

## About one class of special functions

N.M. Atakishiyev\*

Physics Institute, Baku 370143, USSR

S.K. Suslov

Kurchatov Institute of Atomic Energy, Moscow 123182, USSR

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**Abstract.** We prove a theorem which allows to construct in explicit form some particular solutions for difference equations of the hypergeometric type on non-uniform lattices. The main properties of these solutions are derived. We discuss the classical orthogonal polynomials of a discrete variable on non-uniform lattices, the functions of the second kind of a discrete variable, as well as the difference analogues on linear nets for classical special functions of mathematical physics as examples.

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*In the following exposition of the Calculus of Finite Differences, particular attention has been paid to the connection of its methods with those of the Differential Calculus —a connection which in some instances involves far more than a merely formal analogy.*

George Boole [1]

### 1. Introduction

Special functions of mathematical physics (classical orthogonal polynomials, hypergeometric functions, Bessel functions, etc.) are widely used in many branches of physics. They permeate various areas of mathematics and have deep applications in practice. Therefore the properties of these functions have been studied thoroughly (see for example references 2-9 and those therein).

Special functions usually arise as solutions of second-order differential equations. In this connection their theory may be generalized in a natural way, replacing these differential equations by difference ones (with the use of the approximation of derivatives on non-uniform lattices for the general case) and constructing exact solutions of such difference equations by analogy with the well-known solutions of differential ones. The study of the class of special functions originating in this way (the difference analogues of special functions of hypergeometric type) began as early as the last century [1, 10-14]. As it is clear now, in these classical works of Boole [1], Thomae [10, 11], Chebyshev [12], and Heine [13, 14], the

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\* Visitor at Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México-Cuernavaca

first families of such functions were introduced. For further developments in this area see, for example, the profound works of Hahn [15]. But a general theory of difference analogues of special functions in their totality, exposing results as explicitly as possible in all occurring classes of non-uniform lattices is apparently absent in the literature up to now.

The basic idea behind the approach, that we shall discuss in the present paper, follows from the works [9, 17–25] dealing with the classical orthogonal polynomials of a discrete variable, *i.e.* the difference analogues of the Jacobi, Laguerre and Hermite polynomials on uniform and non-uniform lattices. Here the simple method of using classical orthogonal polynomials to develop hypergeometric functions [9] is generalized to the solutions of appropriate difference equations on non-uniform lattices introduced in Ref. 18 (see also references 17 and 26). It turns out that such difference analogues of hypergeometric-type functions can be naturally derived, characterized and classified in this way.

The paper is organized as follows. Section 2 deals with preliminary notions. The main theorem is formulated in Section 3. The sketch of the proof is presented in Section 5, after discussion of the necessary tools in Section 4. Some properties of the aforementioned class of special functions are derived in Section 6 and, finally, Section 7 contains some of the simplest examples including the classical orthogonal polynomials of a discrete variable and the functions of the second kind.

## 2. Preliminary notions and notations

The special functions of mathematical physics are the particular solutions of hypergeometric-type differential equation [9]

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0. \tag{1}$$

In order to generalize their theory we will solve a hypergeometric-type difference equation on a lattice  $x = x(z)$  with the non-uniform step  $\Delta x(z) = x(z + 1) - x(z)$  [17, 18]:

$$\tilde{\sigma}[x(z)] \frac{\Delta}{\Delta x(z - 1/2)} \left[ \frac{\nabla y(z)}{\nabla x(z)} \right] + \frac{\tilde{\tau}[x(z)]}{2} \left[ \frac{\Delta y(z)}{\Delta x(z)} + \frac{\nabla y(z)}{\nabla x(z)} \right] + \lambda y(z) = 0, \tag{2}$$

where  $\Delta y(z) = y(z + 1) - y(z)$ ,  $\nabla y(z) = y(z) - y(z - 1)$ ,  $\tilde{\sigma}(x)$  and  $\tilde{\tau}(x)$  are polynomials of degrees at most two and one, respectively, and  $\lambda$  is a constant.

As it is known [18], for the classes of non-uniform lattices

$$x(z) = \begin{cases} C_1 q^z + C_2 q^{-z} + \beta/(1 - \alpha), & \alpha \neq \pm 1, \\ 4\beta z^2 + C_1 z + C_2, & \alpha = 1; \end{cases} \tag{3}$$

$$\tag{4}$$



( $\alpha$ ,  $\beta$ ,  $C_1$  and  $C_2$  are constants,  $q = q_1^{1/2} = q_2^{-1/2}$ ,  $q_1$  and  $q_2$  are the roots of the equation  $q^2 - 2\alpha q + 1 = 0$ ), the *fundamental property* is valid, i.e. the difference differentiation of the original equation (2) on a non-uniform lattice results in an equation of the same type.

