Revista Mexicana de Física 31 No. 4 (1985) 611-622

FOURIER METHODS APPLIED TO THE FRAUNHOFER DIFFRACTION THEORY FOR ANY BIDIMENSIONAL REGULAR ARRAY OF IDENTICAL APERTURES

Román Castañeda

Departamento de Física
Universidad Nacional de Colombia
A.A. 3840 Medellín, Colombia

(recibido julio 19, 1984; aceptado diciembre 5, 1984)

ABSTRACT

Fourier Transform provides elegant methods to study the Fraunhofer diffraction for many types of apertures, not only for the calculation of the amplitude and intensity distributions of the pattern but also the understanding of the diffraction phenomenon. In this paper it is shown that both amplitude and intensity distributions of the pattern are Fourier Transforms of the specific functions. The results are applied to the diffraction produced by any bidimensional array of identical apertures and are extended to the well known cases, *i.e.*, the diffraction grating, the Young experiment, the rectangular slit and the long slit.

RESUMEN

La transformada de Fourier aporta métodos elegantes para el estudio de la difracción de Fraunhofer producida por muchos tipos de aberturas, no sólo para el cálculo de las distribuciones de amplitud e intensidad del patrón sino también, para la comprensión del fenómeno de la difracción. En este artículo se muestra la relación entre estas distribuciones y las transformadas de Fourier de funciones específicas. Los re sultados obtenidos son aplicados a los casos de rejilla de difracción, experimento de Young, abertura rectangular y ranura larga.

INTRODUCTION

The bidimensional Fourier transform of a function[1] $f(x,y)$ is defined as

$$F(P,Q) = \iint_{-\infty}^{\infty} f(x,y) \exp i\,[Px + Qy]\,dxdy. \tag{1}$$

On the other hand, the amplitude distribution for the bidimensional pattern of Fraunhofer diffraction of a monochromatic plane wave which is incident on any bidimensional dispersor[2] (Fig. 1) is given by

$$E(\xi,\eta) = \iint_{\text{DISPERSOR}} \varepsilon(x,y) \exp i\left[\frac{k\xi}{R}\ x + \frac{k\eta}{R}\ y\right]dxdy, \tag{2}$$

where $\varepsilon(x,y)$ is the superficial amplitude density on the dispersor or aperture function[3] and k is the wave number of the plane wave. Let us define the spacial frequencies[4] P and Q of the diffraction pattern as

$$P = \frac{k\xi}{R} \quad , \qquad Q = \frac{k\eta}{R} \quad , \tag{3a,b}$$

and extend the limits of the double integral in Eq.(2) over the interval $(-\infty, \infty)$. That is posible because the dispersor is transparent only into a finite region of the space. Then, Eq.(2) shows the same form than Eq.(1), and we conclude that the amplitude distribution of the pattern is the bidimensional Fourier transform of the aperture function.

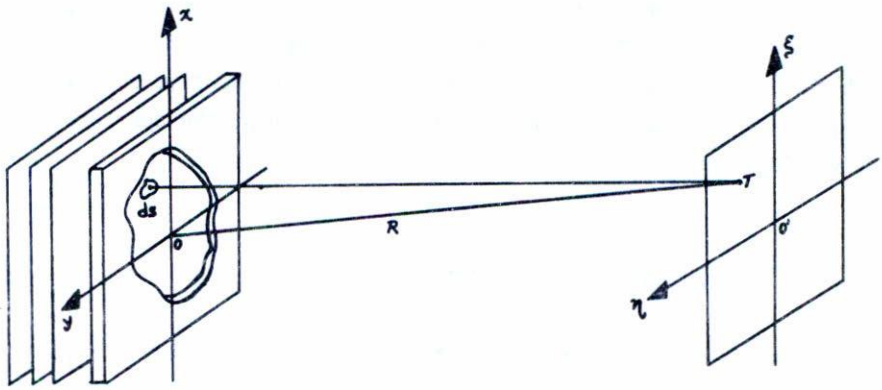

Fig. 1 Illustrating the Fraunhofer diffraction.

