (See section 3).

A comparison with Eq. (10) shows now that

$$\frac{\partial x_\beta(t)}{\partial \xi_n(t')} = \frac{\partial x_\beta(t)}{\partial x_\lambda(t')} g_{\lambda n}(\vec{x}',t') \quad . \tag{22}$$

With this result, Eq. (12) transforms into

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} f_\alpha W = \frac{\partial}{\partial x_\alpha} g_{\alpha n} \frac{\partial}{\partial x_\beta} \langle R \int_0^t dt' \frac{\partial x_\beta(t)}{\partial x_\lambda(t')} g_{\lambda n}(\vec{x}',t') \psi_{Kn}(t,t') \rangle \quad . \tag{23}$$

This is an exact result, but also a formal one, since it still contains the non-averaged distribution function $R(\vec{x},t)$; in fact, it is an equation of infinite order for W, as we shall see below. Since our formal procedure has implied everywhere the usual rules of calculus, it must lead in the white-noise limit to the results of the Stratonovich treatment[10,11]; we will show that this is indeed the case (see Eq. (34a) below).

3. GENERALIZED FOKKER-PLANCK EQUATION

To obtain a master equation for $W(\vec{x},t)$ it is necessary to perform explicitly the average expressed in the r.h.s. term of Eq. (12) or (23). This step is straighforward when ζ is a sure function, since then

$$\langle \zeta_{\kappa\beta} R \rangle = \zeta_{\kappa\beta} W \tag{24}$$

and Eq. (12) reduces to

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} f_\alpha W = \frac{\partial}{\partial x_\alpha} g_{\alpha\kappa} \frac{\partial}{\partial x_\beta} \zeta_{\kappa\beta} W \quad . \tag{25}$$

This result can be cast into the usual form of a Fokker-Planck equation:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} B_\alpha W = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\alpha\beta} W \tag{26a}$$

with

$$B_\alpha(\vec{x},t) = f_\alpha + \frac{\partial g_{\alpha k}}{\partial x_\beta} \zeta_{\kappa\beta} \tag{26b}$$

and

$$D_{\alpha\beta}(\vec{x},t) = g_{\alpha\kappa}\zeta_{\kappa\beta} \quad . \tag{26c}$$

Owing to the symmetry of the second derivative in Eq. (26a) we may use instead of $D_{\alpha\beta}$ the symmetrized diffusion coefficient:

$$D^s_{\alpha\beta}(\vec{x},t) = \frac{1}{2}\left(g_{\alpha\kappa}\zeta_{\kappa\beta} + g_{\beta\kappa}\zeta_{\kappa\alpha}\right) \quad . \tag{26d}$$

Eq. (24) is exactly true only in particular instances. One is the linear (so-called Langevin) case, when Eq. (1) reduces to

$$\dot{x}_\alpha = a_{\alpha\beta}(t)\, x_\beta + b_\alpha(t) + g_{\alpha\kappa}(t)\xi_\kappa(t) \quad . \tag{27}$$

Assuming the matrix M(t) to exist, such that

$$M^{-1}\dot{M} = -a \quad , \tag{28}$$

the solution of Eq. (27) is

$$x_\alpha = (M^{-1})_{\alpha\lambda}\int_{t'}^{t} dt_1 M_{\lambda\nu}(t_1)[b_\nu(t_1) + g_{\nu K}(t_1)\xi_K(t_1)] + [M^{-1}(t)M(t')]_{\alpha\nu}x_\nu(t')$$

whence

$$\frac{\partial x_\alpha(t)}{\partial x_\nu(t')} = [M^{-1}(t)M(t')]_{\alpha\nu}$$

Introducing this result into Eq. (21) we obtain a sure value for ζ:

$$\zeta_{K\beta}(t,t) = \int_0^t dt'\psi_{kn}(t,t')\, M^{-1}_{\beta\nu}(t)M_{\nu\lambda}(t')g_{\lambda n}(t') \quad . \tag{29}$$