The following *basic equations* are satisfied

$$\frac{1}{2}[x(z + \mu) + x(z)] = \alpha(\mu)x\left(z + \frac{\mu}{2}\right) + \beta(\mu), \quad (5)$$

$$x(z + \mu) - x(z) = \alpha(\mu)\nabla x\left(z + \frac{\mu + 1}{2}\right), \quad (6)$$

$$\frac{1}{4}[x(z + \mu) - x(z)]^2 = \gamma^2(\mu)\left[(\alpha^2 - 1)x^2\left(z + \frac{\mu}{2}\right) + 2(\alpha + 1)\beta x\left(z + \frac{\mu}{2}\right) + C_3\right], \quad (7)$$

where

$$\alpha(\mu) = \begin{cases} \frac{q^{\mu/2} + q^{-\mu/2}}{2}, \\ 1; \end{cases} \quad \beta(\mu) = \begin{cases} \beta \frac{1 - \alpha(\mu)}{1 - \alpha}, \\ \beta\mu^2; \end{cases}$$

$$\gamma(\mu) = \begin{cases} \frac{q^{\mu/2} - q^{-\mu/2}}{q^{1/2} - q^{-1/2}}, \\ \mu; \end{cases} \quad C_3 = \begin{cases} \frac{\alpha + 1}{\alpha - 1}\beta^2 - 4(\alpha^2 - 1)C_1C_2, \\ \frac{1}{4}C_1^2 - 4\beta C_2 \end{cases}$$

for the lattices (3) and (4), respectively.

**DEFINITION.** *By analogy with the solutions (1) (see Ref. 9), solutions of equation (2) will be called difference functions of hypergeometric type.*

For  $\lambda = \lambda_n = -\gamma(n)[\alpha(n-1)\tilde{\tau}' + \gamma(n-1)\tilde{\sigma}''/2]$ ,  $n = 0, 1, 2, \dots$ , the partial solutions of equation (2) are classical orthogonal polynomials of a discrete variable on non-uniform lattices. The general approach to the theory of these polynomials was investigated in Ref. 18 on the basis of equation (2). For further developments in their theory, see Refs. 9 and 19 to 25, while Ref. 27 contains considerable simplifications in the proofs. For the alternative approach based on the use of the theory of the basic hypergeometric series, see, for example, Refs. 26 and 28 to 34. The present paper is concerned with the explicit construction of exact solutions of (2) for arbitrary values of  $\lambda$  and the derivation of their main properties.

### 3. Formulation of the main theorem

The particular solutions of (1) can be constructed by generalization of the Rodrigues formula for the Jacobi, Laguerre and Hermite polynomials, which leads in a natural way to the well-known integral representations for classical special functions of mathematical physics [9]. In Ref. 35 similar exact solutions were constructed for the difference equation (2) in the case of the linear net  $x(z) = z$ . The present paper deals with the general case of the non-uniform lattices (3) and (4). The main result is as follows.

**THEOREM (S.K. Suslov).** *Let a lattice  $x(z)$  satisfy the equations (5)–(7). Then the equation (2) has the particular solutions*

$$y = y_\nu(z) = \frac{C_\nu}{\rho(z)} \varphi_{\nu\nu}(z), \tag{8}$$

where  $C_\nu$  is a constant,  $\varphi_{\nu\nu}(z) = \varphi_{\nu\mu}(z)|_{\mu=\nu}$ ,

$$\varphi_{\nu\mu}(z) = \begin{cases} \sum_{s=a}^{b-1} \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\mu+1)}}, & (9) \\ \int_C \frac{\rho_\nu(s) \nabla x_{\nu+1}(s) ds}{[x_\nu(s) - x_\nu(z)]^{(\mu+1)}}, & (10) \end{cases}$$

$C$  is a contour in the complex  $S$  plane, and  $x_\nu(s) = x(s + \nu/2)$ , provided that the four conditions below are satisfied. They are the properties that generalize to non-uniform lattices the well-known identities  $(s - z)(s - z)^\mu = (s - z)^\mu(s - z) = (s - z)^{\mu+1}$ .

