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A THERMAL ENGINE CONNECTED TO FINITE RESERVOIRS

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ABSTRACT

The performance of a turbine-type engine connected to finite heat sources and the environment is optimized, under the criterion of maximum work delivered during a fixed period of operation. Time is introduced considering heat transfers through walls with finite conductivities.

RESUMEN

Se optmimiza el desempeño de un motor tipo turbina conectado a fuentes de calor finitas y el medio ambiente, bajo el criterio de máximo trabajo entregado durante un período fijo de operación. Se introduce el tiempo considerando transferencias de calor a través de paredes con conductividades finitas.

I. THE PROBLEM

The study of thermal processes taking into account their finite duration and avoiding microscopic equations for the fluxes involved has attracted attention lately, with the main lines of argument centering around the optimization of cyclical engines whose heat transfers take place through walls with finite conductivities[1]. Our problem deals with the optimization of a continuously operating engine, a turbine, under the criterion of maximal work output when it undergoes a process whose duration and initial state are known.

States of the system are defined by the temperatures T, T_1, T_h, T_ℓ of its different parts (see Fig. 1), and , given the time of operation and the values of T and T_1 at the beginning of the process, the work output W can be affected through manipulation from outside the system of the conductivities k, k_h, k_ℓ, the heat capacities C, C_1, and, most importantly for us, the engine temperatures T_h and T_ℓ. Our task consists then of determining the externally controlled parameters that will maximize the work output; with finite reservoirs involved they will be functions of time.

II. CALCULATIONS AND RESULTS

Our system is characterized by the heat conduction equations (cf. Fig. 1 for the meaning of symbols),

$$\left.\begin{aligned} \dot{Q} &= k(T_1 - T) \\ \dot{Q}_1 &= k_h(T - T_h) \\ \dot{Q}_2 &= k_\ell(T_\ell - T_0) \end{aligned}\right\} \qquad (1)$$

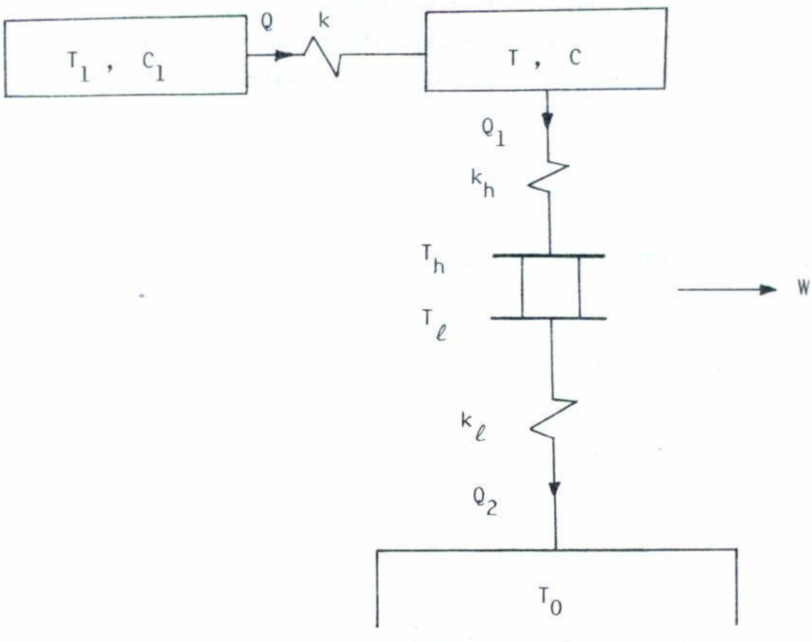

Fig. 1. The system.

and the equations for the reservoirs' temperatures,

$$\left.\begin{aligned} C_1\dot{T}_1 &= k(T - T_1) \\ C\dot{T} &= k(T_1 - T) - k_h(T - T_h) \end{aligned}\right\} \quad . \tag{2}$$

The quantity we want to maximize is the work

$$W = \int_0^{t_f} (\dot{Q}_1 - \dot{Q}_2)dt = \int_0^{t_f} [k_h(T - T_h) - k_\ell(T_\ell - T_0)]\, dt \quad , \tag{3}$$

with given t_f, $T_i = T(t = 0)$ and $T_{1i} = T_1(t = 0)$.

