THE GRAVITATIONAL FREQUENCY SHIFT FOR A CLASS OF NON-METRIC THEORIES OF GRAVITY

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ABSTRACT

A formalism is presented in which a class of non-metric theories of gravity is defined. Within this formalism an analysis of the predictions for the gravitational frequency shift effect (up to third order in the Newtonian gravitational potential U) is carried out in a spherically symmetric, static gravitational field. A particular hypothetical experimental situation is considered and the relevant predictions computed. The experimental limits of the actual tests performed to date show that

the theories under consideration must coincide with their metric counter parts up to first order in U. The formalism also constitutes a framework within which forthcoming improved gravitational redshift experiments can be discussed.

RESUMEN

Se presenta un formalismo para una clase de teorías gravita cionales no-métricas, definidas como teorías métrico-afines, dentro del cual se realiza un análisis en términos de los valores que predicen dichas teorías para el corrimiento gravitacional en frecuencia, a tercer orden en el potencial gravitacional Newtoniano U, en un campo esféricamente simétrico y estático. Dentro de este marco teórico se considera una situación experimental hipotética para la cual se calculan las predicciones relevantes. Las restricciones experimentales derivadas de las pruebas realizadas muestran que las teorías bajo consideración deben coin cidir a primer orden en U con sus contrapartes métricas. El formalismo constituye así mismo un marco teórico para la discusión de los experimentos que con mayor precisión se realicen en el futuro.

1. INTRODUCTION

The gravitational frequency shift is usually calculated in general relativity using the equation governing ideal clocks in a gravitational field. However, such an equation is a theoretical idealization. Real atomic clocks are complex physical systems and in principle if the laws governing such systems were understood in sufficient detail we would be able to determine gravitational frequency shift directly from the underlying physics. In particular, it should be possible to calculate the frequency shift of radiation (regarded as a wavelike phenomenon) directly from the gravitationally generalized laws of electromagnetism in a given theory of gravity.

In non-metric theories of gravity there is no universal equation for calculating the gravitational frequency shift; in order to calculate this effect a complete set of laws of gravity in the theory are needed, including a detailed theory of measurement and a knowledge of the measuring instruments involved. The work described here is a first step in this direction. In section 2 a formalism is presented in which a class of non-metric theories of gravity, called metric-affine theories of gravity,

is defined. We can then calculate the gravitational frequency shift within this formalism (*i.e.*, for the particular theories of gravity under consideration).

In this formalism equations of motion are derived from a Lagrangian and an equation governing atomic clocks is given (all equations are given in a spherically symmetric and static gravitational field). In section 3 the formalism is used and the frequency shift is calculated in a hypothetical experimental situation within the solar system. Finally, in section 4, the results are discussed and compared to similar calculations that have been done in general relativity.

2. A CLASS OF NON-METRIC THEORIES OF GRAVITY

We wish to analyze the gravitational redshift effect within a general class of non-metric theories of gravity, called metric-affine theories of gravity (MATG's). MATG's have been studied extensively elsewhere[1,2,3]. Essentially, in an MATG the gravitational field is described (completely) by a metric tensor field g, and a connection Γ. The measuring process, and hence the laws describing ideal clocks, are governed by g, while the motion of freely falling test particles is governed by Γ through the path equation

$$\frac{d^2x^i}{d\eta^2} + \Gamma^i_{jk} \frac{dx^j}{d\eta} \frac{dx^k}{d\eta} = 0 \quad . \tag{2.1}$$

Since we are interested in experiments measuring the gravitational redshift in the solar system we shall want to analyze the laws of gravity in a spherically symmetric, static (SSS) gravitational field. In such a field Γ and g take on the symplified forms:

$$\Gamma^\sigma_{\mu\nu} = \alpha,_\nu \delta^\sigma_\mu + \bar{\alpha},_\mu \delta^\sigma_\nu + \beta,_\sigma \delta_{\mu\nu} \quad ,$$

$$\Gamma^\sigma_{00} = \gamma,_\sigma \quad , \quad \Gamma^0_{0\nu} = \delta,_\nu \quad , \quad \Gamma^0_{\mu 0} = \bar{\delta},_\mu \tag{2.2}$$

and

$$g_{00} = f \quad , \qquad g_{\mu\nu} = - g\delta_{\mu\nu} \quad , \tag{2.3}$$

where $f, g, \alpha, \bar{\alpha}, \beta, \gamma, \delta$ and $\bar{\delta}$ are arbitrary functions of the Newtonian gravitational potential U. In addition Eq. (2.1) becomes

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}(\gamma) - [\vec{\nabla}(\alpha + \bar{\alpha} - \delta - \bar{\delta}) \cdot \vec{v}]\vec{v} - \vec{\nabla}(\beta) \; \vec{v}^2 \quad , \tag{2.4}$$

where $t = x^0$ is the time coordinate associated with the static nature of gravitational field, $\vec{\nabla}$ is the usual gradient operator, $\vec{x}$ is the spatial 3-position of the test particle and $\vec{v} = d\vec{x}/dt$ its coordinate 3-velocity.

