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On the moments of classical and related polynomials

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Abstract. We consider the moments of weight functions for the classical discrete orthogonal polynomials.

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1. Introduction

Recently considerable progress has been made in the theory of classical orthogonal polynomials of a discrete variable on non-uniform lattices ([1–8] and references therein). In these papers, a class of polynomials was singled out and many of its properties were established. Among the topics that were discussed we may mention several equations, the classification of lattices, the orthogonality property of polynomials and their various derivatives, series expansions, Rodrigues-type formulas, etc. Since the results obtained in [1–5] in fact amount to a further generalization of Hahn's approach to the theory of orthogonal polynomials [9], it is natural to expect that in this case some analogue of Hahn's characterization theorem should also exist, in consequence of which any one of five proper conditions are known to determine a unique class of orthogonal polynomials (Hahn's class, see reference 10). Moreover, fragments of the proof of such a theorem are essentially contained in the works mentioned above. But, as far as we know, the question concerning the properties of the moments for weight functions still remains untouched. This will be the subject of our discussion in the present paper.

2. The moments of the classical orthogonal polynomials

Before obtaining the relation between the moments of the classical orthogonal polynomials of a discrete variable, it is convenient to give a simple derivation of an analogous relation for the Jacobi, Laguerre, and Hermite polynomials, which is then easy to generalize.

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From the Pearson equation $(\sigma\rho)' = \tau\rho$ for the function $\rho_\nu(s) = \sigma^\nu(s)\rho(s)$ we have

$$[\sigma(s)\rho_\nu(s)]' = \tau_\nu(s)\rho_\nu(s), \tag{1}$$

where $\tau_\nu(s) = \tau(s) + \nu\sigma'(s)$. Using the equation (1) and the expansions

$$\sigma(s) = \sigma(z) + \sigma'(z)(s-z) + \tfrac{1}{2}\sigma''(s-z)^2,$$
$$\tau_\nu(s) = \tau_\nu(z) + \tau'_\nu(s-z),$$

it is easy to verify the identity

$$\begin{aligned}\frac{d}{ds}[\sigma(s)\rho_\nu(s)(s-z)^\mu] &= \mu\sigma(z)(s-z)^\mu\rho_\nu(s)\\ &+ [\tau_\nu(z) + \mu\sigma'(z)](s-z)^\mu\rho_\nu(s) + (\tau'_\nu + \tfrac{1}{2}\mu\sigma'')(s-z)^{\mu+1}\rho_\nu(s),\end{aligned} \tag{2}$$

where μ and ν are arbitrary complex numbers.

If the boundary conditions

$$\sigma(s)\rho_\nu(s)(s-z)^\mu\Big|_a^b = 0 \tag{3}$$

are satisfied and the integrals exist, then from (2) for the quantities

$$C_{\nu\mu}(z) = \int_a^b (s-z)^\mu \rho_\nu(s)\,ds$$

follows the relation

$$\mu\sigma(z)C_{\nu,\mu-1}(z) + [\tau_\nu(z) + \mu\sigma'(z)]C_{\nu\mu}(z) + \left(\tau'_\nu + \tfrac{1}{2}\mu\sigma''\right)C_{\nu,\mu+1}(z) = 0. \tag{4}$$

Hence when $\nu = 0$, $\mu = p = 0, 1, 2, \ldots$, $z = 0$ and $\sigma(0) = 0$, we arrive at the desired two-term relation

$$[\tau(0) + p\sigma'(0)]C_p + \left(\tau' + \tfrac{1}{2}p\sigma''\right)C_{p+1} = 0 \tag{5}$$

for the classical moments

$$C_p \equiv C_{0p}(0) = \int_a^b s^p\rho(s)\,ds.$$

The derivation of the relation (5) can be generalized to the case of the classical orthogonal polynomials of a discrete variable on non-uniform lattices.