i) functions  $\rho(z)$  and  $(\rho_\nu(z))$  are solutions of the equations

$$\Delta(\sigma\rho) = \rho\tau\nabla x_1, \quad \Delta(\sigma\rho_\nu)\rho_\nu\tau_\nu\nabla x_{\nu+1} \tag{11}$$

with

$$\sigma = \tilde{\sigma}(x) - \frac{1}{2}\tilde{\tau}(x)\nabla x_1, \quad \tau = \tilde{\tau}(x); \tag{12}$$

$$\tau_\nu(z)\nabla x_{\nu+1}(z) = \sigma(z + \nu) - \sigma(z) + \tau(z + \nu)\nabla x_1(z + \nu), \tag{13}$$

and  $\nu$  is a root of the equation

$$\lambda + \alpha(\nu - 1)\gamma(\nu)\tilde{\tau}' + \frac{1}{2}\gamma(\nu - 1)\gamma(\nu)\tilde{\sigma}'' = 0; \tag{14}$$

ii) for the “generalized power”, i.e. for the function  $[x_\nu(s) - x_\nu(z)]^{(\mu)}$ , the following properties hold

$$\begin{aligned}
 & [x_\nu(s) - x_\nu(z)][x_\nu(s) - x_\nu(z-1)]^{(\mu)} \\
 &= [x_\nu(s) - x_\nu(z)]^{(\mu)}[x_\nu(s) - x_\nu(z-\mu)] = [x_\nu(s) - x_\nu(z)]^{(\mu+1)}, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 & [x_{\nu-1}(s) - x_{\nu-1}(z)]^{(\mu)}[x_{\nu-\mu}(s) - x_{\nu-\mu}(z)] \\
 &= [x_{\nu-\mu}(s+\mu) - x_{\nu-\mu}(z)][x_{\nu-1}(s) - x_{\nu-1}(z)]^{(\mu)} \\
 &= [x_\nu(s) - x_\nu(z)]^{(\mu+1)}; \quad (16)
 \end{aligned}$$

iii) the difference differentiation of the functions  $\varphi_{\nu\mu}(z)$  with  $\mu = \nu - 1$  and  $\mu = \nu$  may be derived by the formula

$$\frac{\nabla \varphi_{\nu\mu}(z)}{\nabla x_{\nu-\mu}(z)} = \gamma(\mu + 1)\varphi_{\nu,\mu+1}(z); \quad (17)$$

iv) in the cases (9) and (10) the ‘boundary conditions’ are valid, respectively:

$$\left. \frac{\sigma(s)\rho_\nu(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z+1)]^{(\nu+1)}} \right|_a^b = 0, \quad (18)$$

$$\int_C \Delta_s \left\{ \frac{\sigma(s)\rho_\nu(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z+1)]^{(\nu+1)}} \right\} ds = 0. \quad (19)$$

This theorem was first formulated in Ref. 23 and it plays the central role in the approach under consideration (cf. Ref. 9).

#### 4. Necessary lemmas

In this section we introduce lemmas which will be required for the proof of the main theorem.

LEMMA 0. *Let the properties (6), (15) and (16) be valid for the functions  $x(z)$  and  $[x_\nu(s) - x_\nu(z)]^{(\mu)}$ , respectively. Then*

$$\frac{\Delta_z}{\Delta x_{\nu-\mu+1}(z)} [x_\nu(s) - x_\nu(z)]^{(\mu)} = - \frac{\nabla_s}{\nabla x_{\nu+1}(s)} [x_{\nu+1}(s) - x_{\nu+1}(z)]^{(\mu)}$$



$$= -\gamma(\mu)[x_\nu(s) - x_\nu(z)]^{(\mu-1)}, \tag{20}$$

$$\begin{aligned} & \frac{\nabla_z}{\nabla x_{\nu-\mu+1}(z)} \left\{ \frac{1}{[x_\nu(s) - x_\nu(z)]^{(\mu)}} \right\} \\ &= -\frac{\Delta_s}{\Delta x_{\nu-1}(s)} \left\{ \frac{1}{[x_{\nu-1}(s) - x_{\nu-1}(z)]^{(\mu)}} \right\} = \frac{\gamma(\mu)}{[x_\nu(s) - x_\nu(z)]^{(\mu+1)}}. \end{aligned} \tag{21}$$