Before going any further it will be convenient to write all expressions in terms of dimensionless quantities, using ratios of temperature, conductivities and heat capacities, and measuring time and energy in units t_0, W_0 that are natural to the system:

$$t_0 = \frac{C}{k_\ell} \quad , \qquad W_0 = CT_0 \quad . \tag{4}$$

Our dimensionless quantities will be

$$x = \frac{T}{T_0} \quad , \quad x_1 = \frac{T_1}{T_0} \quad , \quad x_\ell = \frac{T_\ell}{T_0} \quad , \quad x_h = \frac{T_h}{T} \quad , \tag{5}$$

$$\alpha = \frac{k_h}{k_\ell} \quad , \quad \beta = \frac{k}{k_\ell} \quad , \quad \gamma = \frac{C}{C_1} \quad , \tag{6}$$

$$\tau = \frac{t}{t_0} \quad , \quad \omega = \frac{W}{W_0} \quad , \tag{7}$$

in terms of which Eqs. (2) and (3) become

$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= \beta\gamma(x - x_1) \\ \frac{dx}{d\tau} &= \beta(x_1 - x) - \alpha x(1 - x_h) \end{aligned}\right\} \quad , \tag{8}$$

$$\omega = \int_0^{t_f} \left[\alpha x(1 - x_h) + 1 - x_\ell\right] d\tau \quad . \tag{9}$$

In a physical process all heat transfers are such that entropy increases with time. Applied to the engine, this leads to a further condition on its internal temperatures,

$$\frac{\dot{Q}_1}{T_h} - \frac{\dot{Q}_2}{T_\ell} \geq 0$$
$$\Longrightarrow \alpha\left(\frac{1}{x_h} - 1\right) + \frac{1}{x_\ell} - 1 \geq \quad . \tag{10}$$

It has become customary to use the equality sign in (10), and so introduce the requirement of "endoreversibility" for the engines considered[1]:

$$\alpha\left(\frac{1}{x_h} - 1\right) + \frac{1}{x_\ell} - 1 = 0 \quad . \tag{11}$$

This will be our condition, too, although any other combination of T_h and T_ℓ that satisfied (10), $\alpha\left(\frac{1}{x_h} - 1\right) + 2\left(\frac{1}{x_\ell} - 1\right) = 0$ for example, would have been equally acceptable.

The work output was maximized under constraints (8) and (11) using optimal control theory, which involves the following steps[2]:

(i) Define a "Hamiltonian" (cf. Eqs. (8),(9),(11))

$$H(\vec{x}(\tau), \vec{u}(\tau), \vec{\psi}(\tau), \mu(\tau)) = \left\{\alpha x(1 - x_h) + 1 - x_\ell\right\} +$$
$$+ \psi_1\left[\beta\gamma(x - x_1)\right] + \psi\left[\beta(x_1 - x) - \alpha x(1 - x_h)\right] - \mu\left[\alpha\left(\frac{1}{x_h} - 1\right) + \frac{1}{x_\ell} - 1\right] \tag{12}$$

with "state variables" $\vec{x} = (x, x_1)$, "control variables" $\vec{u} = (x_h, x_\ell, \alpha, \beta, \gamma)$, "co-state variables" $\vec{\psi} = (\psi, \psi_1)$ and a Lagrange multiplier μ.

(ii) Determine the optimal control parameters on which H depends linearly using Pontryagin's maximum principle,

$$H(\vec{x}, \vec{u}^*, \vec{\psi}, \mu) \geq H(\vec{x}, \vec{u}, \vec{\psi}, \mu) \qquad , \tag{13}$$

where $\vec{u}^*$ denotes optimal values.