3. The moments of the classical orthogonal polynomials of a discrete variable

We start with the Pearson-type equation for weight functions

$$\Delta(\sigma\rho_\nu) = \rho\tau_\nu \nabla x_{\nu+1}, \tag{6}$$

and with the "generalized power" introduced in reference 11. It can be shown that for all complex values μ and ν the identity

$$\begin{aligned} &\Delta s\left\{\sigma(s)\rho_\nu(s)\Big[x_{\nu+1}(s-1) - x_{\nu+1}(\xi)\Big]^{(\mu)}\right\} \\ &\qquad = S(s)\Big[x_\nu(s) - x_\nu(\xi)\Big]^{(\mu-1)} \rho_\nu(s)\nabla x_{\nu+1}(s) \end{aligned} \tag{7}$$

is valid, where $S(s) = \gamma(\mu)\sigma(s) + \tau_\nu(s)[x_{\nu-\mu}(s+\mu) - x_{\nu-\mu}(\xi+1)]$. Using the expansions

$$\begin{aligned} S(s) &= \gamma(\mu)\tilde{\sigma}_\nu[x_\nu(s)] + \tilde{\tau}_\nu[x_\nu(s)]\,[\alpha(\mu)x_\nu(s) + \beta(\mu) - x_{\nu-\mu}(\xi+1) \\ &= D_0 + D_1[x_\nu(s) - x_\nu(\xi-\mu+1)] \\ &\qquad + D_2[x_\nu(s) - x_\nu(\xi-\mu+1)]\,[x_\nu(s) - x_\nu(\xi+1)], \end{aligned}$$

where

$$D_0 = \gamma(\mu)\sigma(\xi-\mu+1), \qquad D_2 = \alpha(\mu)\tilde{\tau}'_\nu + \tfrac{1}{2}\gamma(\mu)\tilde{\sigma}''_\nu,$$

$$D_1 = \frac{\sigma(\xi+\nu+1) + \tau(\xi+\nu+1)\nabla x_1(\xi+\nu+1) - \sigma(\xi-\mu+1)}{\Delta x_{\nu-\mu+1}(\xi)}$$

we come to the identity analogous to (2),

$$\begin{aligned} &\Delta_s\left\{\sigma(s)\rho_\nu(s)[x_{\nu+1}(s-1) - x_{\nu+1}(\xi)]^{(\mu)}\right\} = \rho_\nu(s)\nabla x_{\nu+1}(s) \\ &\quad\times\left\{D_0[x_\nu(s) - x_\nu(\xi)]^{(\mu-1)} + D_1[x_\nu(s) - x_\nu(\xi)]^{(\mu)}\right. \\ &\qquad\left. + D_2[x_\nu(s) - x_\nu(\xi+1)]^{(\mu+1)}\right\}. \end{aligned} \tag{8}$$

From (8), for the quantities

$$C_{\nu\mu}(\xi) = \sum_{s=a}^{b-1}[x_\nu(s) - x_\nu(\xi)]^{(\mu)}\rho_\nu(s)\nabla x_{\nu+1}(s) \tag{9}$$

under the conditions similar to (3), follows the three-term recurrence relation

$$D_0 C_{\nu,\mu-1}(\xi) + D_1 C_{\nu\mu}(\xi) + D_2 C_{\nu,\mu+1}(\xi) = 0. \tag{10}$$

If we put in (10) $\xi = a + \mu - 1$ and choose $\sigma(a) = 0$, then the equality

$$\begin{aligned} &\frac{\sigma(a+\mu+\nu) + \tau(a+\mu+\nu)\nabla x_1(a+\mu+\nu)}{\nabla x_{\mu+\nu+1}(a)} C_{\nu\mu}(a+\mu-1) \\ &\quad + [\alpha(\mu)\tilde{\tau}'_\nu + \tfrac{1}{2}\gamma(\mu)\tilde{\sigma}''_\nu] C_{\nu,\mu+1}(a+\mu) = 0 \end{aligned} \tag{11}$$

arises. As a result when $\nu = 0$ and $\mu = p = 0, 1, 2, \ldots$, we find the two-term relation for the moments of distributions that we have been seeking

$$C_p \equiv C_{0p}(a+p-1) = \sum_{s=a}^{b-1} [x(s) - x(a+p-1)]^{(p)} \rho(s) \nabla x_1(s), \tag{12}$$

where

$$[x(s) - x(a+p-1)]^{(p)} = \prod_{k=0}^{p-1} [x(s) - x(a+k)] = \sum_{k=0}^{p} S_p^{(k)}(a) x^k(s).$$

Here we have chosen as an example the moments for the discrete orthogonality property. Actually the same line of reasoning remains valid for the continuous orthogonality property and, in particualr, for the Askey-Wilson polynomials.

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Resumen. Se consideran los momentos de las funciones de peso para los polinomios ortogonales discretos clásicos.