These equations generalize to non-uniform lattices the well-known differentiation formulas

$$\begin{aligned} \frac{d}{dz}(s-z)^\mu &= -\frac{d}{ds}(s-z)^\mu = -\mu(s-z)^{\mu-1}, \\ \frac{d}{dz} \left[ \frac{1}{(s-z)^\mu} \right] &= -\frac{d}{ds} \left[ \frac{1}{(s-z)^\mu} \right] = \frac{\mu}{(s-z)^{\mu+1}}. \end{aligned}$$

**LEMA 1.** *Let a lattice  $x(z)$  satisfy the identities (5)-(7). Then functions  $\sigma_\nu(z)$  and  $\tau_\nu(z)$  defined by the equations*

$$\begin{aligned} \sigma_\nu(z) &= \sigma(z) + \frac{1}{2}\tau_\nu(z)\nabla x_{\nu+1}(z), \\ \tau_\nu(z)\nabla x_{\nu+1}(z) &= \sigma(z+\nu) - \sigma(z) + \tau(z+\nu)\nabla x_1(z+\nu), \end{aligned}$$

(for  $\sigma(z)$  and  $\tau(z)$ , see (12)) are polynomials of degrees at most two and one in  $x_\nu = x(z + \nu/2)$ , respectively:

$$\begin{aligned} \sigma_\nu(z) &= \tilde{\sigma}_\nu(x_\nu) = \tilde{\sigma}_\nu(0) + \tilde{\sigma}'_\nu(0)x_\nu + \frac{1}{2}\tilde{\sigma}''_\nu x_\nu^2, \\ \tau_\nu(z) &= \tilde{\tau}_\nu(x_\nu) = \tilde{\tau}_\nu(0) + \tilde{\tau}'_\nu x_\nu, \end{aligned}$$

with

$$\begin{aligned} \tilde{\sigma}''_\nu &= 2(\alpha^2 - 1)\gamma(2\nu)\tilde{\tau}' + \alpha(2\nu)\tilde{\sigma}'', \\ \tilde{\tau}'_\nu &= \alpha(2\nu)\tilde{\tau}' + \frac{1}{2}\gamma(2\nu)\tilde{\sigma}''; \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}'_\nu(0) &= (\alpha^2 - 1)\gamma(\nu)\tilde{\tau}(0) + \left[ \frac{3}{2}(\alpha + 1)\beta\gamma(2\nu) + (\alpha^2 - 1)\beta(\nu)\gamma(\nu) \right] \tilde{\tau}' \\ &+ \alpha(\nu)\tilde{\sigma}'(0) + [\alpha(\nu)\beta(\nu) + (\alpha + 1)\beta\gamma^2(\nu)]\tilde{\sigma}'', \\ \tilde{\tau}_\nu(0) &= \alpha(\nu)\tilde{\tau}(0) + [\alpha(\nu)\beta(\nu) + (\alpha + 1)\beta\gamma^2(\nu)]\tilde{\tau}' \\ &+ \gamma(\nu)\tilde{\sigma}'(0) + \beta(\nu)\gamma(\nu)\tilde{\sigma}''; \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}_\nu(0) &= (\alpha + 1)\beta\gamma(\nu)\tilde{\tau}(0) + \gamma(\nu)[(\alpha + 1)\beta\beta(\nu) + \alpha(\nu)C_3]\tilde{\tau}' \\ &\quad + \tilde{\sigma}(0) + \beta(\nu)\tilde{\sigma}'(0) + \frac{1}{2}[\beta^2(\nu) + \gamma^2(\nu)C_3]\tilde{\sigma}'' . \end{aligned}$$

These equations generalize the equations found in Refs. 18, 9, and 21, for  $\nu = n = 0, 1, 2, \dots$  to arbitrary values of  $\nu$ .

LEMMA 2. *Let the conditions of the main theorem be satisfied. Then the function*

$$P(s) = \gamma(\mu + 1)\sigma(s) - \tau_\nu(s) \left[ x_{\nu-\mu} \left( s - \frac{1}{2} \right) - x_{\nu-\mu} \left( z + \frac{1}{2} \right) \right] \quad (22)$$

has the form

$$\begin{aligned} P(s) &= A_0 + A_1[x_\nu(s) - x_\nu(z - 1)] \\ &\quad + A_2[x_\nu(s) - x_\nu(z + 1)][x_\nu(s) - x_\nu(z - \mu)] , \end{aligned} \quad (23)$$

