

General Equations of Motion in the Solar System: Gravitational Frequency Shift for Weyl-affine Theories of Gravity

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Abstract. A formalism in which a class of non-metric theories of gravity can be analyzed, is extended so that the nature of possible constraints arising from the outcome of a planned solar probe experiment can be investigated. The modelling of the planned solar mission allows the gravitational frequency shift effect predicted by these theories to be calculated (up to third order in the Newtonian gravitational potential). The comparison with general relativity theory predictions places strong limits on the possible non-metric nature of spacetime within the solar system.

Resumen. Se extiende un formalismo para el análisis de una clase de teorías no-métricas con el objetivo de investigar la naturaleza de las posibles restricciones que puedan resultar de un experimento solar planeado. Mediante un modelo de la misión solar, se calcula el efecto del corrimiento gravitacional en frecuencia que predicen estas teorías (a tercer orden en el potencial gravitacional newtoniano). Al comparar los cálculos con las predicciones de la teoría general de la relatividad, se obtienen severos límites sobre la posible naturaleza no-métrica del espacio-tiempo en el sistema solar.

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1. Introduction

In a recent article [1], hereinafter referred to as paper I, a formalism for the analysis of a class of non-metric theories of gravity was presented. The formalism, accurate to third order in the Newtonian gravitational potential U , was used to compute the gravitational frequency shift effect predicted by Weyl-affine theories of gravity (WATG's) in a spherically symmetric and static (SSS) gravitational field for a hypothetical experimental situation in which both an artificial satellite and the Earth followed circular coplanar orbits around the Sun. Using available experimental limits it was then shown that (WATG's) had to be "metric" to first order in U .

It is the aim of this article to investigate in greater depth the nature of possible constraints on WATG's arising from the outcome of a planned solar probe experiment in which both the satellite and the Earth follow general (non-circular) orbits. For the sake of brevity we shall often refer to equations in paper I (*e.g.*, Eq. (I.2.1) refers to equation (2.1) in paper I). The notation is identical to that used in paper I.

A brief review of the class of non-metric theories of gravity called Weyl-affine theories is presented in section 2.

A formalism is then designed and applied, in section 3, to the modelling of a planned solar mission in which the Earth and a satellite move around the Sun following the coplanar orbits determined by the gravitational solar field.

The calculations of the gravitational frequency shift up to third order in U are then carried out both for Weyl-affine theories in which λ is strictly non constant and for general relativity theory (an example of a metric theory of gravity in which $\lambda = 1$). Finally, in section 4 the results are presented and discussed and the level at which the possible non-metric nature of spacetime would affect the outcome of the planned experiment is analyzed.

2. A Class of Non-Metric Theories of Gravity and their Description in a Spherically Symmetric and Static Gravitational Field

As was discussed in paper I, there exists a class of non-metric theories of gravity, called Weyl-affine theories of gravity (WATG's), which satisfy all the current conditions of viability. Essentially, a WATG is a geometric theory of gravity (*i.e.*, spacetime is characterized by a four-dimensional, Hausdorff, differentiable manifold of signature -2), in which spacetime is endowed with a Weyl connection Γ_W and a metric tensor g , and satisfies the following two conditions:

1. The motion of freely falling test particles is governed by the path equation (I.2.1) with respect to the Weyl connection. That is, the equations of motion governing test bodies are of a "metric" form with respect to a tensor λg_{ab} , where λ is a scalar field depending on the gravitational field only.
2. Measurements made with rods and clocks are governed by the metric tensor g_{ab} through equation (I.2.8).

We note that a WATG reduces to a metric theory of gravity (MTG) if the scalar function λ is a constant (unity). Thus a non-constant λ represents the non-metric nature of the theories under investigation.

