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Comment on gauge transformations in classical mechanics

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Abstract. The invariance of the electromagnetic field tensor is used as a necessary and sufficient condition for constructing the gauge transformations in Electrodynamics.

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When dealing with gauge transformations in Electrodynamics some text books, as those of Panofski and Phillips (1961) and Kompaneyets (1978), indicate that the allowed form of such transformations is found by inspection, whereas other authors like Reitz, Milford and Christy (1980), Laudau and Lifshitz (1962) and Sommerfeld (1952) simply establish the transformations and then show they are consistent with the invariance of the fields or, equivalently, with the electromagnetic field tensor. There exists a third group of authors, like Jackson (1975) and Marion and Heald (1980) who correctly analyze the transformation for the vector potential and then guess the corresponding transformation for the scalar potential.

Since the uniqueness of such type of transformation is never mentioned, the question about the existence of another type often arises. It can be rigorously shown, that the invariance of the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \tag{1}$$

under a gauge transformation is a necessary and sufficient condition to determine its general form.

To prove this statement we first note that equation (1) is linear in the 4-potential and hence the transformation must be of the type

$$A'_\mu = A_\mu + Z_\mu \tag{2}$$

where the 4-vector Z_μ is, so far, arbitrary.

When the new electromagnetic field tensor $F'_{\mu\nu}$ is computed with the help of equation (2) one gets

$$\begin{aligned} F'_{\mu\nu} &= \partial_\nu A'_\mu - \partial_\mu A'_\nu \\ &= \partial_\nu A_\mu - \partial_\mu A_\nu + \partial_\mu Z_\mu - \partial_\mu Z_\nu \\ &= F_{\mu\nu} + \partial_\nu Z_\mu - \partial_\mu Z_\nu \end{aligned} \tag{3}$$

and, in order to make the field tensor invariant under the transformation (2) we must have

$$\partial_\nu Z_\mu - \partial_\mu Z_\nu = 0 \tag{4}$$

This equation is the expression of the components of curl Z in four dimensions and, since it vanishes, the 4-vector Z_μ must, therefore, be the 4-gradient of an arbitrary scalar function f, that is

$$Z_\mu = \partial_\mu f \tag{5}$$

Hence, the gauge transformations which leave $F_{\mu\nu}$ invariant are, from (2) and (5)

$$A'_\mu = A_\mu + \partial_\mu f$$

or, in non-covariant notation

$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$$
$$\mathbf{A}' = \mathbf{A} + \nabla f$$

and these are the only ones.

References

1. J.D. Jackson, Classical Electrodynamics, Wiley, New York, USA, 1975, p. 220.
2. A.S. Kompaneyets, A course of theoretical physics, Vol. 1, MIR, Moscow, USSR, 1978, p. 154.
3. L.D. Landau and E.M. Lifshitz, The classical theory of fields, Pergamon, Oxford, UK, 1962, p. 54.
4. J.B. Marion and M.A. Heald, Classical electromagnetic radiation, Academic Press, New York, USA, 1980, p. 113.
5. W.K.H. Panofski and M. Phillips, Classical electricity and magnetism, Addison-Wesley, Reading, Mass., USA, 1962, p. 241.
6. J.R. Reitz, F.J. Milford and R.W. Christy, Foundations of electromagnetic theory, 3rd. Edition, Addison-Wesley, Reading, Mass., USA, 1980, p. 353.
7. A. Sommerfeld, Electrodynamics, Academic Press, New York, USA, 1952, p. 102.

Resumen. Se utiliza la invariancia del tensor electromagnético como condición necesaria y suficiente para construir las transformaciones de norma de la electrodinámica.