Now, the intensity of the diffraction pattern is defined as[5]

$$I(P,Q) = E(P,Q)E^*(P,Q) \quad . \tag{4}$$

Then, according with Eq.(1) we can write

$$I(P,Q) = \iiiint_{-\infty}^{\infty} \varepsilon(x,y)\,\varepsilon^*(u,v)\,\exp i\,[P(x-u) + Q(y-v)]\,dxdydudv \ ,$$

and after the change of variable x-u=z, y-v=w we obtain

$$I(P,Q) = \iint_{-\infty}^{\infty} \left[\iint_{-\infty}^{\infty} \varepsilon(x,y)\,\varepsilon^*(x-z,y-w)\,dxdy \right] \exp i[\,Pz+Qw)\,]\,dzdw \quad ,$$

$$I(P,Q) = \iint_{-\infty}^{\infty} \Gamma_{\varepsilon\varepsilon}(z,w)\,\exp i[\,Pz + Qw]\,dzdw \quad , \tag{5}$$

where $\Gamma_{\varepsilon\varepsilon}(z,w)$ is the autocorrelation of the aperture function[6]. Eq.(5) is the Fourier transform of the bidimensional Winer-Khintchine theorem[7].

REGULAR BIDIMENSIONAL ARRAY OF IDENTICAL APERTURES

Any regular bidimensional array of identical apertures (Fig.2) is composed of N columns and M rows which are regularly apart, with a and b the separation constants for each pair of consecutive rows and each pair of consecutive columns respectively. Then, the aperture function of the array is

$$\varepsilon_T(x,y) = \sum_{n=0}^{N=1} \sum_{m-0}^{M-1} \varepsilon(x\text{-}na,\ y\text{-}mb) \quad , \tag{6}$$

with $\varepsilon(x\text{-}na,\ y\text{-}mb)$ the aperture function for the nm-aperture. From Eq.(1) the amplitude distribution of the pattern will be

$$E_T(P,Q) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} E_{nm}(P,Q)$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \iint_{-\infty}^{\infty} \varepsilon(x\text{-}na,\ y\text{-}mb) \exp i[Px+Qy]\, dxdy \quad , \tag{7}$$

where $E_{nm}(P,Q)$ is the contribution from the nm-aperture. Replacing $s = x\text{-}na$ and $t = y\text{-}mb$ into Eq.(7) and extending the integral only over the nm-aperture, we obtain

$$E_T(P,Q) = \iint_{\text{nm-apt.}} \varepsilon(s,t) \exp i[Ps + Qt]\, dsdt \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \exp i[nPa+mQb]$$

$$= E(P,Q) \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \exp i[nPa + mQb] \quad . \tag{8}$$