(iii) Solve the "equations of motion"

$$\frac{\partial H}{\partial x} = -\frac{d\psi}{d\tau} \quad , \quad \frac{\partial H}{\partial x_1} = -\frac{d\psi_1}{d\tau} \quad , \quad \frac{\partial H}{\partial x_h} = 0 \quad , \quad \frac{\partial H}{\partial x_\ell} = 0 \quad , \tag{14}$$

subject to the boundary conditions

$$\psi(T_f) = 0 \qquad , \qquad \psi_1(\tau_f) = 0 \quad . \tag{15}$$

Hence the solution to Eqs. (8), (11), (14) will involve the values of the state variables at the boundary opposite that on which they are known. We will deal with this problem later.

The equations of motion are

$$\begin{aligned} \frac{d\psi}{d\tau} &= \alpha(1 - x_h)(\psi - 1) + \beta(\psi - \gamma\psi_1) \quad , \\ \frac{d\psi_1}{d\tau} &= -\beta(\psi - \gamma\psi_1) \quad , \\ x(\psi - 1) &+ \left(\frac{x_\ell}{x_h}\right)^2 = 0 \quad , \\ \mu &= x^2_\ell \quad . \end{aligned} \tag{16}$$

It was found convenient to express the system to be solved, Eqs. (8), (11), (16), in terms of two variables,

$$r_1 = \frac{x_1}{x} = \frac{T_1}{T}$$

and (17)

$$r_2 = \frac{x_\ell}{x_h} \quad \frac{T\,T_\ell}{T_0\,T_h} \qquad ,$$

one of which will afterwards be eliminated using the expression for H.

The only difficulty in carrying through this change of variables comes from the equation for $\frac{d\psi_1}{d\tau}$, which was integrated starting from the observation that combining Eqs. (8) with the first pair in (16),

$$\frac{d}{d\tau}\left[x_1\psi_1 + x\psi - x - \frac{x_1}{\gamma}\right] = 0 \tag{18}$$

$$\Longrightarrow x_1\psi_1 + x\psi - x - \frac{x_1}{\gamma} = \lambda = \text{constant}. \tag{19}$$

Hence our system becomes

$$
\begin{aligned}
x_h &= \frac{\alpha}{1 + \alpha}\left(1 + \frac{1}{\alpha r_2}\right) \qquad , \\
x_\ell &= \frac{1 + \alpha r_2}{1 + \alpha} \qquad , \\
x_1 &= x r_1 \qquad , \\
\frac{d(\ln x)}{d\tau} &= \beta(r_1 - 1) - \frac{\alpha}{1 + \alpha}\left(1 - \frac{1}{r_2}\right) \qquad , \\
\psi &= 1 - \frac{r^2{}_2}{x} \\
\psi_1 &= \frac{1}{\gamma} + \frac{1}{x r_1}\,(\lambda + r_2^2) \qquad ,
\end{aligned} \tag{20}
$$

in which only one is a differential equation.

To eliminate r_2, use Eqs. (20) to obtain

$$H = \frac{\alpha}{1+\alpha}(r_2 - 1)^2 + \beta(1 - r_1)\left[\frac{\gamma\lambda}{r_1} + r_2^2\left(1 + \frac{\gamma}{r_1}\right)\right] \quad , \tag{21}$$

which is a second degree equation for r_2, with solutions

$$r_2 = \frac{B_1}{2A_1} \pm \frac{1}{2A_1}\sqrt{B_1^2 - 4A_1D_1} \quad , \tag{22}$$

where

$$A_1 = \frac{\alpha}{1+\alpha} + \beta(1 - r_1)\left(1 + \frac{\gamma}{r_1}\right) \quad ,$$

$$B_1 = \frac{2\alpha}{1+\alpha} \quad , \tag{23}$$

$$D_1 = \frac{\alpha}{1+\alpha} + \frac{\beta\gamma\lambda}{r_1}(1 - r_1) - H \quad .$$

The correct sign in Eq. (22) will be determined later, using the boundary conditions (Eq. (15)).