where

$$\begin{aligned} A_0 &= \gamma(\mu + 1)\sigma(z + 1) , \\ A_1 &= \frac{\Delta\sigma(z) - \tau_{\nu-\mu}(z)\nabla x_{\nu-\mu+1}(z)}{\Delta x_{\nu-\mu}(z)} , \\ A_2 &= -\alpha(\mu + 1)\tilde{\tau}'_\nu + \gamma(\mu + 1)\frac{\tilde{\sigma}''_\nu}{2} = -\chi_{2\nu-\mu} \\ &= -\alpha(2\nu - \mu - 1)\tilde{\tau}'_\nu - \frac{1}{2}\gamma(2\nu - \mu - 1)\tilde{\sigma}'' , \end{aligned}$$

where  $\chi_\nu = \alpha(\nu - 1)\tilde{\tau}' + (1/2)\gamma(\nu - 1)\tilde{\sigma}''$ .

Proofs of these lemmas are obtained by direct analogy with the method used in Ref. 35.

### 5. Sketch of the proof

We will derive the theorem of Section 3 by analogy with the proof of the main theorem for the differential equation (1) in Ref. 9 (see also Refs. 35 and 36).

Let us multiply the Pearson-type equation

$$\Delta_s[\sigma(s)\rho_\nu(s)] = \tau_\nu(s)\rho_\nu(s)\nabla x_{\nu+1}(s)$$

by  $1/[x_{n\mu-1}(s+1) - x_{\nu-1}(z+1)]^{(\mu+1)}$  and transform the left-hand side using the identity

$$\Delta_s[f(s)g(s)] = f(s)\Delta_s g(s) + g(s+1)\Delta_s f(s)$$

with  $f(s) = \sigma(s)\rho_\nu(s)$  and  $g(s+1) = 1/[x_{\nu-1}(s+1) - x_{\nu-1}(z+1)]^{(\mu+1)}$ . Then

$$\begin{aligned} \Delta_s \left\{ \frac{\sigma(s)\rho_\nu(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z+1)]^{(\mu+1)}} \right\} + \gamma(\nu+1) \frac{\sigma(s)\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z+1)]^{(\mu+2)}} \\ = \frac{\tau_\nu(s)\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_{\nu-1}(s+1) - x_{\nu-1}(z+1)]^{(\mu+1)}}. \end{aligned}$$

Assuming the particular solution is in the form of (9), we put here  $s = a, a+1, \dots, b-1$  and sum over  $s$  taking into account (18). As a result, one can write

$$\sum_s \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z+1)]^{(\mu+2)}} P(s) = 0. \tag{24}$$

With the aid of (22) and (23) we obtain from (24)

$$\begin{aligned} \sigma(z+1) \frac{\Delta\varphi_{\nu\mu}(z)}{\Delta x_{\nu-\mu}(z)} + \frac{\Delta\sigma(z) - \tau_{\nu-\mu}(z)\nabla x_{\nu-\mu+1}(z)}{\Delta x_{\nu-\mu}(z)} \varphi_{\nu\mu}(z) \\ = \chi_{2\nu-\mu} \varphi_{\nu,\mu-1}(z), \end{aligned} \tag{25}$$

Using the equations  $\Delta(\sigma\rho_\nu) = \tau_\nu\rho_\nu\nabla x_{\nu+1}$  and  $\rho_{\nu-\mu}(z)\vartheta_{\nu\mu}(z) = C_{\nu\mu}\varphi_{\nu\mu}(z)$ , it is not hard to derive that the left-hand side of (25) is equal to

$$C_{\nu\mu}^{-1} \sigma(z+1)\rho_{\nu-\mu}(z+1) \frac{\Delta\vartheta_{\nu\mu}(z)}{\Delta x_{\nu-\mu}(z)}.$$

Thus, we have

$$\sigma(z+1)\rho_{\nu-\mu}(z+1) \frac{\Delta\vartheta_{\nu\mu}(z)}{\Delta x_{\nu-\mu}(z)} = \chi_{2\nu-\mu} C_{\nu\mu} \varphi_{\nu,\mu-1}(z). \tag{26}$$

The action of operator  $(1/\nabla x_1)\nabla$  on the both sides of (26) leads to the equation

$$\frac{\Delta}{\Delta x_{\nu-\mu-1}(z)} \left[ \sigma(z)\rho_{\nu-\mu}(z) \frac{\nabla\vartheta_{\nu\mu}(z)}{\nabla x_{\nu-\mu}(z)} \right] + \mu(\nu)\rho_{\nu-\mu}(z)\vartheta_{\nu\mu}(z) = 0, \tag{27}$$

where  $\mu(\nu) = -\gamma(\mu)\chi_{2\nu-\mu}$ . Putting  $\mu = \nu$ , we finally come to the equation (2) in the self-adjoint form. Using the same considerations *mutatis mutandis* one can derive the solutions in the integral form (10).