Hence Eq. (26a) applies, with

$$B_\alpha(\vec{x},t) = a_{\alpha\beta}x_\beta + b_\alpha \quad , \tag{30a}$$

$$D_{\alpha\beta}(t) = g_{\alpha K}(t)M^{-1}_{\beta\nu}(t)\int_0^t dt'\psi_{Kn}(t,t')M_{\nu\lambda}(t')g_{\lambda n}(t') \quad . \tag{30b}$$

It is worth mentioning that these results apply as well to the generalized Langevin equation (with memory):

$$\dot{x}_\alpha = \int_0^t dt'\; b_{\alpha\beta}(t-t')x_\beta(t') + g_{\alpha K}(t)\xi_K(t) \tag{31}$$

since this equation can be cast into the form (27) by means of a Laplace transformation.[4,5,12] Note that the only condition on Eq. (27) was that the distribution of ξ is Gaussian with $\langle\xi\rangle = 0$.

Let us now consider another particular instance in which Eq. (24) holds, namely, the white-noise problem, characterized by a correlation function of the form

$$\psi_{kn}(t,s) = 2\gamma_{kn}(t)\delta(t-s) \quad . \tag{32}$$

In this case Eq. (21) reduces to

$$\zeta_{K\beta}(t,t) = \gamma_{Kn}(t) g_{\beta n}(\vec{x},t) \tag{33}$$

and hence, according to Eqs. (26b,c),

$$B_\alpha(\vec{x},t) = f_\alpha + \gamma_{Kn} \frac{\partial g_{\alpha K}}{\partial x_\beta} g_{\beta n} = f_\alpha + \frac{\partial D_{\alpha\beta}}{\partial x_\beta} - \gamma_{Kn}\, g_{\alpha K} \frac{\partial g_{\beta n}}{\partial x_\beta} \quad , \tag{34a}$$

$$D_{\alpha\beta}(\vec{x},t) = \gamma_{Kn} g_{\alpha K}\, g_{\beta n} \; . \tag{34b}$$

Notice that the drift coefficient contains the additional contribution that arises from the multiplicative nature of the stochastic force. This is the well-known extra term appearing in the Stratonovich treatment and absent in Itô's treatment of stochastic differential equations.[10,11] These particular results have been obtained through various procedures; see, *e.g.*, Ref. 6.

In the more general case, in which the external force is not linear and the noise spectrum is coloured, Eq. (24) does not hold exaclty. In order to find in this case an expression for $<\zeta_{K\beta}R>$ in terms of W, we apply the Bochkov, Dubkov and Malakhov (BDM) formula[2]

$$<PQ> = <P><Q> + \sum_{m=1}^{\infty} \frac{1}{m!} \underbrace{\int \cdots \int}_{2m} < \frac{\delta^m P}{\delta\xi(t_1)\cdots\delta\xi(t_m)}> <\frac{\delta^m Q}{\delta\xi(\tau_1)\cdots\delta\xi(\tau_m)}>$$

$$\prod_{i=1}^{m} \psi(t_i,\tau_i) dt_i d\tau_i \quad ,$$

valid for any pair of functionals P,Q of the Gaussian random variable ξ. For $P = \zeta_{K\beta}$ and $Q = R$, we obtain

$$<\zeta_{K\beta}R> - <\zeta_{K\beta}>W = \iint dt'd\tau'\psi_{ij}(t',\tau') < \frac{\delta\zeta_{K\beta}}{\delta\xi_i(t')} > < \frac{\delta R}{\delta\xi_j(\tau')} > + \cdots$$

$$= \iint dt'd\tau'\psi_{ij}(t',\tau') < \frac{\partial\zeta_{K\beta}}{\partial x_\lambda(t)} \frac{\delta x_\lambda(t)}{\delta\xi_i(t')} > < \frac{\partial R}{\partial x_\mu(t)} \frac{\delta x_\mu(t)}{\delta\xi_j(\tau')} > + \cdots$$

$$= -\frac{\partial}{\partial x_\mu} \int_0^t dt' < \frac{\partial\zeta_{K\beta}}{\partial x_\nu(t')} g_{\nu i}(\vec{x}',t') > < \zeta_{i\mu}(t',t)\, R > + \cdots \tag{36}$$

In writing the last equality we have used Eqs. (3), (21) and (22), and the fact that R is the only function within the integral that depends on x_μ.