Since all the experiments testing the nature of gravity, and in particular the experimental configuration to be discussed below, are performed in the spherically symmetric and static (SSS) gravitational field of the Sun, hereafter we shall restrict our attention to this type of field. In particular, in an SSS gravitational field the forms of Γ_W and g simplify greatly (see equations (I.2.2), (I.2.3) and (I.2.9) and references 4 and 5) and the laws (1) and (2) above can be restated as:

- 1'. The equations of motion for test particles are derived from the Lagrangian given by

$$L = (\lambda f - \lambda g \mathbf{v}^2)^{1/2} dt. \quad (2.1)$$

2'. Atomic clocks are governed by

$$ds = (f - g\mathbf{v}^2)^{1/2} dt. \quad (2.2)$$

In the above equations $f(\equiv g_{00})$, $g(\equiv -g_{11} = -g_{22})$ and λ are arbitrary functions of the Newtonian gravitational potential U , and $\mathbf{v} = d\mathbf{x}/dt$ is the coordinate 3-velocity of the test particle.

The analysis that follows is designed to investigate the possible forms for λ by predicting the gravitational frequency shift for the class of theories of gravity under consideration.

The predictions for the frequency shift effect presented in paper I were calculated in terms of the signal's motion alone since both the Earth and satellite were forced to follow circular, uniform orbits around the Sun. This proved adequate for the actual experiments performed to date.

In this article both bodies are left to move freely (*i.e.*, to follow the geodesics determined by the Sun's gravitational field which are coplanar but noncircular) in an attempt to model the improved experiments to be performed in the near future. In the more realistic model corresponding to the experimental set up the predicted values of the frequency shift will be more seriously affected by the influence of the function λ on the motion of the Earth and satellite and as a result more severe experimental limits will be placed on λ .

3. Modelling of a Planned Solar Mission Experiment

We assume the gravitational field of the Sun to be SSS.

Any entity than can be regarded as a test particle, will follow a trajectory determined by the solar field. This is the case for a solar probe experiment in which the gravitational frequency shift of light signals connecting an artificial satellite to the earth is to be measured. The motion of both the Earth and the satellite will then be given by [11]

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \left[\frac{g}{\lambda r^4} \right]^{1/2} \times \left\{ \frac{\lambda_- r_-^2 [(\lambda f)^{-1} - (\lambda_- f_-)^{-1}] - \lambda_+ r_+^2 [(\lambda f)^{-1} - (\lambda_+ f_+)^{-1}]}{\lambda_+ \lambda_- r_+^2 r_-^2 [(\lambda_+ f_+)^{-1} - (\lambda_- f_-)^{-1}]} - \frac{1}{\lambda r^2} \right\}^{-1/2} dr, \tag{3.1a}$$

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \left[\frac{g}{\lambda f^2} \right]^{1/2} \times \left\{ \frac{1}{\lambda f} - \frac{\lambda_- f_- [(\lambda r^2)^{-1} - (\lambda_- r_-^2)^{-1}] - \lambda_+ f_+ [(\lambda r^2)^{-1} - (\lambda_+ r_+^2)^{-1}]}{\lambda_+ \lambda_- f_+ f_- [(\lambda_+ r_+^2)^{-1} - (\lambda_- r_-^2)^{-1}]} \right\}^{-1/2} dr, \tag{3.1b}$$

where r , ϕ , and t are spherical polar coordinates for a reference system based on the centre of the Sun and oriented in such a way that the $\theta = \pi/2$ plane coincides with the orbits' plane, f and g are the tensor field components defined by equations (I.2.3) (g_{33} has been set equal to $-r^2$ so that r is interpreted as the distance from the source [6]), and the subindices $+$ and $-$ indicate that the function is to be evaluated at perihelion (r_-) or at aphelion (r_+). The motion of the light signals will also be governed by the field according to [11]

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \left[\frac{g}{r^2} \right]^{1/2} \frac{dr}{[(f_0/r_0^2)(r^2/f) - 1]^{1/2}}, \tag{3.2a}$$

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \left[\frac{g}{f} \right]^{1/2} \frac{dr}{[1 - (r_0^2/f_0)(f/r^2)]^{1/2}}, \tag{3.2b}$$

where the subindex 0 means the tensor field component is to be evaluated at r_0 , the distance of nearest approach of the signal to the Sun.