**6. Some properties of the difference hypergeometric-type functions**

The representation (8) allows us to derive some main properties of the functions under consideration.

1. In view of (26) we have

$$\frac{\Delta y_\nu(z)}{\Delta x(z)} = \frac{C_\nu^{(1)}}{\sigma(z+1)\rho(z+1)} \varphi_{\nu,\nu-1}(z) \tag{28}$$

with  $C_\nu^{(1)} = \chi_\nu C_\nu = \left[ \alpha(\nu-1) + \frac{1}{2}\gamma(\nu-1)\tilde{\sigma}'' \right] C_\nu$ . It is thus clear that

$$\Delta^{(k)} y_\nu(z) = \frac{C_\nu^{(k)}}{\rho_k(z)} \varphi_{\nu,\nu-k}(z) \tag{29}$$

where

$$\Delta^{(k)} = \left( \frac{\Delta}{\Delta x_{k-1}} \right) \cdots \left( \frac{\Delta}{\Delta x_0} \right), \quad C_\nu^{(k)} = \prod_{p=0}^{k-1} \chi_{\nu+p} C_\nu, \tag{30}$$

$$\rho_k(z) = \rho(z+k) \prod_{p=1}^k \sigma(z+p).$$

2. From (8) and (28) one can derive the *difference differentiation formula*

$$\sigma(z) \frac{\nabla y_\nu(z)}{\nabla x(z)} = \frac{\chi_\nu}{\tau'_\nu} \left[ \gamma(\nu+1) \frac{C_\nu}{C_{\nu+1}} y_{\nu+1}(z) - \tau_\nu(z) y_\nu(z) \right]. \tag{31}$$

The proof goes similarly as in Ref. 9 with proper modifications.

3. Under appropriate conditions the *three-term recurrence relation*

$$x(z)y_\nu(z) = \alpha_\nu y_{\nu+1}(z) + \beta_\nu y_\nu(z) + \gamma_\nu y_{\nu-1}(z) \tag{32}$$

is also valid. The constants  $\alpha_\nu$ ,  $\beta_\nu$ , and  $\gamma_\nu$  are equal to

$$\alpha_\nu = \frac{\gamma(\nu+1)\chi_\nu C_\nu}{\tilde{\tau}'_\nu \tilde{\tau}'_{\nu-1/2} C_{\nu+1}},$$

$$\beta_\nu = \gamma(\nu) \frac{\tilde{\tau}_{\nu-1}(0)}{\tilde{\tau}'_{\nu-1}} - \gamma(\nu+1) \frac{\tilde{\tau}_\nu(0)}{\tilde{\tau}'_\nu} - [\gamma(\nu)D_\nu - \gamma(\nu+1)D_{\nu+1}],$$

$$\gamma_\nu = -\frac{\tilde{r}'_{\nu-1}\tilde{\sigma}_{\nu-1}(-\tilde{r}_{\nu-1}(0)/\tilde{r}'_{\nu-1})C_\nu}{\tilde{r}'_{\nu-1/2}C_{\nu-1}}.$$

Here,  $D_\nu = 0$  for  $\beta = 0$  and  $D_\nu = (\nu^2 - 1/3)$  for  $\alpha = 1$ .

### 7. Examples

Let us use the main theorem to give the simplest solutions of the equation (2).

#### a) Classical orthogonal polynomials of a discrete variable

For  $\nu = n = 0, 1, 2, \dots$  we may derive solutions of the equation (2) in the form (9):

$$y = y_n(z) = \frac{\ln q}{q^{1/2} - q^{-1/2}} [\gamma(n)]^{(n)} \frac{B_n}{2\pi i \rho(z)} \int_C \frac{\rho_n(s) \nabla x_{n+1}(s) ds}{[x_n(s) - x_n(z)]^{(n+1)}}, \quad (33)$$

where  $[\gamma(n)]^{(n)} = \gamma(1)\gamma(2)\dots\gamma(n)$ ,  $B_n$  is a constant,

$$[x_n(s) - x_n(z)]^{(m)} = \begin{cases} \prod_{k=0}^{m-1} [x_n(s) - x_n(z - k)], & m > 0, \\ 1, & m = 0; \end{cases}$$

the function  $\rho_n(s)$  is defined by (26);  $C$  is a contour in the complex  $s$  plane which circumflexes the points  $s = z, z - 1, \dots, z - n$ , does not include some other singularities of the integrand and may be shifted by unity.