It is clear that each time m increases by one, an additional noise factor g and an additional correlation ψ (through the function ζ) appear. This means that if ψ has a finite correlation time τ_c, every new term in Eq. (36) is of relative order $|g|\tau_c$. This factor is equivalent to the Kubo number encounttered in the literature[13]. Whenever $|g|\tau_c \ll 1$, it is legitimate to neglect these terms and to use the approximate result

$$<\zeta_{K\beta}R> \doteq <\zeta_{K\beta}>W \tag{37}$$

Under these circumstances, Eq. (12) (or (23)) reduces to the approximate Fokker-Planck equation

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} B_\alpha W = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\alpha\beta} W \tag{38}$$

with

$$B_\alpha(\vec{x},t) = f_\alpha + <\zeta_{K\beta}> \frac{\partial g_{\alpha K}}{\partial x_\beta} = f_\alpha + \frac{\partial D_{\alpha\beta}}{\partial x_\beta} - g_{\alpha K} \frac{\partial <\zeta_{K\beta}>}{\partial x_\beta} \tag{39a}$$

and

$$D_{\alpha\beta}(\vec{x},t) = g_{\alpha K} <\zeta_{K\beta}> \tag{39b}$$

The diffusion coefficient may be written once more in its symmetrized form:

$$D^s_{\alpha\beta}(\vec{x},t) = \frac{1}{2} \left(g_{\alpha K}<\zeta_{K\beta}> + g_{\beta K}<\zeta_{K\alpha}>\right) \tag{39c}$$

The averaged function $<\zeta_{K\beta}>$ appearing in Eqs. (39) is, according to Eq. (21).

$$<\zeta_{K\beta}> = \int_0^t dt' \psi_{Kn}(t,t') < \frac{\partial x_\beta(t)}{\partial x_\lambda(t')} g_{\lambda n}(\vec{x}',t')> \quad ; \tag{40}$$

hence, its explicit calculation requires further approximations. If the effects of the stochastic force on $\vec{x}(t')$ are assumed to be small during the time interval τ_c, the variable $\vec{x}(t')$ in the integrand may be replaced by $\tilde{\vec{x}}(t')$, which is the value of $\vec{x}$ at time $t' \leq t$ such that it evolves deterministically towards $\vec{x}(t) = \vec{x}$. This is true, in particular, within our approximation ($|g|\tau_c \ll 1$) and if the system has had a long time to interact with the stochastic field ($t \gg \tau_c$). Under these circumstances Eq. (40) reduces to

$$\langle\zeta_{K\beta}\rangle = \int_0^t dt' \varphi_{Kn}(t,t') \frac{\partial x_\beta(t)}{\partial \tilde{x}_\lambda(t')} g_{\lambda n}(\tilde{\vec{x}}',t') \tag{41}$$

and Eqs. (39a, b) transform into

$$B_\alpha(\vec{x},t) = f_\alpha + \int_0^t dt' \varphi_{Kn}(t,t') \frac{\partial g_{\alpha K}(\vec{x},t)}{\partial \tilde{x}_\lambda(t')} g_{\lambda n}(\tilde{\vec{x}}',t') \tag{42a}$$

and

$$D_{\alpha\beta}(\vec{x},t) = \int_0^t dt' \varphi_{Kn}(t,t')\, g_{\alpha K}(\vec{x},t)\, g_{\lambda n}(\tilde{\vec{x}}',t') \frac{\partial x_\beta(t)}{\partial \tilde{x}_\lambda(t')} \tag{42b}$$