All our variables are now in terms of r_1 and the constants λ and H, and these will be related to T_f and T_{1f} using one of the boundary conditions:

$$\psi_1(\tau_f) = 0 \quad \Longrightarrow \quad \lambda = -\left[\frac{T_f}{T_0} + \frac{1}{\gamma}\,\frac{T_{1f}}{T_0}\right] \quad . \tag{24}$$

Evaluating H for $\tau = \tau_f$ then leads to

$$H = \frac{\alpha}{1+\alpha}\left[\left(\frac{T_f}{T_0}\right)^{\frac{1}{2}} - 1\right]^2 \quad . \tag{25}$$

The other boundary condition implies that at the end of the interval of operation the optimal engine works maximizing the instantaneous power output, as if connected to an infinite heat source:

$$\psi(\tau_f) = 0 \quad \Longrightarrow \quad r_{2f}^{\;2} = x_f \tag{26}$$

$$\Longrightarrow \quad \frac{T_{\ell f}}{T_{hf}} = \left(\frac{T}{T_f}\right)^{\frac{1}{2}} \quad . \tag{27}$$

If one considers a turbine connected to an infinite reservoir at temperature T, a straightforward calculation gives $\frac{T_\ell}{T_h} = \left(\frac{T_\ell}{T}\right)^{\frac{1}{2}}$ for its internal temperatures that will maximize instantaneous power output.

The above is not a surprising result, because according to Eqs. (2) changes in T will be irrelevant for times much shorter than the smaller of $\left\{\frac{C}{k}, \frac{C}{k_h}\right\}$.

From Eqs. (20) the results for the temperatures are

$$\begin{aligned}
T &= T_i \exp\left\{\int_{r_{1i}}^{r_1(\tau)} dr_1 \left[\frac{dr_1}{d\tau}\right]^{-1} \left[\beta(r_1 - 1) - \frac{\alpha}{1+\alpha}\left(1 - \frac{1}{r_2}\right)\right]\right\} \quad , \\
T_1 &= T r_1 \quad , \\
T_h &= T\left[1 - \frac{r_2 - 1}{r_2(1+\alpha)}\right] \quad , \\
T_\ell &= T_0 \left[1 + \frac{\alpha}{1+\alpha}(r_2 - 1)\right] \quad .
\end{aligned} \tag{28}$$

All these temperatures are functions of time, through r_2 given by Eq. (22) and $r_1(\tau)$ obtained from

$$\int_{r_{1i}}^{r_1(\tau)} dr_1 \left[\frac{dr_1}{d\tau}\right]^{-1} = \int_0^\tau d\tau' = \tau \quad , \tag{29}$$

where $r_{1i} = \frac{T_{1i}}{T_i}$ and

$$\frac{dr_1}{d\tau} = \beta(1 - r_1)(\gamma + r_1) + \frac{\alpha r_1}{1+\alpha}\left(1 - \frac{1}{r_2}\right) \tag{30}$$

from Eqs. (8).

As mentioned before, the solution involves T_f and T_{1f}, the values of T and T_1 at the end of the interval of operation. This calls for a trial and error procedure in which T_f and T_{1f} are assigned arbitrary values, the functions $T(\tau)$ and $T_1(\tau)$ calculated from Eqs. (22)-(25), (28)-(30) with the correct sign in (22) coming from Eq. (26), and the whole process repeated until $T(\tau_f)$ and $T_1(\tau_f)$ coincide with the chosen values of T_f and T_{1f}.

This is a straighforward but time consuming enterprise, which can be considerably shortened if one is interested in analyzing optimal processes with given T_i and T_{1i}, the exact value of τ_f being of little concern. In this case one has to search for only one correct parameter, T_f or T_{1f}; the other one in the pair fixes τ_f through Eq. (29). The graphs in Figs. 2 and 3 were obtained this way.

When one is considering cases with $\beta = \frac{k}{k_\ell} << 1$, that is, small coupling between the upper reservoirs, a perturbative calculation gives the temperatures explicitly as functions of time, in the form $Y(\tau) = Y_0(\tau) + \beta Y_1(\tau) + \beta^2 Y_2(\tau) + \ldots$, where Y is a temperature Y_0 its value for $\beta = 0$, *i.e.*, with the upper reservoirs decoupled. The broken lines in Fig. 3 correspond to a solution to first order in β; the errors involved are of order β because one is expanding from a known function $Y_0(\tau)$, not an extremal solution with respect to variations of β. The fit does not seem impressive for $\beta = 0.1$, but the approach is still useful considering the fact that one gets explicit time dependences, and the time involved in obtaining a numerical solution, even when searching for only one of T_f and T_{1f} according to the preceding paragraph. The details and results are in the Appendix.