Using the identity

$$\nabla_z^{(n)} \left[ \frac{1}{[x_n(s) - x_n(z)]} \right] = \frac{[\gamma(n)]^{(n)}}{[x_n(s) - x_n(z)]^{n+1}}, \quad \nabla^{(n)} = \left( \frac{\nabla}{\nabla x_1} \right) \dots \left( \frac{\nabla}{\nabla x_n} \right),$$

which holds for lattices (3) and (4), and Cauchy's theorem, we obtain from (33) the *Rodrigues-type formula* for classical orthogonal polynomials of a discrete variable on non-uniform lattices [18, 26]:

$$y_n(z) = \frac{B_n}{\rho(z)} \nabla^{(n)} [\rho_n(z)]. \quad (34)$$

Hence, the relation (33) is the *integral representation* for these polynomials.

For additional properties of the polynomials under consideration, some of their applications and further references, see references 9, 12, 15, 17-26 and 28-34.

b) *The functions of the second kind of a discrete variable*

The second linearly-independent solution of equation (2) for  $\nu = n$  may be written in the form:

$$y = Q_n(z) = \frac{B_n[\gamma(n)]^{(n)}}{\rho(z)} \sum_{s=a}^{b-1} \frac{\rho_n(s) \nabla x_{n+1}(s)}{[x_n(s) - x_n(z)]^{(n+1)}}. \quad (35)$$

The constants  $a$  and  $b$  are chosen here in accordance with the condition

$$\sigma(s)\rho(s)x^k\left(s - \frac{1}{2}\right) \Big|_{s=a,b} = 0 \quad (k = 0, 1, 2, \dots), \quad (36)$$

which holds for the polynomials (34) in the case of discrete orthogonality relations (see, for example, Refs. 9 and 21). The functions (35) will be called the *functions of the second kind of a discrete variable*.

With the aid of (35) and (34) we can obtain a relationship between the functions  $Q_n(z)$  and the polynomials  $y_n(z)$ :

$$Q_n(z) = \frac{1}{\rho(z)} \sum_{s=a}^{b-1} \frac{y_n(s)\rho(s)\nabla x_1(s)}{x(s) - x(z)} \quad (z \neq a, a+1, \dots, b-1).$$

In the points  $z = a, a+1, \dots, b-1$  one can define the functions  $Q_n(z)$  by a proper limit [35]. It is also not hard to define the functions  $Q_n(z)$  in the case of continuous orthogonality relations for the polynomials (34), studied in Refs. 22, 24–26, 28, 30, 33, and 34. An example of such function of the second kind is discussed in Refs. 37. For additional properties of the aforementioned functions of the second kind, see Ref. 36.

c) *Some analogues of hypergeometric-type special functions on linear nets*

In the case  $x(z) = z$  with the aid of linear transformations the equation (2) may be reduced to the following canonical forms

$$\begin{aligned} (z - \gamma)(z - \delta)\Delta\nabla u + [(\alpha + \beta + 1)z - \gamma\delta]\Delta u + \alpha\beta u &= 0, \\ (z - \gamma)\Delta\nabla u + (\gamma - tz)\Delta u - \alpha tu &= 0, \\ (z - \gamma)\Delta\nabla u + \gamma\Delta u - tu &= 0, \\ z\Delta\nabla u + (t - z)\Delta u + \nu u &= 0. \end{aligned}$$



The appropriate partial solutions are [1, 10, 11, 38]

$$u = F\left(\begin{matrix} z, \alpha, \beta \\ \gamma, \delta \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(z)_k (\alpha)_k (\beta)_k}{(\gamma)_k (\delta)_k k!},$$

$$u = F(z, \alpha, \gamma, t) = \sum_{k=0}^{\infty} \frac{(z)_k (\alpha)_k t^k}{(\gamma)_k k!},$$

$$u = F(z, \gamma, t) = \sum_{k=0}^{\infty} \frac{(z)_k t^k}{(\gamma)_k k!},$$

$$u = t^{-z/2} W_{\frac{1}{2}(z+\nu+1), \frac{1}{2}(z-\nu)}(t),$$

respectively. Here  $W_{k,\mu}(t)$  is the Whittaker function. For details, see Ref. 38.

d) *Some solutions of difference hypergeometric-type equation on non-uniform lattices*

With the aid of the main theorem the simplest solutions of the equation (2) were constructed above. A number of further examples can be found in Refs. 15 and 39 to 41. Naturally, it is of interest to construct an analogue the Gauss hypergeometric function on non-uniform lattices. In this connection we discuss in short two more types of solutions.