Assuming now that the field equations of the non-metric theories under consideration admit asymptotically flat vacuum solutions, one can expand the tensor field components f and g (Eqs. (I.2.3)) and

the function λ (Eq. (2.1)) in terms of the Newtonian gravitational potential $U = m/r$, where $m = GM/c^2$ represents the mass of the field's source and r the distance to its centre. Suitable expansions accurate to third order in U are given by

$$f(r) = g_{00} = 1 + f_1U + f_2U^2 + f_3U^3, \quad (3.3a)$$

$$g(r) = -g_{11} = 1 + g_1U + g_2U^2 + g_3U^3, \quad (3.3b)$$

$$\lambda(r) = 1 + \lambda_1U + \lambda_2U^2 + \lambda_3U^3, \quad (3.3c)$$

and, as already mentioned, a formalism for MTG's can be obtained by setting $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (in particular, the values for the other parameters in the case of general relativity theory are $f_1 = -2$, $g_1 = 2$, $g_2 = 4$, $g_3 = 8$, and $f_2 = f_3 = 0$).

For the bound orbits of the Earth and satellite one can apply the Virial theorem and recover a familiar result of Newtonian mechanics: the typical kinetic energy and the typical potential energy are of the same order of magnitude, *i.e.*, $v^2 \sim m/r$, where v is the typical value of the spatial velocity of the test particle under consideration. Therefore, if the expansion of equation (2.1) is to be homogeneous, terms involving the square of the spatial velocity of the particles, $(dx^\nu/dt)^2$, need only be expanded up to second order in m/r .

Using the expansions given by equations (3.3) one can integrate equations (3.1-2) and obtain analytical expressions accurate to third order in m/r for the trajectories of test particles and light rays moving in a plane [12].

In order to calculate the gravitational frequency shift effect, we shall model an experimental situation in which the Earth and satellite are joined by a light signal when both follow orbits determined by the Sun's gravitational field. The Earth's orbits is specified by $r_i = r_-$, $\phi_i = -\pi/2$, $r_+ = 1.520935 \times 10^{13}$ cm., and $r_- = 1.471035 \times 10^{13}$ cm. while that of the satellite is specified by $r_i = 5.548 \times 10^{11}$ cm., $\phi_i = 0$, $r_+ = 7.78318 \times 10^{13}$ cm., and $r_- = 2.78392 \times 10^{11}$ cm. According to (I.3.4c) the gravitational frequency shift experienced by a signal travelling from the Earth to the

satellite and back is given by

$$Z = 1 - \frac{dS_e/dS_b}{dS_r/dS_b}, \quad (3.4a)$$

which can be rewritten as [12]

$$Z = 1 - \frac{\lambda_e}{\lambda_r} f_e f_b \left[\frac{1}{f_b} - \frac{l_b}{r_b^2} \left(\frac{dp}{dr_0} / \frac{dq}{dr_0} \right)_0 \right] \left[\frac{1}{f_r} - \frac{l_r}{r_r^2} \left(\frac{dp}{dr_0} / \frac{dq}{dr_0} \right)_i \right], \quad (3.4b)$$

where l is the constant angular momentum per unit mass, p is the coordinate time t it takes for a wave signal to go from emitter to reflector or back again, q is the angular displacement of a wave signal when going from emitter to reflector (or viceversa), the subscripts e , b , r , 0 and i indicate emitter, reflector, receiver, outgoing and incoming signal.

4. Results and Discussion

The results presented in Figs. 1–3 are those given by Eq. (3.4b) when the Earth and satellite follow the orbits determined by the Sun's gravitational field and the conditions given in the previous section are satisfied. Both bodies orbit in the same sense (counterclockwise), and while the Earth starts at its perihelion the satellite begins moving half way to its perihelion.

The calculations cease at the moment the satellite is eclipsed by the Sun and the communication is interrupted.

We consider that the calculations for a whole orbit of the satellite or more should be performed once the actual solar probe experiment has been planned and the orbital parameters are known. For the derivation of possible constraints on WATG's, the purpose of this article, the results in Figs. 1–3 suffice.