The plots in Figs. 2 and 3 represent optimal processes for the case of fixed conductivities and heat capacities. If one is able to modifiy these quantities during the time of operation then the work output will be further increased varying them according to the results of Pontryagin's maximum principle, which requires that these quantities alternatively take their maximum or minimum possible values during certain periods along the process:

Spelling out condition (13) it implies, from Eq. (21),

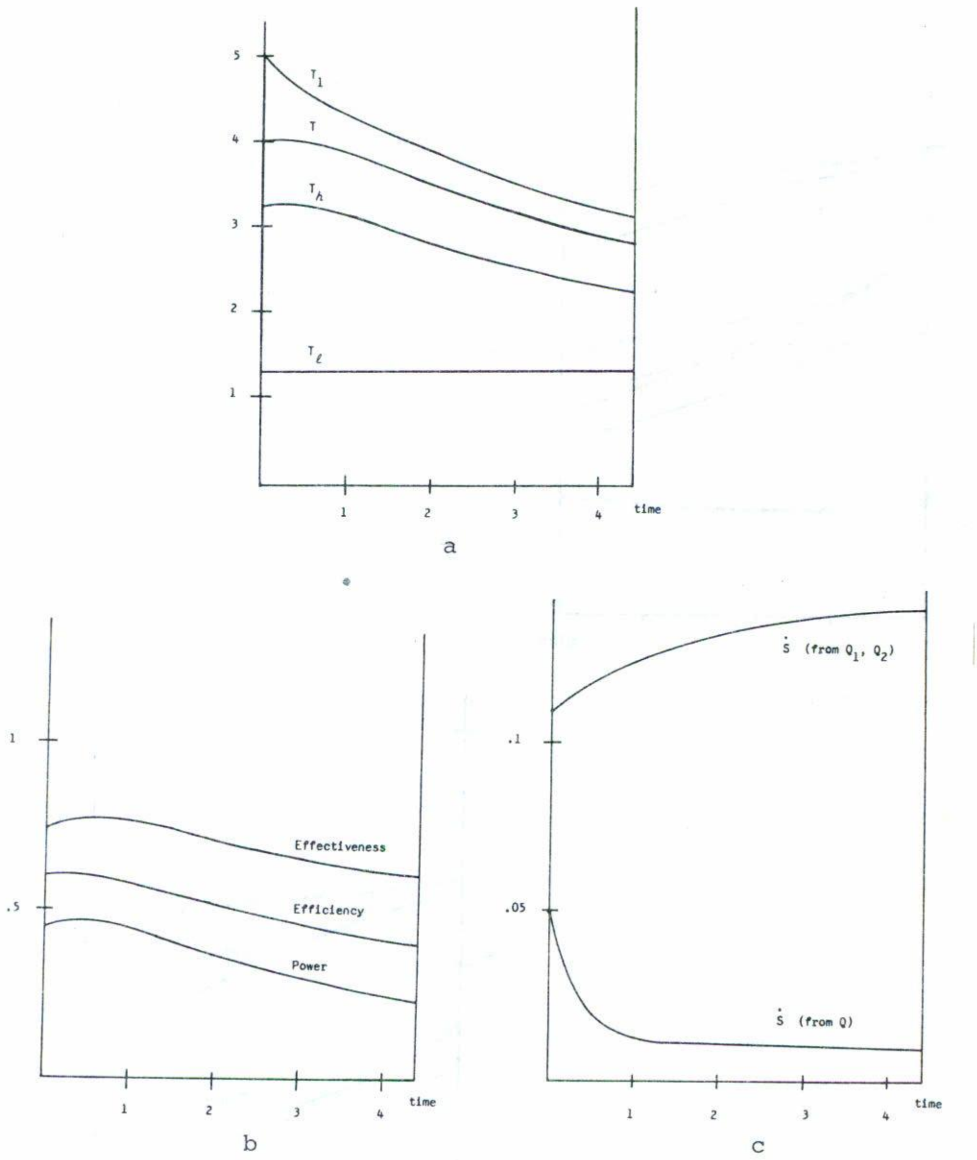