These approximate results have been obtained following other procedures, by various authors.[4,6-8,10,14,15] One must recall that they hold for $t \gg \tau_c$, which means that Eq. (38) with the drift and diffusion coefficients given by (42) is not a true Fokker-Planck equation, since it does not apply at short times; in fact, for $t \lesssim \tau_c$ the solution $W(\vec{x},t)$ is not necessarily positive definite.[14]

4. RELATIONSHIP BETWEEN CORRELATION AND DIFFUSION COEFFICIENTS

Let us derive an alternative expression for the diffusion coefficient appearing in the approximate Fokker-Planck equation (38). For this purpose we introduce the correlation functions, given as usual by

$$\Gamma_{A(t)B(t')} = \langle A(t)B(t')\rangle - \langle A(t)\rangle \langle B(t')\rangle \quad , \tag{43}$$

for arbitrary functions $A(\vec{x},t)$, $B(\vec{x}',t')$. By Eqs. (10) and (39b) we have

$$D_{\alpha\beta} = g_{\alpha K} \int_0^t dt' \psi_{Kn}(t,t') < \frac{\delta x_\beta(t)}{\delta \xi_n(t')} > \qquad (44)$$

By Novikov's theorem, Eq. (8), we realize that this expression can be written in the form

$$D_{\alpha\beta} = g_{\alpha K} < \xi_K(t)\; x_\beta(t) > - \; g_{\alpha K} < \xi_K\;(t) > < x_\beta(t) > \quad .$$

The last term is actually zero, since $<\xi_K> = 0$. We now rewrite this equation by adding and subtracting the same terms with the factor $g_{\alpha K}$ within the angle brackets, and using Eq. (1); the result is

$$D_{\alpha\beta} = < (\dot{x}_\alpha - f_\alpha) x_\beta > - < \dot{x}_\alpha - f_\alpha > < x_\beta > + \; [< (\Delta g_{\alpha K})\, \xi_K x_\beta > - < (\Delta g_{\alpha K}) \xi_K > < x_\beta >] \; . \qquad (45)$$

The term within square brackets vanishes whenever

$$\Delta g_{\alpha K} = g_{\alpha K}(\vec{x},t) - g_{\alpha K}\;(\vec{x}(t),t) \qquad (46)$$

is equal to zero. This is true, in particular, for additive noise, *i.e.*, when $g_{\alpha K}$ is a function of time only. When this is not the case, the contribution of this term to $D_{\alpha K}$ is of relative order $|g\tau_c|$ and hence, can be neglected within the approximation used to derive Eqs. (39). (See Appendix for the demonstration). This allows us to write Eq. (45) in terms of correlations as

$$D_{\alpha\beta} = \Gamma_{\dot{x}_\alpha x_\beta} - \Gamma_{f_\alpha x_\beta} \quad . \qquad (47)$$

Since the time derivative of $\Gamma_{x_\alpha x_\beta}$ is

$$\dot{\Gamma}_{x_\alpha x_\beta} = \Gamma_{\dot{x}_\alpha x_\beta} + \Gamma_{x_\alpha \dot{x}_\beta} \quad ,$$

the symmetrized diffusion coefficient takes on the form

$$D^s_{\alpha\beta} = \frac{1}{2} \left[\dot{\Gamma}_{x_\alpha x_\beta} - (\Gamma_{f_\alpha x_\beta} + \Gamma_{f_\beta x_\alpha}) \right] \quad . \qquad (48)$$

This result is a generalization of the formula derived by Stratonovich[10]:

$$D^s_{\alpha\beta} = \frac{1}{2}\dot{\Gamma}_{x_\alpha x_\beta}$$

to the case in which the deterministic force $\vec{f}$ is different from zero.