In order to simplify the comparison with previous calculations the results are presented in a way similar to that of paper I; *i.e.*, Fig. 1 shows the resulting values of subtracting the predictions made with different values of λ_2 from the general relativity theory (GRT)

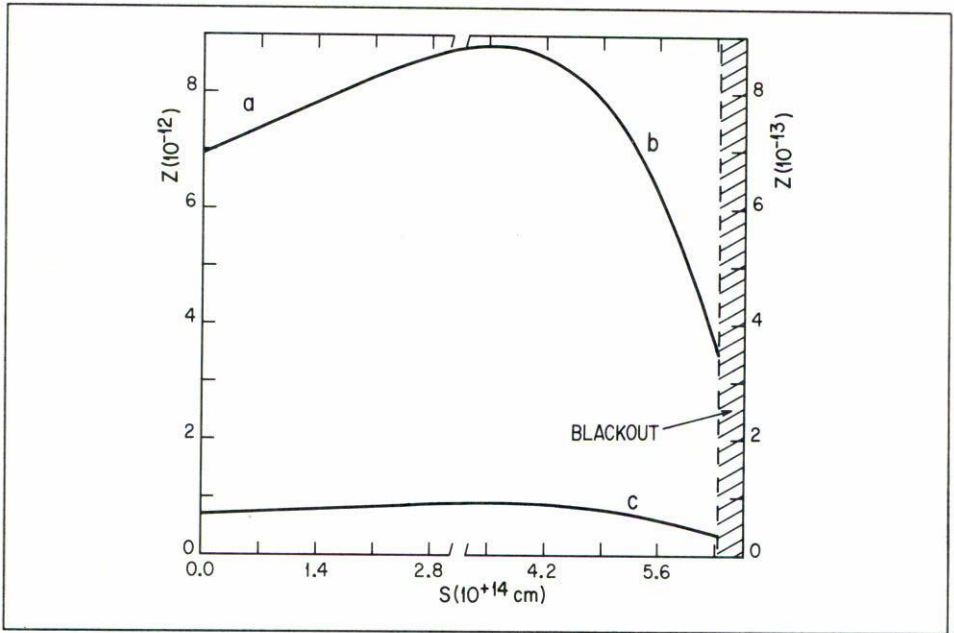


FIGURE 1. Differences in the frequency shift calculations made with GRT and those made with a MATG where $\lambda_2 \neq 0$, $\lambda_1 = \lambda_3 = 0$. Curve (a) corresponds to $\lambda_2 = 10^{-1}$ and has been plotted according to the scale on the left hand side, while curves (b): $\lambda_2 = 10^{-2}$ and (c): $\lambda_2 = 10^{-3}$ have been plotted according to the scale on the right hand side. Curve (a) differs from curve (b) by a factor which, at the resolution allowed by the graph becomes exactly 10. The curves have been obtained by subtracting the calculations with $\lambda_2 \neq 0$ from GRT calculations.

predictions, Fig. 2 contains similar graphs for different λ_3 's, and the results obtained with $\lambda_1 = 10^{-5}$ are compared to those obtained with $\lambda_2 = 10^{-1}$ in Fig. 3. In all cases the frequency shift predicted by the MATG's under consideration is smaller than that predicted by GRT. The largest corrections to GRT, those coming from the calculations with $\lambda_1 = 10^{-5}$, are two orders of magnitude greater than the corrections predicted in the circular orbits case of paper I for the same value of λ_1 . Thus, they are an order of magnitude greater