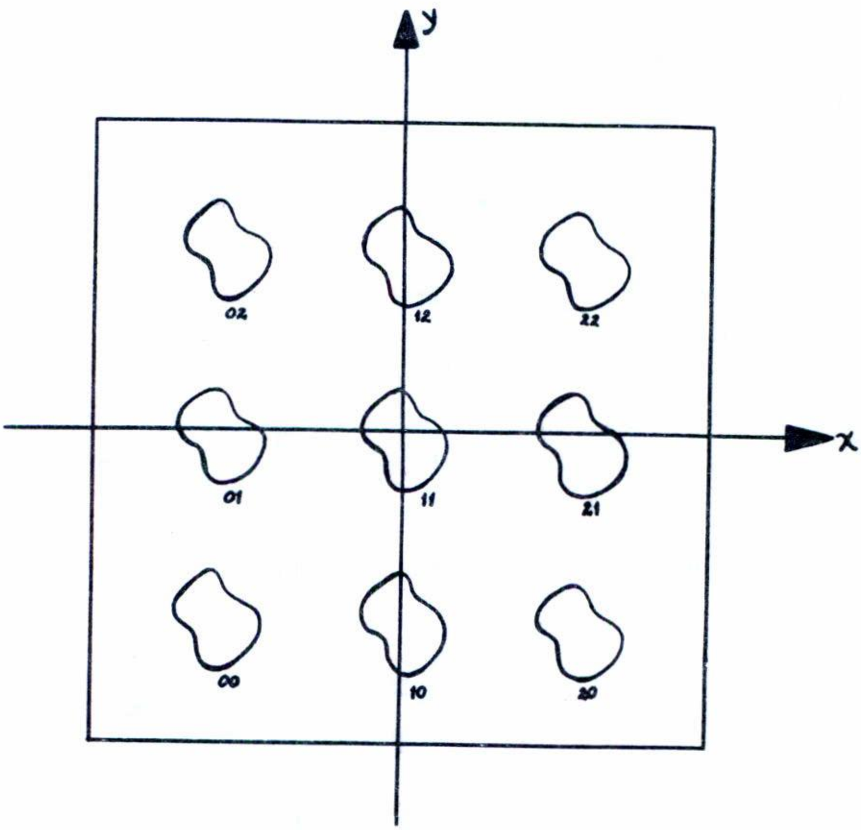

Fig. 2 Bidimensional array of identical apertures. The numeration indicates the nm-order of each aperture. The origin of the frame has been dislocated by the Dirac's delta function $\delta(x-a, y-b)$ for convenience.

Note that into Eq.(8) the nm-subindex is dropped because the amplitude contributions from all apertures of the array are identical. To solve the summatory in Eq.(8) we call $\alpha = Pa/2$ and $\beta = Qb/2$ and apply the relationship[8]

$$\sum_{j=0}^{J-1} (\exp i2x)^j = \exp i[(J-1)x] \, . \quad \frac{\sin Jx}{\sin x} \quad ,$$

then

$$E_T(P,Q) = E(P,Q) \cdot \exp i[(N-1)\alpha+(M-1)\beta] \frac{\sin N\alpha}{\sin \alpha} \cdot \frac{\sin M\beta}{\sin \beta} . \quad (9)$$

And according with Eq.(4), the intensity distribution of the diffraction pattern will be

$$I_T(P,Q) = E(P,Q)E^*(P,Q) \left[\frac{\sin N\alpha}{\sin \alpha}\right]^2 \left[\frac{\sin M\beta}{\sin \beta}\right]^2 . \quad (10)$$

An interesting remark is obtained after the application of inverse Fourier transform[9] to Eq. (8). Let us symbolize this operation by means of the letter F^{-1}. From Eq.(8) we have

$$\varepsilon_T(x,y) = F^{-1}\left[E_T(P,Q)\right] = F^{-1}\left[E(P,Q) \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \exp i[nPa+mQb]\right] .$$

And using the properties of the convolution product[9] symbolized by *, we can write

$$\varepsilon_T(x,y) = F^{-1}[E(P,Q)] * F^{-1}\left[\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \exp i[nPa + mQb]\right] .$$

But, $F^{-1}[E(P,Q)] = \varepsilon(u,v)$ is the aperture function for any aperture of the array and $F^{-1}[\sum_{n,m} \exp i[nPa+mQb]] = \sum_{n.m} \delta(u+na, v+mb)$, where each Dirac's delta function[10] specifies the location of a particular aperture into the array. After the substitution of these results into the expresion above, we obtain an alternative form for Eq.(6):

$$\varepsilon_T(x,y) = \iint_{-\infty}^{\infty} \varepsilon(u,v) \left[\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \delta(x-na-u, y-mb-v)\right] du dv, \quad (11)$$

where the summatory of Dirac's delta functions is called the bidimensional comb function. Thus, any bidimensional regular array of identical apertures is the convolution between a specific aperture function and a bidimensional regular comb function. And, taking into account Eq. (5), the Fraunhofer diffraction realizes the bidimensional Wiener-Khintchine theorem for such array.