5. A SIMPLE, PERTURBATIVE DERIVATION

We have seen that the Fokker-Planck equation (38) is a good approximation to the master equation whenever $|g\tau_c| \ll 1$. From Eqs. (39b) and (40), it is evident that the coefficient $D_{\alpha\beta}$ is of order $|g^2\tau_c|$, which means that Eq. (38) is valid up to second order in g.

If one knows beforehand that this approximation is sufficient, it is actually unnecessary to go through such a complex procedure to derive the approximate Fokker-Planck equation; working to second order in g from the beginning the derivation can be considerably simplified. Here we present a simple and direct method to achieve this task.

Let us rewrite Eq. (1) introducing a constant parameter β to keep track of the smallness of the stochastic force:

$$\dot{x}_\alpha = F_\alpha(\vec{x},t) + \beta^2 f^d_\alpha(\vec{x},t) + \beta g_{\alpha K}(\vec{x},t)\xi_K(t) \tag{49}$$

In the present terminology the Kubo number is $\beta\tau_c$, which we shall assume to be small. The deterministic force $\vec{f}$ has been separated into the external contribution $\vec{F}$ and the dissipative term $\beta^2\vec{f}^d$, which is assumed, as usual, to be of order β^2.

We now construct the stochastic continuity equation and take its average, thus obtaining, for $W = \langle R \rangle$,

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha}(F_\alpha + \beta^2 f^d_\alpha)W = -\beta\frac{\partial}{\partial x_\alpha}\langle g_{\alpha K}\xi_K R\rangle \tag{50}$$

Since R is of the form given by Eq. (3), we may write the term in angle brackets as before:

$$<g_{\alpha K}(\vec{x}(t),t)\xi_K(t)R(t)> = g_{\alpha K}(\vec{x},t)<\xi_K(t)R(t)> \quad . \tag{51}$$

Our task is to calculate this term to first order in β, since the r.h.s. term in Eq. (50) already contains one factor β. We therefore perform a perturbative treatment, which consists in developing R(t) as a power series in β and keeping only the first two terms of the series. Let $\vec{x}_c(t)$ be the solution of Eq. (49) with $\beta = 0$: $\dot{x}_{c\alpha} = F_\alpha(\vec{x}_c,t)$; then

$$R(\vec{x},t;\,\beta) = R(\vec{x},t;\,0) + \beta \left.\frac{\partial R(\vec{x},t;\beta)}{\partial \beta}\right|_{\beta=0} \quad , \tag{52a}$$

where

$$R(\vec{x},t;0) = \delta(\vec{x}_c(t) - \vec{x}) \quad . \tag{52b}$$

Since R depends on β through $\vec{x}(t)$, we obtain from Eq. (3):

$$\left.\frac{\partial R(\vec{x},t;\beta)}{\partial \beta}\right|_{\beta=0} = \left.\frac{\partial R}{\partial x_\lambda(t)}\frac{\partial x_\lambda(t)}{\partial \beta}\right|_{\beta=0} = -\left.\frac{\partial}{\partial x_\lambda} R \frac{\partial x_\lambda(t)}{\partial \beta}\right|_{\beta=0} \quad . \tag{52c}$$

Introducing Eqs. (52) into (51) we are left with

$$<g_{\alpha K}\xi_K\, R> = -\,\beta g_{\alpha K}\frac{\partial}{\partial x_\lambda}<\xi_K(t)\frac{\partial x_\lambda(t)}{\partial \beta}>_{\beta=0} \; W(\vec{x},t;0) \quad , \tag{53}$$

since $<\xi_K R(\vec{x},t;0)> = 0$. The product $<\xi_K \frac{\partial x_\lambda}{\partial \beta}> W$ can be calculated to zero order in β, since it is already multiplied in Eq. (53) by an additional factor β. This allows us to write $W = W(\vec{x},t;\beta)$ instead of $W(\vec{x},t;0)$, and hence

$$<g_{\alpha K}\xi_K R> = -\,\beta g_{\alpha K}\frac{\partial}{\partial x_\lambda}\, y_{K\lambda} W \quad , \tag{54}$$

with

$$y_{K\alpha} = <\xi_K \frac{\partial x_\lambda}{\partial \beta}>_{\beta=0} \quad . \tag{55}$$

To evaluate the function y we perform a formal time integration of Eq.