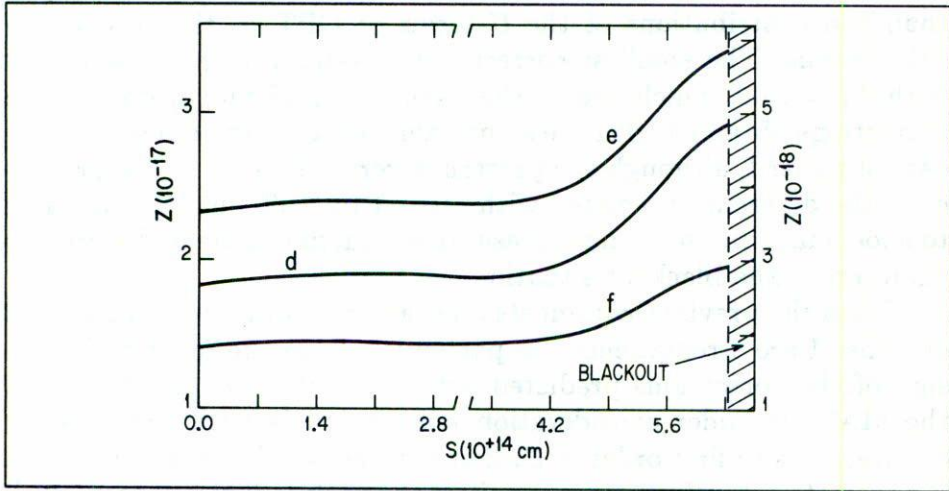


FIGURE 2. Analogue of Fig. 1 with $\lambda_3 \neq 0$, $\lambda_1 = \lambda_2 = 0$. Curve (d) represents the differences obtained using $\lambda_3 = 0.5$ and follows the scale on the left hand side while curves (e): $\lambda_3 = 0.1$ and (f): $\lambda_3 = 0.05$ follows the right hand side scale.

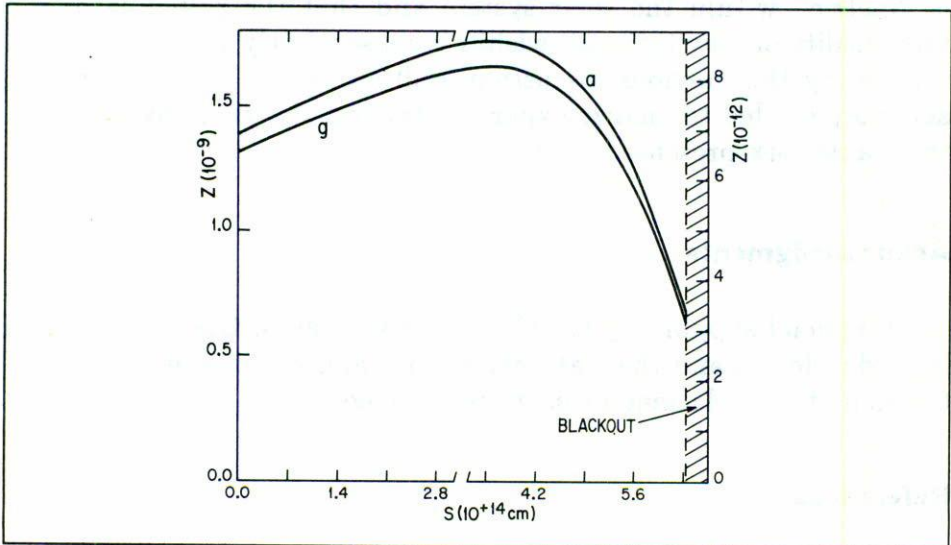


FIGURE 3. Comparison of the differences obtained with $\lambda_1 = 10^{-5}$, $\lambda_2 = \lambda_3 = 0$ (plot g) and those obtained with $\lambda_2 = 10^{-1}$, $\lambda_1 = \lambda_3 = 0$ (plot a).

than the contributions of the U -terms of GRT to the Newtonian calculations. The smallest corrections to GRT are those predicted with $\lambda_3 = 0.05$, which are of the same order of magnitude as the corrections obtained with the same value for λ_3 in the circular orbits case of paper I, although the pattern is very different. In the present case, the difference predicted with respect to GRT initially increases to a local maximum, then decreases to a local minimum and increases again up to the blackout situation.

From the previous comments one can see that all the constraints of paper I are strengthened. In particular, from the magnitude and sign of the corrections predicted with $\lambda_1 = 10^{-5}$, we conclude that the MATG's under consideration must coincide with their metric counterparts to first order in U . If spacetime within the solar system is non-metric its effects must manifest themselves through very small contributions at the U^2 -level ($\lambda_2 \leq 10^{-3}$) or at the U^3 -level. The detection of these effects, if present, is unfortunately still far from present day experiments accuracy. The hypothesis that spacetime is "metric" within the solar system and that the principle of the universality of gravitational redshift is consequently satisfied, though favored by the previous discussion, still awaits the improvement in accuracy needed to analyze experimental and observational data via the framework presented here.

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