APLICATIONS[11]

The single rectangular slit

This is obtained when N=1 and M=1. Further, if the slit is homogeneous and its dimensions are L and L', the corresponding aperture function can be defined as

$$\varepsilon(x,y) = \begin{cases} A \exp i\phi & -\frac{L}{2} < x < \frac{L}{2} \quad , \quad -\frac{L'}{2} < y < \frac{L'}{2} \\ 0 & \text{in other case} \quad , \end{cases} \tag{12}$$

where A and ϕ are real numbers. Then, from Eq.(2),

$$E(P,Q) = \int_{-L/2}^{L/2} \int_{-L'/2}^{L'/2} A \cdot \exp i\phi \cdot \exp i[Px+Qy] \; dxdy$$

$$= A. \exp i\phi \; LL' \; \mathrm{sinc}\, \psi(P) \; \mathrm{sinc}\, \psi'(Q) \quad ,$$

with $\mathrm{sinc}\, \theta = \sin\theta/\theta$, $\psi(P)=PL/2$, $\psi'(Q)=QL'/2$.
Replacing the results above into Eqs.(9) and (10) we abtain

$$E_T(P,Q) = ALL' \exp i\phi \; \mathrm{sinc}\, \psi(P) \; \mathrm{sinc}\, \psi'(Q) \quad , \tag{13}$$

$$I_T(P,Q) = A^2(LL')^2 \; \mathrm{sinc}^2\, \psi(P) \; \mathrm{sinc}^2\, \psi'(Q) \quad . \tag{14}$$

The single long slit

This is a rectangular slit with one of its dimensions very much large (infinity at the limit) in comparison with the other. Thus, from the results above, we have for the amplitude distribution of its diffraction pattern

$$E_T(P,Q) = AL \exp i\phi \operatorname{sinc} \psi(P) \lim_{L'\to\infty} [L' \operatorname{sinc} \psi'(Q)] \quad .$$

But, it is known[12] that

$$\lim_{L'\to\infty} [L' \operatorname{sinc} \psi'(Q)] = \lim_{L'\to\infty} \int_{-L'/2}^{L'/2} \exp iQy \, dy = 2\pi\delta(Q) \quad . \qquad (15)$$

Then,

$$E_T(P,Q) = 2\pi AL \exp i\phi \operatorname{sinc}\psi(P)\delta(Q) \quad .$$

That is, the diffraction pattern will be composed by a series of light fringes defined by sinc $\psi(P)$ in the P-direction (Fig.1) but very narrow (Dirac's delta function at the limit) in the Q-direction. For this reason, it is possible to reduce this case to the unidimensional Fraunhofer diffraction by a single slit by means of the following procedure:

$$E_T(P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(P,Q) \, dQ$$

$$= AL \exp i\phi \operatorname{sinc} \psi(P) \int_{-\infty}^{\infty} \delta(Q) \, dQ \quad .$$

But, by definition[13] the value of the integral into the last expression is the unity, and

$$E_T(P) = AL \exp i\phi \, \mathrm{sinc}\psi(P) \quad ,$$

$$I_T(P) = A^2L^2 \, \mathrm{sinc}^2 \, \psi(P) \quad .$$

The Young experiment

Fraunhofer diffraction produced by two identical slits is called Young experiment. For this case N=2 , M=1 and α is the separation between the slits.

A. With rectangular slits

Let us take two slits whose aperture functions are given by Eq.(12). Thus, from Eq.(9) we obtain for the corresponding pattern

$$E_T(P,Q) = E(P,Q) \exp i\alpha \sin 2\alpha/\sin\alpha \quad ,$$

where E(P,Q) is given by Eq.(13) and $\sin 2\alpha/\sin\alpha = 2\cos\alpha$, $\alpha = Pa/2$ Then,

$$E_T(P,Q) = 2\,ALL' \exp i[\phi+\alpha]\mathrm{sinc}\psi(P) \, \mathrm{sinc} \, \psi'(Q) \, \cos\alpha(P) \quad ,$$

$$I_T(P,Q) = 4\,A^2 \, (LL')^2 \, \mathrm{sinc}^2\psi(P)\,\mathrm{sinc}^2 \, \psi'(Q) \, \cos^2\alpha(P) \quad .$$

B. With long slits

We obtain the description for the pattern relative to this case after the application of the limit process for L'>>L and Eq.(15) into the last results. That is, the amplitude distribution will be

$$E_T(P,Q) = 4\pi \, AL \exp i[\phi+\alpha(P)]\mathrm{sinc}\psi(P) \, \cos\alpha(P) \, \delta(Q) \quad .$$