(49):

$$x_\lambda(t) = x_{0\lambda} + \int_0^t dt' F_\lambda(\vec{x}',t') + \beta^2 \int_0^t dt' f_\lambda^d(\vec{x}',t') + \beta \int_0^t dt' g_{\lambda n}(\vec{x}',t')\xi_n(t')$$

and take the derivative with respect to β, evaluated at $\beta = 0$:

$$\left.\frac{\partial x_\lambda}{\partial \beta}\right|_{\beta=0} = \int_0^t dt' \frac{\partial F_\lambda(\vec{x}',t')}{\partial x_\nu(t')} \left.\frac{\partial x_\nu(t')}{\partial \beta}\right|_{\beta=0} + \int_0^t dt' g_{\lambda n}(\vec{x}',t')\xi_n(t')\Big|_{\beta=0}$$

After multiplying by $\xi_K(s)$ and performing the average we are left with

$$y_{K\lambda}(s,t) = \int_0^t dt' \left.\frac{\partial F_\lambda(\vec{x}',t')}{\partial x_\nu(t')}\right|_{\beta=0} y_{K\nu}(s,t') + \int_0^t dt' \psi_{Kn}(s,t') g_{\lambda n}(\vec{x}',t')\Big|_{\beta=0} \tag{56}$$

where the function

$$y_{K\lambda}(s,t) = \left\langle \xi_K(s) \frac{\partial x_\lambda(t)}{\partial \beta} \right\rangle_{\beta=0} \tag{57}$$

is a generalization to different times s,t of the previous function $y_{K\lambda} = y_{K\lambda}(t,t)$.

Notice that Eq. (56) has the same structure as Eq. (16), except that all functions of $\vec{x}$ are now evaluated at $\beta = 0$, *i.e.*, $\vec{x} \to \vec{x}_c$. Hence, by comparison with Eq. (21) we conclude that the solution of Eq. (56) is

$$y_{K\lambda}(s,t) = \int_0^t dt' \psi_{Kn}(s,t') \frac{\partial x_\lambda(t)}{\partial x_{c\nu}(t')} g_{\nu n}(\vec{x}_c{}',t') \quad . \tag{58}$$

In writing this equation we have taken into account that at time t, $x_{c\lambda}(t) = x_\lambda(t)$. As can be seen from Eq. (54), it is the combination $\beta y_{K\lambda}$ (rather than β alone) which must be small in order to guarantee the validity of our approximation. For $y_{K\lambda}$ not to grow indefinitely for large time intervals, one must assume a finite correlation time, such that $\beta\tau_c \ll 1$. The approximate Fokker-Planck equation valid to second order in β (or to first order in $\beta\tau_c$) is therefore, according to Eqs. (50) and (54),

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} (F_\alpha + \beta^2 f_\alpha^d) W = \beta^2 \frac{\partial}{\partial x_\alpha} g_{\alpha K} \frac{\partial}{\partial x_\lambda} y_{K\lambda} W \quad , \tag{59}$$

where $y_{K\lambda}(\vec{x},t)$ is the sure function given by Eq. (58) with $s = t$. This result can be recast into the traditional form of a Fokker-Planck equation:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} B_\alpha W = \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} D_{\alpha\lambda} W \tag{60a}$$

with

$$B_\alpha = F_\alpha + \beta^2 \left(f_\alpha^d + y_{K\lambda} \frac{\partial g_{\alpha K}}{\partial x_\lambda} \right) \tag{60b}$$

and

$$D_{\alpha\lambda} = \beta^2 g_{\alpha K} y_{K\lambda} \tag{60c}$$

or else, in its symmetrized form,

$$D_{\alpha\lambda}^s = \frac{1}{2} \beta^2 [g_{\alpha K} y_{K\lambda} + g_{\lambda K} y_{K\alpha}] \quad . \tag{60d}$$

