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Classical canonical transformations and changes in reference frame

M. Moreno
Instituto de Física, Universidad Nacional Autónoma de México, apartado postal 20-364, 01000 México, D.F.

J.A. del Río
Area de Física, Universidad Autónoma Chapingo, 56230 Chapingo, Edo. de México
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Abstract. We discuss canonical transformations that correspond to changes in reference frame. We analyze the non-uniqueness of the canonical transformation, and solve the problem of finding the canonical transformation that changes both coordinates and momenta.

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1. Introduction

In this work we discuss the relation between canonical transformations and changes in reference frame in classical mechanics. The aim of retaking this well known problem [1,2,3,4], is to overcome some shortcomings of the standard presentations, in particular, in connection with the transformation of canonical momenta in a changing reference frame.

This problem has been addressed by several authors in the framework of classical field theory and quantum mechanics [5]. We will undertake a more direct approach based on the canonical formalism.

The plan of the work is to present shortly in the second section the theory of canonical transformations, mostly to set up the notation. Then, in section 3, we show how to deal with changes in reference frame in the language of canonical transformations. We solve the problem of transforming the momenta as well as the coordinates with a canonical transformation. Then we point out that a canonical transformation can be defined in such a way as to transform the momenta only. Finally, in section 4, we present some examples. We also discuss the similarities and differences between canonical transformations that represent changes in the reference frame, and those that correspond to electromagnetic gauge transformations.

2. Canonical transformations

Canonical transformations in classical mechanics are transformations in coordinates and momenta that leave Hamilton's equations of motion invariant in form. This means that if such equations can be deduced from an action principle in a given set of phase space variables (q,p),

$$\delta S = \delta \int_{t_1}^{t_2} dt\, L(q,\dot{q},t) = \delta \int_{t_1}^{t_2} dt\, (\dot{q}\cdot p - H) = 0, \tag{1}$$

they should be deduced from an action principle in some other set (Q,P),

$$\delta S' = \delta \int_{t_1}^{t_2} dt\, (P\cdot \dot{Q} - H') = 0, \tag{2}$$

where sums over canonically conjugate variables are indicated by a dot product. A very general relation between S and S' is given by*

$$S = S' + \int_{t_1}^{t_2} \frac{dD}{dt}\, dt, \tag{3}$$

where D is called the generating function of the canonical transformation; it depends on any $2N$ independent variables and time. It turns out that this type of canonical transformation is general enough for the description of changes of reference frame we are considering. Without any loss of generality, one can take

$$D = F - P\cdot Q, \tag{4}$$

in such case from (3) and (4), and taking $F = F(q,P,t)$, we get

$$p = \left.\frac{\partial F(q,P,t)}{\partial q}\right|_{P,t}, \tag{5}$$

$$Q = \left.\frac{\partial F(q,P,t)}{\partial P}\right|_{q,t}, \tag{6}$$

and

$$H' = H + \left.\frac{\partial F}{\partial t}\right|_{q,P}, \tag{7}$$

from these equations we can solve for Q and P in terms of q and p. The last relation gives the new hamiltonian. We can write it down more explicitly

$$H'(Q,P,t) = H(q(Q,P,t),p(Q,P,t),t) + \left.\frac{\partial F(q(Q,P,t),P,t)}{\partial t}\right|_{q,P}. \tag{8}$$

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* The more general canonical transformations includes multiplicative factors in the action.

3. Changes in the reference frame

Let us now apply this standard formalism to changes in reference frame. Consider the case when the new frame is related to the old one by

$$Q_i = q_i + f_i(t); \tag{9}$$

although this is not the most general case, Eq. (9) includes both the change to a linearly accelerated reference frame and, also, the change to a rotating frame if the adequate coordinates are chosen (cartesian or cylindrical respectively).

