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HELMHOLTZ CONDITIONS AND THE TRACE THEOREM IN CLASSICAL MECHANICS*

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ABSTRACT

The Trace Theorem in classical mechanics is that the time derivative of the trace of any power of a certain matrix M vanishes, the matrix M being formed when two Lagrangians L and L' are equivalent in the sense of having the same set of solutions to the equations of motion derived from them. The Helmholtz conditions are conditions on a set of equations which show when they may be derived from a Lagrangian. We show that the Trance Theorem follows from a proper subset of the Helmholtz conditions and discuss some of the implications for the problem of finding a Lagrangian L' which is equivalent to a given Lagrangian L.

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RESUMEN

El teorema de las trazas en mecánica clásica establece que la derivada temporal de la traza de cualquier potencia de la matriz M es cero, donde la matriz M se forma a partir de dos Lagrangianos L y L' que son equivalentes en el sentido de que las ecuaciones de movimiento derivadas de ellos tienen el mismo conjunto de soluciones. Las condiciones de Helmholtz son condiciones en un conjunto de ecuaciones que establecen cuando pueden ser derivadas de un lagrangiano. Mostramos que el teorema de las trazas es implicado por un subconjunto propio de las condiciones de Helmholtz y discutimos algunas de las implicaciones para el problema de encontrar un lagrangiano L' equivalente a un lagrangiano L.

1. INTRODUCTION

The Trace Theorem says that the trace of any power of a certain matrix M must be a constant of the motion, this matrix being formed when two Lagrangians L and L' are equivalent in the sense defined below[1,2]. The purpose of this paper is to elucidate the role which the Helmholtz Conditions[3,4] play in this paper. The Helmholtz conditions are a set of necessary and sufficient conditions on a set of differential equations which ensure that they be derivable from a variational principle. We will show that the Trace Theorem is derived from a proper subset of the Helmholtz conditions.

Consider a set of particles whose configuration space is $\mathbb{R}^N$, with Cartesian coordinates $\{q^i\}$, $i = 1,\ldots,N$. In what follows we will make use of integration theorems which are valid in $\mathbb{R}^N$, and we will not treat the case where the configuration space manifold has a topology different from $\mathbb{R}^N$. A trajectory of the system is defined by the functions $q^i(t)$, t being time, and the components of the velocity are $\dot{q}^i = dq^i/dt$.

A Lagrangian $L(q,\dot{q},t)$, when used in a variational principle, results in equations of motion $E_iL = 0$, where the Euler derivatives are defined by

$$E_iL \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \quad . \tag{1}$$

It will be convenient to write E_iL in the form

$$E_iL = W_{ij}(\ddot{q}^j - f^j) \quad , \tag{2}$$

where the mass matrix W_{ij} and the force vector f^j are defined by

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad , \tag{3}$$

$$W_{ij}fi \equiv \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial t} \quad . \tag{4}$$

We assume that the matrix $W = (W_{ij})$ is non-singular. The inver-verse of W will be written $W^{-1} = (W^{ij})$, so that $W^{is}W_{sj} = \delta^i{}_j$. We call W' the mass matrix defined from L'. The transition matrix $M = (M^i{}_j)$ is defined by[1]

$$M = W' \, W^{-1} \quad . \tag{5}$$

Two Lagrangians L and L' are equivalent if the two sets fo equations of motion $E_iL = 0$ and $E_iL' = 0$ have the same set of solutions. The Trace Theorem states: If L and L' are equivalent, then the trace of any power of M is a constant of the motion.

2. PROOF OF THE TRACE THEOREM

We will prove this theorem by making use of some of the Helmholtz Conditions. These conditions state when a set of differential equations is derivable from a variational principle. We presume that the equations are of the form

$$W_{ij}(\ddot{q}^j - f^j) = 0.$$

The conditions are

$$\text{(H1)}: \; W_{ij} = W_{ji} \quad , \tag{6}$$

$$\text{(H2)}: \; \frac{D}{dt} W_{ij} = -\frac{1}{2}\left[W_{ij}\frac{\partial f^k}{\partial \dot{q}^j} + W_{jk}\frac{\partial f^k}{\partial \dot{q}^i}\right] \quad , \tag{7}$$

$$\text{(H3)}: \; \frac{1}{2}\frac{D}{dt}\left[W_{ik}\frac{\partial f^k}{\partial \dot{q}^j} - W_{jk}\frac{\partial f^k}{\partial \dot{q}^i}\right] = \left[W_{ik}\frac{\partial f^k}{\partial q^j} - W_{jk}\frac{\partial f^k}{\partial q^i}\right] \quad , \tag{8}$$

$$\text{(H4)}: \; \frac{\partial W_{ij}}{\partial \dot{q}^k} = \frac{\partial W_{ik}}{\partial \dot{q}^j} \quad . \tag{9}$$

We have used the derivative $\frac{D}{dt}$ defined by

$$\frac{D}{dt} = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + f^k \frac{\partial}{\partial \dot{q}^k} \quad . \tag{10}$$

It will be convenient to write the first three conditions in matrix form. In addition to the W matrix, we define the matrices $F = (F^i{}_j)$ and $E = (E^i{}_j)$ by

$$F^i{}_j \equiv -\frac{1}{2} \; \partial f^i / \partial \dot{q}^j \quad , \tag{11}$$

$$E^i{}_j \equiv - \partial f^i / \partial q^j \quad . \tag{12}$$

Then (H1), (H2), and (H3) are

$$\text{(H1)}: \; W = W^T \quad ,$$

$$\text{(H2)}: \; \frac{D}{dt} W = WF + F^T W \quad ,$$

$$(H3): \frac{D}{dt}[WF - F^TW] = WE - E^TW \quad ,$$

where the superscript T means matrix transpose.