These results coincide within the present approximation with the results obtained in Section 3, Eqs. (38) and (42), since to zero order in β, $\vec{x}_c(t')$ coincides with $\tilde{\vec{x}}\,(t')$. Also here, we must be careful when applying them at short times, since the smallness of the parameter β does not guarantee the smallness of the dynamical effects of the stochastic force when the system has just started to interact, *i.e.*, at $t \lesssim \tau_c$. Finally, it seems worth noting that the assumption of Gaussianity of the stochastic variable ξ was not necessary in this derivation to second order; this implies that any correction that appears in Eqs. (39) or (42) to account for non-Gaussian noises must be of order higher than β^2.[16]

APPENDIX

Our intention is to demonstrate that the term appearing in Eq. (45):

$$\Delta D_{\alpha\beta} = <(\Delta g_{\alpha K})\xi_K x_\beta> - <(\Delta g_{\alpha K})\xi_K><x_\beta> \quad (A.1)$$

with $\Delta g_{\alpha K}$ given by Eq. (46) represents a negligible contribution to $D_{\alpha\beta}$, when $|g\tau_c| << 1$.

With this purpose, we apply the BDM formula, Eq. (35), with $P = \xi_K \Delta g_{\alpha K}$ and $Q = x_\beta$:

$$\Delta D_{\alpha\beta} = \int_0^t\int_0^t dt'dt''\psi_{ij}(t',t'') < \frac{\delta}{\delta\xi_i(t')}\xi_K\Delta g_{\alpha K}> < \frac{\delta x_\beta}{\delta\xi_j(t'')}> + \cdots$$

Omitting terms of relative order $|g\tau_c|$ and using Eq. (10), this equation transforms into

$$\Delta D_{\alpha\beta} = \int_0^t dt'\psi_{Kj}(t,t')<\Delta g_{\alpha K}> < \frac{\partial x_\beta}{\delta\xi_j(t')}> = <\Delta g_{\alpha K}> <\xi_{K\beta}> . \quad (A.2)$$

In order to evaluate $<\Delta g_{\alpha K}>$ we use once more Eq. (35), with $P = g_{\alpha K}(\vec{x}(t),t)$ and $Q = R(\vec{x},t)$:

$$<g_{\alpha K}(\vec{x}(t),t)R(t)> = <g_{\alpha K}(\vec{x}(t),t)>W + \int_0^t\int_0^t dt'dt''\varphi_{ij}(t',t'')< \frac{\delta g_{\alpha K}}{\delta\xi_i(t')}>$$

$$< \frac{\delta R}{\delta\xi_j(t'')}> + \cdots \quad (A.3)$$

The second r.h.s. term is once more of relative order $|g\tau_c|$ and can be neglected within our working approximation; we are then left with

$$<g_{\alpha K}(\vec{x}(t),t)R(t)> = <g_{\alpha K}(\vec{x}(t),t)> W \quad . \quad (A.4)$$

On the other hand, by using Eq. (3) one obtains

$$<g_{\alpha K}(\vec{x}(t),t)R> = g_{\alpha K}(\vec{x},t)W \quad (A.5)$$

On subtracting Eq. (A.4) from (A.5) we conclude that $\langle \Delta g_{\alpha K} \rangle = 0$ within the approximation $|g\tau_c| \ll 1$ and hence, the term $\Delta D_{\alpha\beta}$ does not contribute to Eq. (45), according to Eq. (A.2).

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