Because a canonical transformation relates, in the general case, coordinates and momenta, it is clear that Eq. (9) does not determine uniquely the canonical transformation. The question therefore is: What is the degree of arbitrariness of the new momenta P? To answer this we observe that Eqs. (5), (6) and (9) require

$$p_i = \frac{\partial F(q,P,t)}{\partial q_i}, \tag{10}$$

$$Q_i = \frac{\partial F}{\partial P_i} = q_i + f_i(t). \tag{11}$$

The last formula is satisfied if

$$F = P \cdot (q + f) + G(q,t), \tag{12}$$

where $G(q,t)$ is an arbitrary function of q and t. Therefore, from Eqs. (10), (12) and (7), we get

$$P_i = p_i - \frac{\partial G}{\partial q_i}, \tag{13}$$

$$H' = H + P \cdot \dot{f} + \frac{\partial G}{\partial t}. \tag{14}$$

We observe that the selection of P_i is quite arbitrary. In order to fix the canonical momenta one must select a function G. A possible choice is $G(q,t) = G_0$, with G_0 a constant, which gives the so called point transformations

$$P_i = p_i, \tag{15}$$

$$H'(P,Q,t) = H(p(P,Q), q(P,Q), t) + P \cdot \dot{f}. \tag{16}$$

This selection of G is not very satisfactory if one is looking for the canonical transformation that relates both the coordinates and the momenta as measured in the two frames of reference.

A different selection, suggested by the cartesian or the polar forms of the kinetic momenta in the new reference frame, would be

$$P = p + \left.\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\right|_{\dot{q}=\dot{f}}. \tag{17}$$

Of course,

$$L(q,\dot{q},t) = \dot{q}\cdot p - H(q,p,t),$$

and in the lagrangian formalism

$$p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t).$$

The physical interpretation of the *ansatz* in Eq. (17) is clear if one notes that $\partial L(q,\dot{q},t)/\partial\dot{q}|_{\dot{q}=\dot{f}}$ are the momenta in the new reference frame of a "particle" at rest in the old reference frame. This connection is obvious if one studies a couple of examples. Calling

$$h(q,t) = \left.\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\right|_{\dot{q}=\dot{f}}, \tag{18}$$

we observe from Eqs. (13) and (17) that

$$G = -\int dq\cdot h + G_0(t), \tag{19a}$$

gives the required momenta, and thus momenta in Eq. (17) are canonical. The last formula reduces to

$$G = -q\cdot h + G_0(t), \tag{19b}$$

if h_i does not depend on q_i . We will restrict ourselves to this case in the following discussion. The corresponding hamiltonian is easily obtained in the canonical formalism from Eq. (14),

$$H' = H + P\cdot\dot{f} - q\cdot\frac{\partial h}{\partial t} + \frac{\partial G_0}{\partial t}. \tag{20}$$

A question that arises naturally in this context is whether or not there exists a canonical transformation that leaving $Q = q$ transforms $P = p + h$. The answer is affirmative and the appropriate generating function is

$$F = (P-h)\cdot q + G_0(t). \tag{21}$$

4. Some examples

Let us now consider a couple of examples. First, take the change in reference frame defined by a non–uniform acceleration, for example

$$X = x + x_0 + v_0 t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3, \tag{22}$$

where we identify

$$f = x_0 + v_0 t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3. \tag{23}$$

We get, for the momenta in Eq. (17)

$$P = p + \left.\frac{\partial L}{\partial \dot{x}}\right|_{\dot{x}=\dot{f}}, \tag{24}$$

in the non–relativistic case

$$L = \tfrac{1}{2}m\dot{x}^2 - V(x) \tag{25}$$

and thus

$$\begin{aligned} P &= p + m\dot{f} \\ &= p + m[v_0 + at + \tfrac{1}{2}bt^2]. \end{aligned} \tag{26}$$

The new hamiltonian is

$$\begin{aligned} H' = &\frac{[P - m\dot{f}]^2}{2m} + V(X - f) \\ &+ P \cdot \dot{f} + m[X - f][a + bt] + \frac{\partial G}{\partial t}, \end{aligned} \tag{27}$$

selecting G_0 adequately we can drop out the terms that depend solely on t; we get

$$H' = \frac{p^2}{2m} + V(X - f) - mX[a + bt]. \tag{28}$$

We observe that this canonical transformation correctly describes the change in reference frame including the fact that the new hamiltonian function is not conserved.