It is interesting to note[4] that (H2) and (H3) may be combined to yield

$$(H3_0): WA_0 = A_0^TW \quad , \tag{13}$$

where

$$A_0 \equiv E - F^2 - \frac{D}{dt}F \quad . \tag{14}$$

By use of (H2) and (H3), it is easy to show iteratively that W obeys the relations

$$(H3_k): WA_k = A_k^TW \quad , \tag{15}$$

where

$$A_{k+1} \equiv \frac{D}{dt}A_k + FA_k - A_kF \quad . \tag{16}$$

These supplementary conditions are useful for studying the problem of finding Lagrangians but will not be of direct use here.

It is a standard proof to show that the Helmholtz conditions (H1), (H2), (H3), and (H4) are necessary and sufficient conditions for the set of differential equations defined by W and f^i to be derivable from a Lagrangian L. If L' is a second Lagrangian, equivalent to L, then its force vector must be identically equal to f^i (in fact, this property defines the equivalence of L and L'). Thus E, F, and A_k are identically equal for L and L'. The Helmholtz conditions must hold for W' also, of course, and we now proceed to prove the Trace Theorem using only (H2).

Proof of the Trace Theorem: Recall that M is defined by $M=W'W^{-1}$. Then the time derivative of M is

$$\frac{D}{dt} M = [\frac{D}{dt}W']W^{-1} - M[\frac{D}{dt}W]W' \quad . \tag{17}$$

It is a direct consequence of (H2) and (H2') (for W') that

$$\frac{D}{dt} M = F^T M - MF^T \quad . \tag{18}$$

It is clear that $tr(F^T M - MF^T) = 0$ (tr meaning trace), so that tr(M) is a constant of the motion. The derivative of any power M^s is a sum of terms of the form

$$\frac{D}{dt} M^s = \sum_{k=0}^{s-1} M^{s-k-1} [\frac{D}{dt}M]M^k \quad . \tag{19}$$

The trace of this equation is therefore a sum of terms of the form $tr(F^T M^s - M^s F^T)$. These terms are all zero, and thus we have shown that $tr(M^s)$ is a constant of the motion. We call such a matrix M an effectively constant matrix.

3. CONCLUSIONS

We thus see that the trace theorem does not use all of the information contained in the Helmholtz conditions. Therefore a particular M whose powers are constants of the motion (for example, M might have entries all of which are constants of the motion) does not guarantee that $MW = W'$ can be used to find an equivalent Lagrangian.

To be more specific, suppose L is used to find the equations of motion

$$E_i L = W_{ij}(\ddot{q}^j - f^j) = 0 \quad , \tag{20}$$

and suppose M is an effectively constant matrix. We define W' by

$$W' = MW \tag{21}$$

and ask whether the equations

$$W'_{ij}(\ddot{q}^j - f^j) = 0 \tag{22}$$

arise from a Lagrangian L'. We do assume that M has been chosen so that W' is symmetric, that is, so that (H1) is satisfied.

Condition (H2) for $W' = MW$ can then be read as a condition on M; this condition is simply

$$\frac{D}{dt} M = F^T M - MF^T \quad , \tag{23}$$

which is stronger than the condition that M be effectively constant. Conditions $(H3_k)$ for W' read as conditions on M:

$$M A_k{}^T = A_k{}^T M \quad , \tag{24}$$

so that M must commute with all of the matrices $A_k{}^T$. In addition, condition (H4) must be satisfied, and it is only under all of these conditions when the Eqs. (22) are derivable from a Lagrangian L'.

Finally it should be noted that although we have concerned ourselves with systems with a finite number of degrees of freedom, the Trace Theorem has been proved for field theory[5,6]. The Helmholtz conditions for field theory are much more difficult to apply than those for the systems we have considered[7]. Further, systems with constraints, especially systems of fields with constraints, pose further difficulties. We are continuing the study of these systems, for the possible nonuniqueness of Lagrangians represents a so-far unresolved difficulty in the relation of classical theories to quantum theories.

REFERENCES

1. S.Hojman and H.Harleston, *J.Math. Phys.* 22 (1981) 1414.
2. S.Hojman and L.C. Shepley, *Rev. Mex. Fis.* 28 (1982) 149.
3. P. Havas, *Supp. Nuovo Cimento* 5 Ser. X (1957) 363.
4. W. Sarlet, *J. Phys. A: Math. Gen.* 15 (1982) 1503
5. J.R. Farias and N.L. Teixeira, *J. Phys. A: Math. Gen.* 16 (1983) 1517.
6. S.Hojman and L.C. Shepley, to appear in *Found. Phys.* (1986). "Equivalent Lagrangians in Classical Field Theory".
7. R.M. Santilli, *Ann. of Phys.* 103 (1977) 354.