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ANOTHER WAY OF USING BERNOULLI'S METHOD

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ABSTRACT

The Bernoulli's method, used in brachistochrone problems, is presented in a different way. Instead of the Snell's law we use the ray equation, whose integration seems easier when some kind of symmetry is present. We apply it in two examples: the classical brachistochrone problem and in the cases with gravitational potential energies of the form $V(r) = \alpha r^n$.

RESUMEN

Se presenta de una manera diferente el método de Bernovilli, que es usado para el problema de la braquistócrona. En lugar de la ley de Snell, usamos la ecuación de los rayos, cuya integración parece ser más fácil cuando existe algún tipo de simetría. Se aplica a dos ejemplos: el problema clásico de la braquistócrona y para potenciales gravitacionales del tipo $V(r) = \alpha r^n$,

Recently, P.K. Aravind[1] used Bernoulli's method to find the brachistochrone of a particle moving inside a homogeneous spherical ball of matter of mass M and radius R. This method is based on an optics-mechanics analogy. The reason is that in geometrical optics light chooses the trajectory which requires the minimal time of travel (Fermat's Principle). Besides its beauty it allows a new way of attacking the problem, which avoid the conventional approach of solving the Euler-Lagrange equations[2]. Instead, all one has to use is the Snell's law and that $n = \frac{c}{v}$, with v calculated from the energy conservation theorem with the adequate potential. In reality this is only a way of shortening the mathematical steps, since Snell's law can be obtained from Fermat's princiñle via the Euler-Lagrange equations.

In this note we apply this method in a different way. We will start with the eikonal equation[3]:

$$(\nabla S)^2 = n^2 \quad . \tag{1}$$

As is well known, even without the knowledge of the eikonal S it is possible from Eq. (1) to obtain the ray equation[4]:

$$\frac{d}{ds}\left(n \frac{d\vec{r}}{ds}\right) = \nabla n \quad , \tag{2}$$

where $\vec{r}$ is the position of an arbitrary point in the trajectory of light and ds is the differential of the arc length. Even though the ray equation is very difficult to solve in the general case, it permits a more systematic way of using the method. The symmetries of each problem can be implemented directly and integration of Eq. (2) often becomes immediate. Let us see in two examples.

We first solve the classic brachistochrone problem of a particle moving in a constant gravitational field. This case has of course a plane symmetry. Thus, $n = \frac{c}{v}$ is a function of one variable only, say y, the vertical distance from a given horizontal plane to the origin O, taken at the initial point of motion. We then write Eq. (2) in the form

$$\frac{d}{ds}\left(n(y)\hat{\tau}\right) = \frac{dn}{dy}\,\hat{j} \qquad , \tag{3}$$

where $\hat{\tau} = \frac{d\vec{r}}{ds}$ and $\hat{j}$ are unit vectors along the ray direction and the vertical direction, respectively. Multiplying both sides of (3) vertically by $\hat{j}$ we obtain

$$\frac{d}{ds}\,(n\hat{j} \wedge \hat{\tau}) = \vec{0} \qquad , \tag{4}$$

which implies

$$n \sin\theta = cte \qquad , \tag{5}$$

where θ is the angle between $\hat{j}$ and $\hat{\tau}$. Substituting $n = \frac{c}{v}$ we have

$$\frac{\sin\theta}{v} = C \qquad (C = cte) \qquad . \tag{6}$$

From the conservation theorem for the energy of the particle we have that

$$v^2 = \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2gy \qquad , \tag{7}$$

where x is the horizontal coordinate of the particle. From (6) and (7) we can write

$$\sin^2\theta = C^2\ 2gy \qquad . \tag{8}$$

Using the trigonometric identity $2\text{sen}^2\theta = 1 - \cos 2\theta$ we get

$$y(\theta) = R(1 - \cos 2\theta), \qquad \text{with } R = \frac{1}{4gC^2} \qquad . \tag{9}$$

The expression for $x(\theta)$ can be obtained from

$$\frac{dx}{dt} = 2g\ Cy \qquad , \tag{10}$$

which derives directly from (6) and (7) if we multiply both numerator and denominator of the L.H.S. of (6) by v. Inserting (9) in (10) we easily see that

$$\frac{dx}{d\theta}\frac{d\theta}{dt} = \left(\frac{1}{2C}\right)\ (1 - \cos 2\theta) \qquad , \tag{11}$$

where we assumed a time parametrization for θ. We are free to choose conveniently this parametrization. Choosing one so that

$$\frac{d\theta}{dt} = A \qquad , \tag{12}$$

we see that

$$x(\theta) = \left(\frac{1}{4CA}\right)\ (2\theta - \text{sen}2\theta) \tag{13}$$

is a solution of (11). All we have to do now is to calculate the constant A. This can be done if we force Eq. (7) to be valid. Inserting Eqs. (9), (12) and (13) in (7) we obtain

$$\begin{aligned} &\frac{1}{4C^4g^2}\{A^2\text{sen}^2(2\theta) + C^2g^2\cos^2(2\theta)\} + \frac{1}{4C^2}\{1 - 2\cos(2\theta)\} = \\ &\qquad = \frac{1}{2C^2}\ (1 - \cos 2\theta) \qquad . \end{aligned} \tag{14}$$

From (14) we see that A must be given by

$$A = gC \qquad . \tag{15}$$

Substituting (15) in (13) we obtain

$$x(\theta) = R(2\theta - \text{sen}2\theta) \qquad , \tag{16}$$

where we used that $R = \frac{1}{4gC^2}$.