The equations outlined above can also be derived from the optical limit of the source-free gravitationally generalized laws of electro magnetism (GGEM), which can be used to investigate the gravitational redshift effect[1,3]. It will not be necessary to consider the laws of GGEM in full generality any further in this article.

We can always decompose Γ according to

$$\Gamma^a{}_{bc} = \{^a_{bc}\} + A^a{}_{bc} \quad , \tag{2.5}$$

where $\{^a_{bc}\}$ denotes the Christoffel symbol constructed from g and A is a tensor representing the "non-metric" part of Γ. We note that a metric theory of gravity (MTG) is a particular case of an MATG in which A is identically zero. In an MTG in an SSS gravitational field we therefore have that

$$\alpha' = \bar{\alpha}' = -\beta' = \frac{g'}{2g} \quad , \qquad \delta' = \bar{\delta}' = \frac{g\gamma'}{f} = \frac{f'}{2f} \tag{2.6}$$

(where a prime denotes differentiation with respect to U).

From an investigation of MATG's[1,2,3] it is found that in general, certain fundamental principles are not satisfied. For example, if we wish that charge be conserved, that the equations governing the motion of light-like particles deduced from the optical limit of the laws of GGEM be equivalent to the mass $\to 0$, speed $\to 1$, limit of the equations of motion governing time-like particles (Eq. (2.1)), and that the weak equivalence principle be satisfied, then severe constraints are imposed on the acceptable form of Γ. Henceforward we shall restrict attention to a viable subclass of the class of all MATG's, in which the above conditions are indeed satisfied, called Weyl-affine theories of gravity (WATG's). (We note that the above conditions demand that an MATG take on the form of a WATG in an SSS gravitational field[3]).

WATG's are characterized by the following:

(i) The equations of motion for test particles are derived from the Lagrangian given by

$$L = (\lambda g_{ab} dx^a dx^b)^{1/2} \ [= (\lambda f - \lambda g\vec{v}^2)^{1/2} dt] \ . \tag{2.7}$$

(ii) Atomic clocks are governed by

$$ds = (g_{ab} dx^a dx^b)^{1/2} \ [= (f - g\vec{v}^2)^{1/2} dt] \tag{2.8}$$

(where the terms in brackets are the respective forms in an SSS gravitational field).

In an SSS gravitational field we note that Eq. (2.7) is equivalent to Eq. (2.4) in which

$$\alpha' = \bar{\alpha}' = -\beta' = \frac{g'}{2g} + \frac{\lambda'}{2\lambda} \ ,$$

$$\delta' = \bar{\delta}' = \frac{g\gamma'}{f} = \frac{f'}{2f} + \frac{\lambda'}{2\lambda} \ . \tag{2.9}$$

Theories of this type have been investigated previously and are also sometimes referred to as 2-tensor theories of gravity (see References in Ref. 3, and Refs. 8 and 9, for examples). The main physical idea behind WATG's is that different units systems may be employed in Eqs. (2.7) and (2.8), represented by a nonconstant λ.

We note that the "non-metricity" is completely contained in the function λ; in MTG's λ is a constant ($\lambda=1$). It is the aim of this article to obtain experimental limits on the form of λ. In an SSS gravitational field λ is assumed to be an arbitrary function of U.

3. GRAVITATIONAL FREQUENCY SHIFT IN THE SOLAR SYSTEM

A) *Equations and approximations*

From the equations given in section 2 it is possible to derive analytical expressions for the planar motions of test particles and wave signals in the gravitational field of the Sun[4]. The motion of wave signals is governed by the following equations:

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \left[\frac{g}{r^2}\right]^{1/2} \frac{dr}{[\,(f_0/r_0^2)(r^2/f) - 1\,]^{1/2}} \ , \qquad (3.1a)$$

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \left[\frac{g}{f}\right]^{1/2} \frac{dr}{[1 - (r_0^2/f_0)(f/r^2)]^{1/2}} \ , \qquad (3.1b)$$

where r, ϕ and t are spherical polar coordinates for a reference system based on the centre of the Sun, f_0 is the tensor field component g_{00} defined by (2.3) and evaluated at r_0 (the distance of nearest approach of the signal to the Sun), and g_{33} has not been expanded but rather taken as $-r^2$ in order to make the results coordinate independent[5].