Fig. 2. Optimal process with $T_i = 4$, $T_{1i} = 5$, $\alpha = \beta = \gamma = 1$. All temperatures in units of T_0. The instantaneous effectiveness was defined as $e = \frac{P}{P + \dot{S}}$, where both the power P and the rate of entropy production $\dot{S}$ are in units of $K_\ell T_0$; this is bases on the usual expression $e = \frac{W}{W + T_0 \dot{S}}$ (cf. Ref. 3). In (c) $\dot{S}$ (from q) means rate of entropy production due to the flow of heat q.

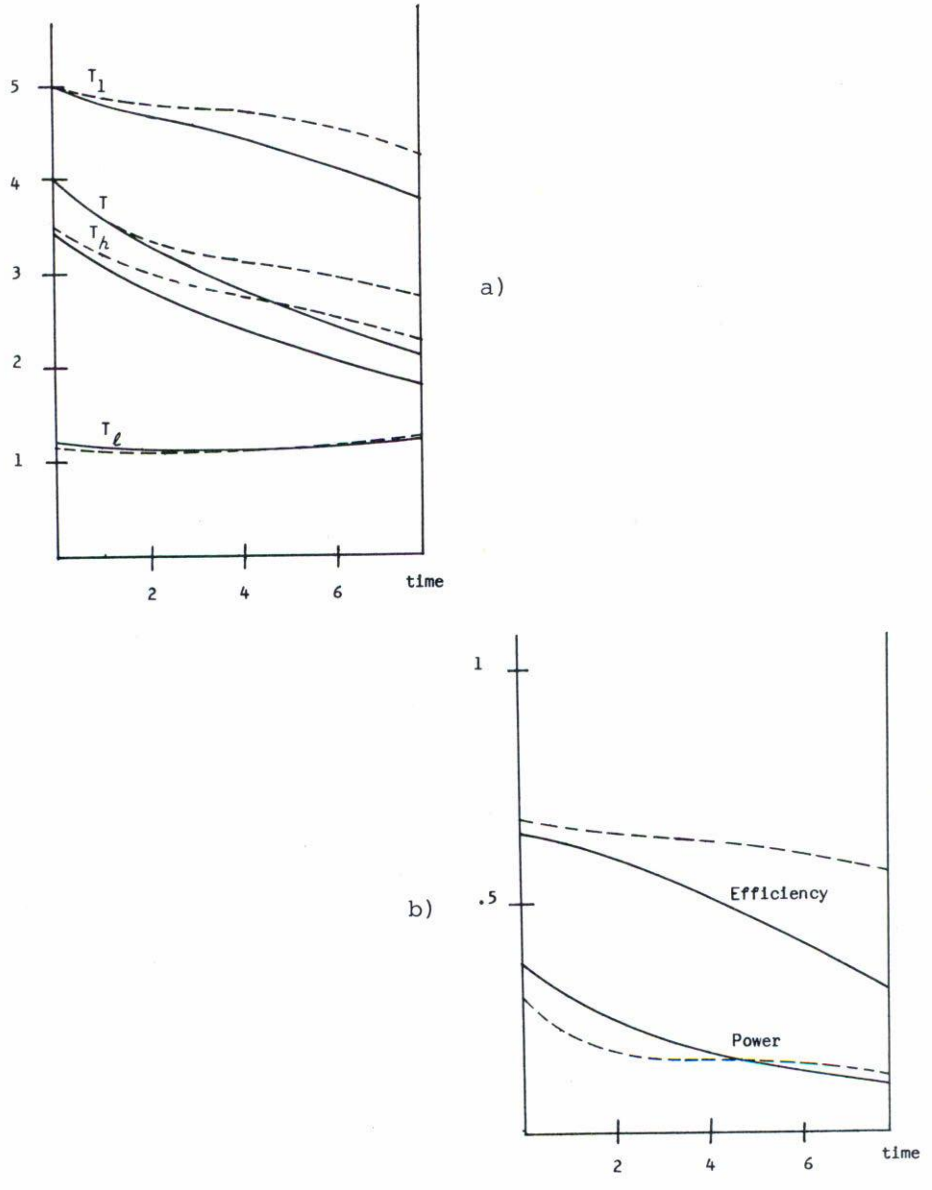

Fig. 3. Optimal process with the same input parameters as in Fig. 2, except that now $\beta = 0.1$. Broken lines refer to a perturbative solution to first order in β (cf. Appendix).