And, by means of the procedure above, we can reduce it to the unidimensional Young experiment and to write for the pattern description

$$E_T(P) = 2AL \exp i[\phi+\alpha(P)] \quad \mathrm{sinc}\psi(P) \cos\alpha(P) \ ,$$

$$I_T(P) = 4A^2L^2 \ \mathrm{sinc}^2\psi(P) \cos^2\alpha(P) \ .$$

The unidimensional array of rectangular slits

We will have that when M=1 and E(P,Q) is given by Eq.(13) for any aperture. According with Eqs.(9) and (10), the diffraction pattern will be described by

$$E_T(P,Q) = ALL' \exp i[\phi+(N-1)\alpha(P)] \,\mathrm{sinc}\ \psi(P) \ \mathrm{sinc}\ \psi'(Q) \ \sin N\alpha(P)/\sin\alpha(P) \ ,$$

$$I_T(P,Q) = A^2(LL')^2 \ \mathrm{sinc}^2\ \psi(P) \ \mathrm{sinc}^2\ \psi'(Q) \ \sin^2 N\alpha(P)/\sin^2\alpha(P) \quad .$$

The diffraction grating

This is an unidimensional array of long slits. To obtain the corresponding amplitude distribution of the diffraction pattern in this case, let us apply the limit process for L'>>L and Eq.(15) into the last results. That is

$$E_T(P,Q) = 2\pi AL \exp i[\phi+(N-1)\alpha(P)] \ \mathrm{sinc}\ \psi(P) \ \sin N\alpha(P)/\sin\alpha(P) \ . \ \delta(Q) \quad .$$

Once more, using the procedure above, it is possible to reduce this result to an unidimensional expression corresponding to the Fraunhofer diffraction pattern by any grating, as it is usual to find

$$E_T(P) = AL \exp i[\phi+(N-1)\alpha(P)] \ \mathrm{sinc}\ \psi(P) \ \sin N\alpha(P)/\sin\alpha(P) \quad ,$$

$$I_T(P) = A^2L^2 \ \mathrm{sinc}^2\ \psi(P) \ \sin^2 N\alpha(P)/\sin^2\alpha(P) \quad .$$

Finally, we would note that the choice of the limit L'>>L for diffraction from arrays of long slits is not arbitrary. In fact both cases, the Young experiment and the diffraction grating, show the arrays oriented on P-direction but the slits oriented on Q-direction Thus, the reverse possibility, L>>L', is senseless.

CONCLUSION

The elegance of the application of Fourier methods on the study of the Fraunhofer diffraction produced by any regular array of identical apertures has two aspects. The first one is the determination of Eqs.(9) and (10) to calculate the amplitude and intensity distributions of the pattern in any case. To handle them we need to know only the aperture function of a single aperture of the array, the total number of aperture and the separation between any pair of consecutive ones, as we showed after their application of the well known cases. The second one is the understanding of the Fraunhofer diffraction as the Fourier transform of the aperture function of the dispersor. In other words, for such phenomenon the physical space behaves as a Fourier transform operator.

ACKNOWLEDGMENTS

The author is sincerely grateful with Dr. Peter Barlai (Group of Optics and Information Theory, Department of Physics, Universidad de Antioquia) for his important comments about the contents of this paper.

REFERENCES

1. H.P. Hsu, *Analisis de Fourier,* Fondo Educativo Interamericano, México (1973) p. 219.
2. M. Born and E. Wolf, *Principles of optics,* 6th ed. Pergamon Press, Oxford, (1980) p. 386.
3. E. Hecht y A. Zajac, *Optica,* Fondo Educativo Interamericano, México (1977) p. 437.
4. Born and Wolf, *op. cit.* p.483.
5. Born and Wolf, *op. cit.* p.386.
6. H.P. Hsu, *op. cit.* p.94
7. Born and Wolf. *op. cit.*p.504.

8. Hecht y Zajac, *op. cit.* p.355
9. H.P. Hsu, *op.cit.* p.88; Born and Wolf. *op. cit.* p. 483, 485
10. H.P. Hsu. *op. cit.* p. 37, 102
11. Born and Wolf. *op. cit.* p. 392 ss
12. H.P. Hsu, *op. cit.* p. 102.
13. H.P. Hsu, *op. cit.* p. 37.