The motion of test particles, on the other hand, is governed by the constant angular velocities[6],

$$\omega = \frac{d\phi}{dt} = \pm \left[\frac{d(\lambda f)/dr}{d(\lambda r^2)/dr} \right]^{\frac{1}{2}}_{r=const} , \qquad (3.2)$$

when the particles are forced to follow circular orbits.

Assuming now that the field equations of any theory in the class specified by conditions (i) and (ii) admit asymptotically flat vacuum solutions, it is possible to expand the functions λ, f and g in terms of the Newtonian gravitational potential $U = m/r$, where $m = GM/C^2$ represents the mass of the field's source and r the distance to its centre.

The corresponding expansions accurate to third order in U are:

$$\lambda(r) = 1 + \lambda_1 U + \lambda_2 U^2 + \lambda_3 U^3 ,$$

$$f(r) = g_{00} = 1 + f_1 U + f_2 U^2 + f_3 U^3 , \qquad (3.3)$$

$$g(r) = -g_{11} = 1 + g_1 U + g_2 U^2 + g_3 U^3 .$$

As mentioned before, setting $\lambda_1 = \lambda_2 = \lambda_3 = 0$ reduces the formalism to one for MTG's. In particular, general relativity theory is obtained by choosing $f_1 = -2$, $g_1 = 2$, $g_2 = 4$, $g_3 = 8$, and $f_2 = f_3 = \lambda_1 = \lambda_2 \lambda_3 = 0$.

B) Frequency shift effect

Solar system experiments can be divided into two classes; those that measure the motion of test particles (e.g., perihelion shifts) and those that measure the effects of gravity on clocks (e.g., time delay in radar propagation). In principle, one should be able to separate these two different manifestations of the gravitational field in non-metric

theories of gravity, especially for the theories we are analyzing where the non-metric nature of spacetime influences only the motion of test particles (Eq. (2.7)) and leaves atomic clocks to operate in exactly the same way as in a metric spacetime (Eq. (2.8)). However, this is not the case for the experimental configuration that will be analyzed here since, as will be discussed, the frequency shift experienced by a wave signal can be calculated in terms of signal's motion alone. The work that follows restricts itself to the frequency shift effect up to third order in U. Other solar system experiments have been discussed in the analysis of MATG's already cited[3].

The frequency shift experienced by a wave signal and defined by

$$Z \equiv \frac{\nu_e - \nu_r}{\nu_e} \quad , \qquad (3.4a)$$

where $\nu_{e(r)}$ denotes the frequency of emission (reception), may be expressed as

$$Z = 1 - \frac{ds_e}{ds_r} \quad , \qquad (3.4b)$$

where $ds_{e(r)}$ denotes the proper time during which the signal is emitted (received), for a one-way trip; and

$$Z = 1 - \frac{ds_e}{ds_b} \bigg/ \frac{ds_b}{ds_r} \quad , \qquad (3.4c)$$

where ds_b is the proper time during which the signal is reflected back to the emitter, for the round trip[6]. Equations (3.4) mean that the measurement of the frequency shift effect consists of comparing how a clock on the receiver runs with respect to an identical clock based on the emitter, *i.e.*, a comparison of the elapsed proper times during emission and reception when the world line of the emitter is mapped onto the world line of the receiver by means of null goedesics.

C) Experimental configuration

The only available detailed calculations and results of the fre‑
quency shift effect for MTG's known to the authors are those made for
wave signals connecting an artificial satellite to the earth, both fol‑
lowing circular coplanar orbits around the Sun at four solar radii
(2.78392 x 10^{11}cm.) and one astronomical unit (1.495985 x 10^{13}cm), respec‑
tively[6]. In order to be able to compare these results with those pre‑
dicted by the class of MTG's we are analyzing, the calculations will be
carried out for the same experimental configuration. For this case
Eqs. (3.4b,c) may be rewritten as[6]