$$\text{(i)} \qquad \alpha = \frac{k}{k_\ell} = \begin{cases} \alpha_{max} \\ \alpha_{min} \\ \text{indeterm.} \end{cases} \qquad \text{for} \quad (r_2 - 1)^2 \longrightarrow \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \tag{31}$$

Physical requirements on the temperatures (cf. last two equations in set (28)) imply $r_2 > 1$, so α must always have its maximum possible value.

$$\text{(ii)} \qquad \beta = \frac{k}{k_\ell} = \begin{cases} \beta_{max} \\ \beta_{min} \\ \text{indet.} \end{cases} \quad \text{for} \quad \left(1 - \frac{T_1}{T}\right) \left[1 - \left(\frac{T_0}{T}\right)\left(\frac{T_h}{T_\ell}\right)^2 \left(\frac{T_{1f} + \gamma T_f}{T + \gamma T}\right)\right] \to \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} .$$

(32)

$$\text{(iii)} \qquad \gamma = \frac{C}{C_1} = \begin{cases} \gamma_{max} \\ \gamma_{min} \\ \text{indet.} \end{cases} \quad \text{for} \quad \left[1 - \left(\frac{T_f}{T_0} + \frac{1}{\gamma}\frac{T_{1f}}{T_0}\right) \left(\frac{T_h}{T}\right)^2 \left(\frac{T_0}{T_\ell}\right)^2 \right] \to \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \tag{33}$$

Summing up: To maximize the work output of the system vary the engine temperatures T_h and T_ℓ according to Eqs. (28), and adjust the conductivities and heat capacities so that conditions (31-33) are satisfied along the process.

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APPENDIX. A perturbative approach

The starting point for a perturbation expansion in powers of β is provided by the differential equations for r_1 and r_2:

$$\frac{dr_1}{d\tau} = \frac{\alpha r_1}{1+\alpha}\left(1 - \frac{1}{r_2}\right) + \beta(1 - r_1)(\gamma + r_1) \quad ,$$

$$\frac{dr_2}{d\tau} = \frac{\beta}{2r_1 r_2}\left[\gamma\lambda + r_2^2(\gamma + r_1^2)\right] \quad , \tag{A-1}$$

where $\frac{dr_1}{d\tau}$ is the same as in Eq. (30) and $\frac{dr_2}{d\tau}$ comes from Eq. (21) and the condition $\frac{dH}{d\tau} = 0$ along an optimal path[2].

Define $r_1^{(0)}$ and $r_2^{(0)}$ through equations (A-1), with $\beta = 0$; then define $r_1^{(n)}$ and $r_2^{(n)}$ for $n > 1$:

$$\frac{dr_1^{(n)}}{d\tau} = \frac{\alpha r_1^{(n-1)}}{1+\alpha}\left[1 - \frac{1}{r_2^{(n-1)}}\right] + \beta\left[1 - r_1^{(n-1)}\right]\left[\gamma + r_1^{(n-1)}\right] \quad ,$$

$$\frac{dr_2^{(n)}}{d\tau} = \frac{\beta}{2r_1^{(n-1)} r_2^{(n-1)}}\left[\gamma\lambda + (r_2^{(n-1)})^2\left(\gamma + (r_1^{(n-1)})^2\right)\right] \quad , \tag{A-2}$$

keeping powers up to β^n on the right hand side. The resulting equations have solutions involving $r_1^{(n)}$ $(\tau = 0)$ and $r_2^{(n)}$ $(\tau = 0)$; determine $r_1^{(n)}(\tau = 0)$ from

$$r_1^{(n)}(\tau = 0) = \frac{T_{1i}}{T_i} \tag{A-3}$$

and, lacking a better handle on it, define

$$r_2^{(n)}(\tau = 0) = r_2^{(0)} \tag{A-4}$$

where $r_2^{(0)}$ is the solution to (cf. Eq. (26))

$$2\ln r_2^{(0)} = \ln\left(\frac{T_i}{T_0}\right) - \frac{\alpha T_f}{1+\alpha}\left(1 - \frac{1}{r_2^{(0)}}\right) \quad , \tag{A-5}$$

$r_1^{(n)}$ and $r_2^{(n)}$ are then obtained in terms of T_i, T_{1i} and τ_f, the input parameters.