Equations (9) and (16) are the parametric equations for a cicloid, confirming a well known result for this problem[3].

Now, we are going to treat more general gravitational fields, whose potentials are given by

$$V(r) = \alpha r^n \quad , \tag{17}$$

where α is a constant, r the distance from the center of force and n a non zero integer number. P.K. Aravind[1] in his article solved a particular case for (17), with $\alpha = \frac{GmM}{2R^3}$ and $n = 2$. The differential equations which determine the brachistochrones for these potentials were obtained by Ramesh Chander[5] via the Euler-Lagrange equations. In this article we reobtain these differential equations with the Bernoulli's method as exposed before.

Since central potentials have spherical symmetry the "index of refraction" in this case is a function of r alone and we can write Eq. (2) in the form

$$\frac{d}{ds}\left(n(r)\hat{\tau}\right) = \frac{dn}{dr}\,\hat{r} \quad , \tag{18}$$

where $\hat{r}$ is the unit vector in the $\hat{r}$ direction. Multiplying both sides of (18) vectorially by $\hat{r}$ we obtain

$$\frac{d}{ds}\left(n\ \vec{r} \wedge \hat{\tau}\right) = \vec{0} \quad . \tag{19}$$

From the last equation we have

$$n\ r\ \sin\beta = cte \quad , \tag{20}$$

where β is the angle between $\vec{r}$ and $\hat{\tau}$. In this case the energy conservation theorem gives the relation

$$n(r) = \frac{c}{v(r)} = c\left\{\frac{2\alpha}{m} \quad (q^n - r^n)\right\}^{-\frac{1}{2}} \quad . \tag{21}$$

Substituting (21) in (20) we obtain

$$\frac{r \sin\beta}{\left\{q^n - r^n\right\}^{\frac{1}{2}}} = k \qquad (k = cte) \quad , \tag{22}$$

where q is the distance from the point where the particle was initially at rest and the center of force (see Fig. 1). To obtain the differential equations for the brachistochrones we easily see that

$$\sin\beta = \frac{rd\phi}{ds} \tag{23}$$

and

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 = 1 \quad . \tag{24}$$

From (23) and (24) we find

$$\sin\beta = \left[1 - \left(\frac{dr}{ds}\right)^2\right]^{\frac{1}{2}} \quad . \tag{25}$$

Inserting (25) into (22) we finally have

$$\frac{dr}{ds} = \frac{1}{r}\left\{r^2 - k^2(q^n - r^n)\right\}^{\frac{1}{2}} \quad , \tag{26}$$

which coincides with Eq. A.5 of R. Chander's[5] paper if we identify our k with his constant of integration σ.

In this article we used Bernoulli's method in a little different manner. Instead of the Snell's law, we started with the eikonal equation and ray equation. It seems easier to use the symmetries of each problem

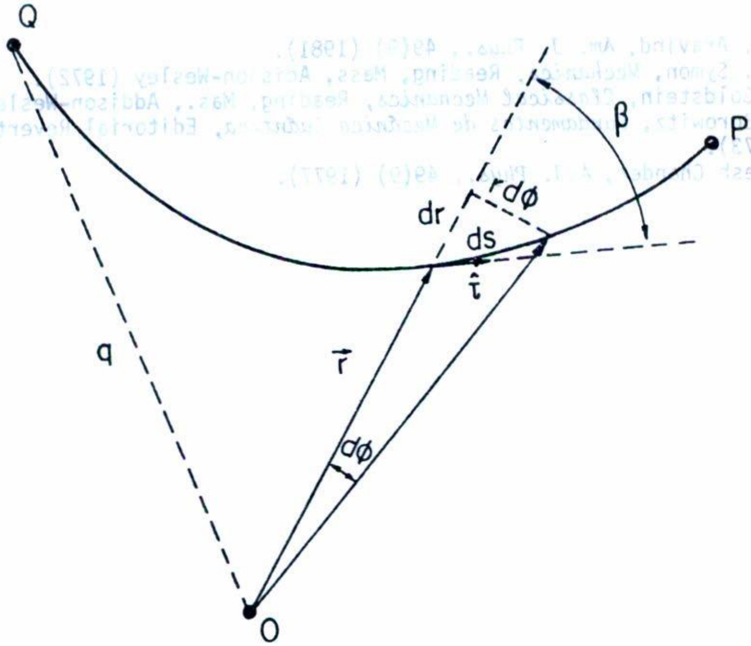

Fig. 1. Brachistochrone between points Q and P in a radial gravitational field with center of force at o.

in the integration of this last equation. Thus, in applying the method to other cases (with axial symmetry for instance) or in more complicated brachistochrone problems this new point of view may also be used. Besides its charm, the simplicity of the Bernoulli's method makes it understandable by any undergraduated student. Therefore, it would be interesting if those students used it more often.

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