$$Z = 1 - \frac{h_e(r,\lambda)}{h_r(r,\lambda)} \left[1 - \omega_r \frac{dp}{dr_0} \bigg/ \frac{dq}{dr_0} \right] \tag{3.5a}$$

for the one-way frequency shift, and

$$Z = \omega_b \frac{\left[\dfrac{dp^0}{dr_0}\right]}{\left[\dfrac{dq^0}{dr_0}\right]} + \omega_r \frac{\left[\dfrac{dp^i}{dr_0}\right]}{\left[\dfrac{dq^i}{dr_0}\right]} - \omega_b\omega_r \frac{\left[\dfrac{dp^0}{dr_0}\right]\left[\dfrac{dp^i}{dr_0}\right]}{\left[\dfrac{dq^0}{dr_0}\right]\left[\dfrac{dq^i}{dr_0}\right]} \tag{3.5b}$$

for the round trip frequency shift effect. The function $h(r,\lambda)$ relates
the proper time s to the coordinate time t through ds = $h(r,\lambda)$dt, the
function p represents the coordinate time t it takes for a wave signal
to go from the emitter (subindex e) to the receiver (subindex r) or back
again, and the function q denotes the angular displacement of the wave
signal when going from receiver (emitter) to emitter (receiver). For
the round trip effect, the constant angular speeds (Eq. (3.2)) of the
emitter (= receiver and reflector) are denoted by ω_e (=ω_r) and ω_b, the
superindices 0 and i respectively indicate the outgoing and incoming
parts of the trip, and r_0 is the distance of closest approach of the wave
signal to the Sun. It is possible to derive analytical expressions for

all the functions entering into expressions(3.5)$^{(6)}$.

D) Calculations and results

Before doing the calculations one final condition is imposed on the equations of motion, namely that these equations take the correct Newtonian limit when gravity is weak and typical velocities of test bodies are small. Both assumptions, weak-field and slow-motion, are valid in the solar system. Thus we demand that $\lambda_1 + f_1 = -2$. Since the first order gravitational redshift experiment demands $f_1 = -2$ (at least within the experimental limits of the actual tests performed[1]),the previous condition implies $\lambda_1 = 0$. Nonetheless, calculations have been made with a very small value for $\lambda_1 (10^{-5})$. The results presented in Figs. 1-3 show the distinct contributions from the different values of the λ coefficients to the calculations of the frequency shift effect during a whole orbit of the satellite around the Sun. The resulting values of substracting general relativity predictions[6] from these made by MATG's with different values for λ_2 are plotted in Fig. 1. Similar graphs for different λ_3's are presented in Fig. 2. Figure 3 shows the results of the calculations for $\lambda_1 = 10^{-5}$ (plot g) compared to those with $\lambda_2 = 10^{-1}$ (plot a).

Almost all of the graphs follow the same pattern, the only exception being that for a non-zero value of λ_1 (curve g). The common pattern followed by the value of the frequency shift (Z) as the angular distance between satellite and earth (ϕ) increases, is as follows. Z is initially positive (redshift) and increases up to a point where it reaches a maximum value ($\phi \sim 90°$), it then starts decreasing, vanishes ($\phi \sim 117°$), and being negative (blueshift) goes on decreasing up to the beginning of the blackout situation due to the eclipse of the satellite. As the satellite comes out from behind the Sun, Z is still negative (blueshift), decreases to a local minimum value ($\phi \sim 273°$), then increases, vanishes ($\phi \sim 299°$), and taking positive values (redshift) goes on increasing up to the point that marks the completion of a satellite's revolution around the Sun. Graph g (Fig. 3) on the other hand, is positive (redshift) and increasing
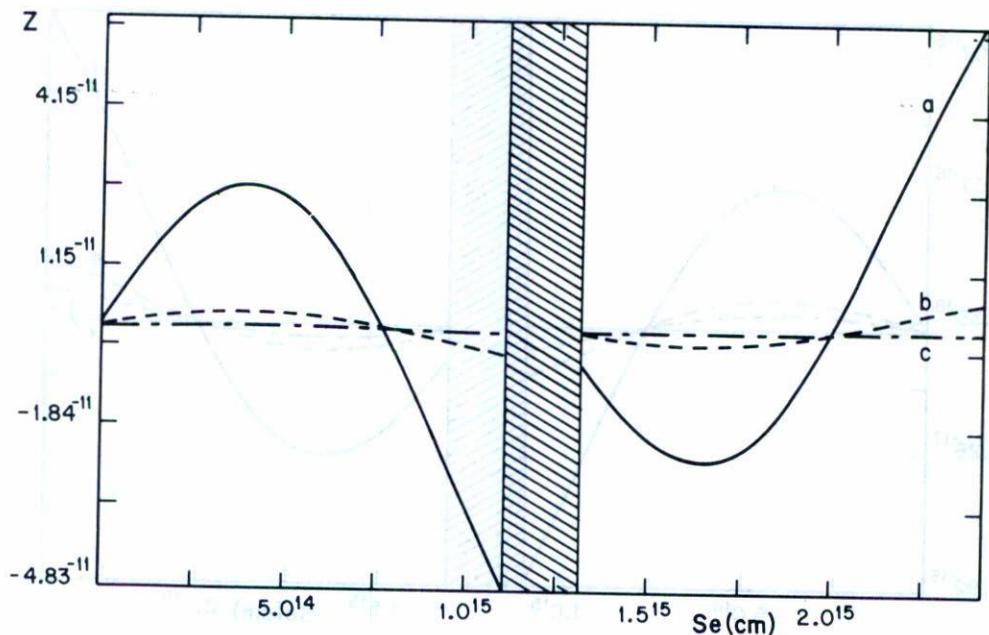
Curve	λ_1	λ_2	λ_3
a	0	10^{-1}	0
b	0	10^{-2}	0
c	0	10^{-3}	0