Define now $T^{(n)}$, $T_1^{(n)}$, $T_h^{(n)}$ and $T_\ell^{(n)}$ substituting $r_1^{(n)}$ and $r_2^{(n)}$ into Eqs. (28) and keeping powers up to β^n. The results to first order in β are:

$$T^{(1)} = T^{(0)} + \beta T^{(0)} L \quad ,$$

$$T_1^{(1)} = T_1^{(0)} + \beta T_i \left[r_{1i} L + D e^{-A\tau} \right] \quad ,$$

$$T_h^{(1)} = T_h^{(0)} + \beta \left[\frac{T^{(0)}}{1 + \alpha} \right] \left[\left(\alpha + \frac{1}{r_2^{(0)}} \right) L - \frac{E}{(r_2^{(0)})^2} \right] \quad , \qquad \text{(A-6)}$$

$$T_\ell^{(1)} = T_\ell^{(0)} + \beta \left[\frac{\alpha T_0 E}{1 + \alpha} \right] \quad ,$$

with

$$T^{(0)} = T_i e^{-A\tau} \quad ,$$

$$T_1^{(0)} = T_{1i} \quad ,$$

$$T_h^{(0)} = T^{(0)} \left[1 - \frac{r_2^{(0)} - 1}{r_2^{(0)} (1 + \alpha)} \right] \quad , \qquad \text{(A-7)}$$

$$T_\ell^{(0)} = T_0 \left[1 + \frac{\alpha (r_2^{(0)} - 1)}{1 + \alpha} \right] \quad ,$$

and

$$A = \frac{\alpha (r_2^{(0)} - 1)}{r_2^{(0)} (1 + \alpha)} \quad ,$$

$$D = \gamma\tau + \frac{(1 - \gamma) r_{1i}}{A} \left(e^{A\tau} - 1 \right) - \frac{r_{1i}^2}{2A} \left(e^{2A\tau} - 1 \right) \quad , \qquad \text{(A-8)}$$

$$E = \left[\frac{1}{2Ar_2^{(0)}}\right]\left[\left(1 - e^{-A\tau}\right)\left(\frac{\gamma}{r_{1i}}\right)\left(\lambda + (r_2^{(0)})^2\right) + r_{1i}(r_2^{(0)})^2\left(e^{A\tau} - 1\right)\right] \quad ,$$

$$L = -\left[1 + \left(\frac{\alpha}{1+\alpha}\right)\left(\frac{1}{2A(r_2^{(0)})^3}\right)\left[\frac{\gamma}{r_{1i}}\left(\lambda + (r_2^{(0)})^2\right) - r_{1i}(r_2^{(0)})^2\right]\right]\tau +$$

$$+ \left(e^{A\tau} - 1\right)\left[\frac{r_{1i}}{A} - \left(\frac{\alpha}{1+\alpha}\right)\frac{r_{1i}(r_2^{(0)})^2}{2A^2(r_2^{(0)})^3}\right] + \qquad \text{(A-9)}$$

$$+ \left(1 - e^{-A\tau}\right)\left(\frac{\alpha}{1+\alpha}\right)\left[\frac{\gamma\left(\lambda + (r_2^{(0)})^2\right)}{2A^2 r_{1i}(r_2^{(0)})^3}\right] \quad .$$

The power $P = \alpha x(1 - x_h) + 1 - x_\ell$ (in units of $k_\ell T_0$; cf. Eq. (9)) and the efficiency $\eta = 1 - \frac{T_\ell}{T_h}$ then readily follow to first order in β.

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