As a second example, let us consider the case of a point charge with velocity $v = v\hat{z}$ moving in an electric field E and in a uniform magnetic field $B = B\hat{z}$. The hamiltonian is

$$H = \frac{1}{2m}\left[p + \frac{eA}{c}\right]^2 + e\Phi, \tag{29}$$

where e is the particle's charge, m the mass of the point charge, c the speed of light, A the vector potential, and Φ the scalar potential. The lagrangian is

$$L = \frac{mv^2}{2} - \frac{me}{c} v \cdot A - e\Phi. \tag{30}$$

Now, if our objective were to simplify the lagrangian or the hamiltonian, a conceivable option would be to use electromagnetic gauge transformations, *i.e.*, the new potentials being [6]

$$\begin{aligned} A' &= A + \nabla l, \\ \Phi' &= \Phi - \frac{1}{c}\frac{\partial l}{\partial t}. \end{aligned} \tag{31}$$

Eq. (30) is transformed to

$$L' = \frac{mv^2}{2} - \frac{me}{c} v \cdot A' - e\Phi' + \frac{me}{c} v \cdot \nabla l - \frac{2}{c}\frac{\partial l}{\partial t}. \tag{32}$$

We observe that the two last terms are a total derivative, therefore the electromagnetic gauge transformation is canonical with a generating function $e\,l(q,t)/c$ at the level of Eq. (3), depending solely on the coordinates and time.

We can rewrite Eq. (29), using $A = B \times r/2$, and get

$$H = \frac{p^2}{2m} + \frac{e}{2mc} p_z B + \frac{e^2}{4mc^2} B^2 + e\Phi. \tag{33}$$

Of course, under gauge transformations the magnetic field never vanishes. Thus, to simplify the Eq. (29) or (30) in this way is impossible. An appropriate way to simplify the problem is to change the reference frame. Let us consider now the case in which the two frames are related by a uniform rotation along a fixed axis which we take as the z-axis. If we choose the angular velocity as

$$\omega = \frac{e}{2mc} B. \tag{34}$$

Then in polar coordinates

$$z' = z, \qquad \rho' = \rho, \qquad \varphi' = \varphi + \omega t. \tag{35}$$

Thus the canonical momenta are

$$p_{z'} = p_z, \qquad p_{\rho'} = p_\rho, \qquad p_{\varphi'} = p_\varphi.$$

Using Eq. (21) and (34) we get the new hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{e^2}{4mc^2}B^2 + e\Phi. \tag{36}$$

Notice that the linearly dependent term on the magnetic field has been eliminated, as expected from Larmor's theorem [7].

It is worthwhile to note that in principle a change in reference frame requires to change also A and Φ together with p. This is not made in this work because the change in the electromagnetic potentials is negligible in the non relativistic case. This change has been discussed by Schmutzer and Plebański [5].

5. Final remarks and conclusions

We can conclude that the canonical transformation determined by Eqs. (12) and (19) properly describes the change in reference frame. This is so because it not only changes the coordinates but also the momenta. The generating function was shown to be

$$F(q,P,t) = -P \cdot (q - f) + q \cdot h + G_0(t), \tag{37}$$

with h defined in Eq. (18) (and $\partial h_i/\partial q_i = 0$). This transformation is not necessarily more convenient when solving a specific problem as was shown in the second example of Section 3. There we have eliminated the magnetic field using point canonical transformations. It is therefore useful to keep in mind the freedom implied in Eq. (12) by the arbitrariness of $G(q,t)$. An immediate consequence for the quantum formalism is that the equivalent unitary transformation is not uniquely fixed. This feature can be used to simplify specific problems as in the analogous classical case [8,9]. We have shown that it is feasible to take up a problem without formal contradictions and sometimes to simplify it considering observables in different reference frames. We should remember that gauge transformations in electromagnetism depend only on q and t. We may add that the apparent restriction to a particular set of coordinates implied by Eq. (9) can be easily lifted by an additional canonical transformation. This has not been presented as it tends to obscure the discussion. But of course many cases of physical interest can be cast into the form of Eq. (19*b*).

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Resumen. Se discuten las transformaciones canónicas que corresponden a cambios de sistema de referencia. Se analiza la no unicidad de la transformación canónica y se resuelve el problema de encontrar la transformación canónica que cambia coordenadas y momentos.