Fig. 1. Frequency shift calculations made with non-zero values for λ_2. Curve (a) corresponds to $\lambda_2 = 10^{-1}$, (b) corresponds to $\lambda_2 = 10^{-2}$, and (c) to $\lambda_2 = 10^{-3}$. The values predicted by general relativity theory[6] have been substracted so that the plots only show non-metric contributions.

from the beginning up to the blackout situation, and negative (blueshift) and decreasing from the restoration of communication up to the end of the satellite's turnaround.

If we compare the results obtained here with available calculations for general relativity theory (GRT)[6] one sees that the largest
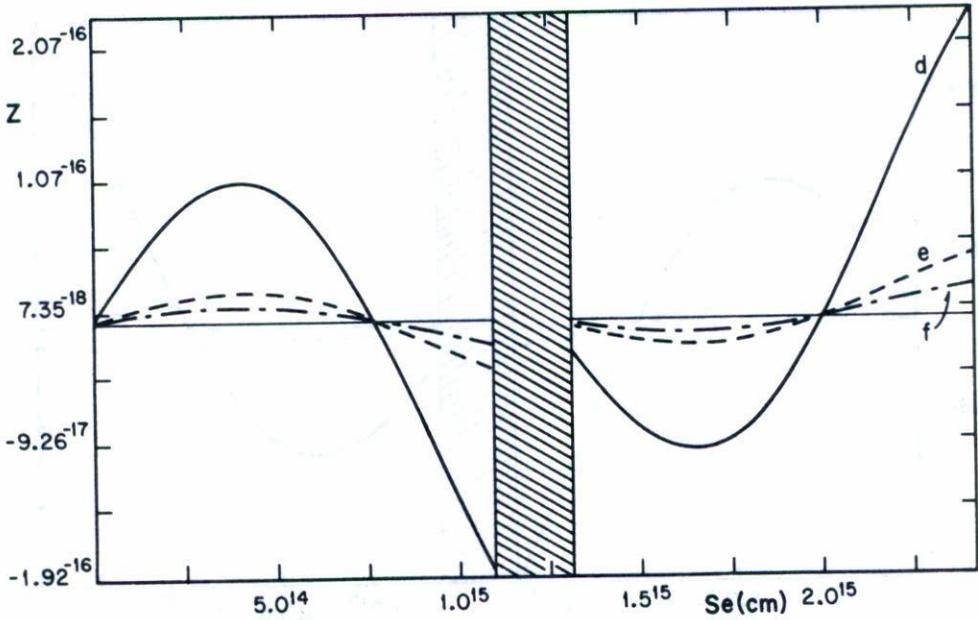
Curve	λ_1	λ_2	λ_3
d	0	0	0.5
e	0	0	0.1
f	0	0	0.05

Fig. 2. Analogue of Fig. 1 with non-zero values for λ_3. Curve (d) represents the values of $\lambda_3 = 0.5$, (e) those of $\lambda_3 = 0.1$, and (f) for $\lambda_3 = 0.05$.

corrections to GRT, those coming from the calculations with $\lambda_1 = 10^{-5}$ or $\lambda_2 = 10^{-1}$, are approximately two orders of magnitude smaller than the contributions of the U-terms of general relativity theory to the Newtonian calculations. Smallest corrections to GRT are those predicted with $\lambda_3 = 0.05$ which in turn are three orders of magnitude smaller than the contributions of the U^2 terms of GRT to the Newtonian calculations.
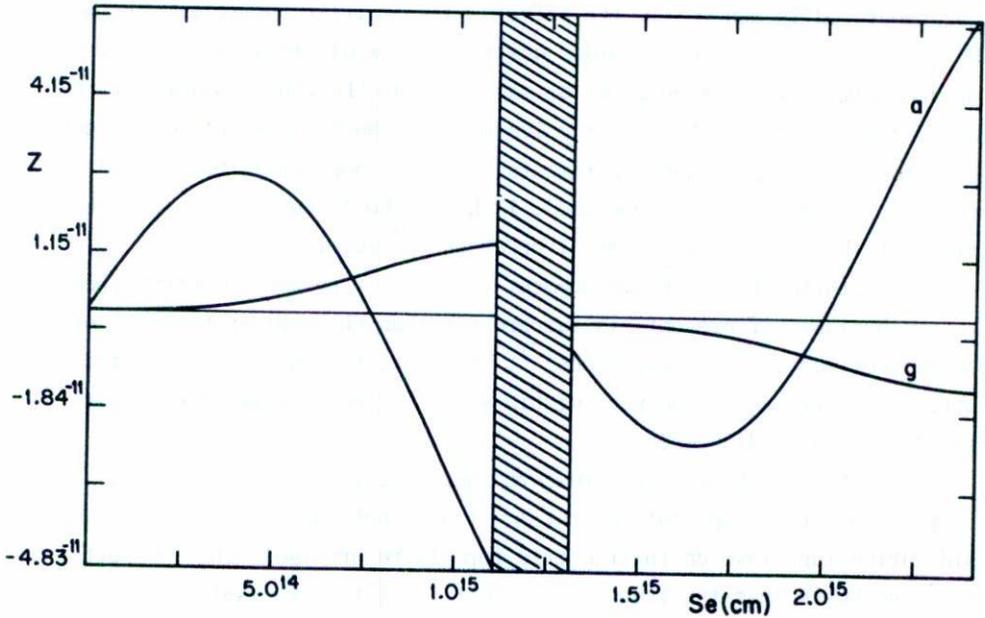
Curve	λ_1	λ_2	λ_3
a	0	10^{-1}	0
g	10^{-5}	0	0

Fig. 3. Comparison of the frequency shift values calculated with $\lambda_1 = 10^{-5}$, $\lambda_2 = \lambda_3 = 0$ (plot g) and those with $\lambda_2 = 10^{-1}$, $\lambda_1 = \lambda_3 = 0$ (plot a). As previously done the values predicted by general relativity theory[6] have been substracted from both curves.

4. DISCUSSION

From the comments at the end of the previous section we conclude that $\lambda_1 = 0$ so that to $o(U)$ the MATG's under consideration must coincide with their metric counterparts. Therefore if spacetime within the solar

neighbourhood is non-metric the effects must manifest themselves at the U^2 or U^3 level. A value of order unity for λ_2 would imply a correction of approximately the same order of magnitude as the contributions of the U-terms in GRT to the Newtonian calculations. Such a correction has not been found in present day experiments[7]. If a non-metric nature is present in the solar neighbourhood, it has to be through small contributions at the U^2 level $(\lambda_2 \lesssim 10^{-3})$ or at the U^3 level.

Unfortunately the improvement in accuracy needed to detect these contributions, if present, is far greater than that obtainable with present experiments and observations[7]. However, the analysis presented here will serve as a framework within which forthcoming improved experiments[7] can be discussed.

Although the results presented here are far from conclusive, they do support the postulate that $\lambda = 1$ and that the theories under consideration must take on their metric form (more precisely they support the hypothesis that the theories are "metric" within the solar system; that is $\lambda = 1$ to present observational levels of accuracy).

This is precisely what would be expected from Will's[10] principle of the universality of gravitational redshift (UGR), which states that "the gravitational redshift between a pair of identical ideal clocks at two events in spacetime is independent of their structure and composition", and in this sense the results can be thought of as supporting UGR. If the principle of UGR is satisfied in the theories under investigation λ must be unity and the theories "metric".

ACKNOWLEDGEMENTS

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