1. For the lattice  $x(z) = z^2$  owing to (12) we have

$$\sigma(z) = (z - a)(z - b)(z - c)(z - d),$$

where  $a, b, c,$  and  $d$  are arbitrary complex numbers. Let us choose the following solutions of the equations (11):

$$\rho_\nu(s) = C_0 \Gamma(a + \nu + s) \Gamma(a - s) \Gamma(b + \nu + s) \Gamma(b - s) \\ \times \Gamma(c + \nu + s) \Gamma(c - s) \Gamma(d + \nu + s) \Gamma(d - s) \sin 2\pi \left[ s + \frac{1}{2}(\nu + 1) \right],$$

$$\rho(z) = \frac{\rho_0(z)}{\sin 2\pi \left( s + \frac{1}{2} \right)}, \quad C_0^{-1} = (-1)^{\nu+1} \frac{\sin \pi (s - z + \nu + 1)}{\sin (s - z) \pi}.$$

Defining the "generalized power" in the form [36]

$$[x_\nu(s) - x_\nu(z)]^{(\mu)} = \frac{\Gamma(s - z + \mu) \Gamma(s + z + \nu + 1)}{\Gamma(s - z) \Gamma(s + z + \nu - \mu + 1)}, \quad x(z) = z^2,$$

from (8) and (10) we obtain

$$\begin{aligned}
 y_\nu(z) &= \frac{A}{\rho(z)} \int_C \frac{\rho_\nu(s) \nabla x_{\nu+1}(s) ds}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}} \\
 &= \frac{A'}{\rho(z)} \int_{-i\infty}^{i\infty} dt \frac{\Gamma(a + \frac{\nu}{2} + t) \Gamma(a + \frac{\nu}{2} - t) \Gamma(b + \frac{\nu}{2} + t) \Gamma(b + \frac{\nu}{2} - t)}{\Gamma(2t) \Gamma(-2t)} \\
 &\quad \times \frac{\Gamma(c + \frac{\nu}{2} + t) \Gamma(c + \frac{\nu}{2} - t) \Gamma(d + \frac{\nu}{2} + t) \Gamma(d + \frac{\nu}{2} - t) \Gamma(z - \frac{\nu}{2} + t) \Gamma(z - \frac{\nu}{2} - t)}{\Gamma(1 + z + \frac{\nu}{2} + t) \Gamma(1 + z + \frac{\nu}{2} - t)}.
 \end{aligned} \tag{37}$$

As was shown in Ref. 41, this integral representation leads to the function  ${}_7F_6(1)$ , which is represented in accordance with Ref. 42 as a sum of two functions of the  ${}_4F_3(1)$  type.

2. In the case when  $x(z) = \cosh 2\omega z = (1/2)(q^z + q^{-z})$ ,  $q = e^{2\omega}$ , according to (12) it is possible to write in the very general form

$$\sigma(z) = q^{-2z} (q^z - a)(q^z - b)(q^z - c)(q^z - d).$$

Therefore for this case we have

$$\rho(z) = \rho(z, a, b, c, d) = f_q(z) \prod_{\vartheta=a,b,c,d} g(z, \vartheta),$$

where  $f_q^{-1}(z) = \Gamma_q(2z) \Gamma_q(-2z) (q^z - q^{-z})$ ,  $|q| < 1$ ;

$$g^{-1}(z, \vartheta) = \prod_{k=0}^{\infty} [1 - \vartheta(q^z + q^{-z})q^k + \vartheta^2 q^{2k}], \quad |\vartheta| < 1,$$

and  $\Gamma_q(w)$  is the  $q$ -gamma function (see for example Ref. 39). Taking into account the arbitrariness in the choice of normalization and periodic factors it is not also difficult to verify that

$$\begin{aligned}
 \rho_\nu\left(t - \frac{1}{2}\nu\right) &= \text{constant } \rho\left(t, aq^{\nu/2}, bq^{\nu/2}, cq^{\nu/2}, dq^{\nu/2}\right), \\
 \left[x_\nu\left(t - \frac{1}{2}\nu\right) - x_\nu(z)\right]^{(\nu+1)} &= \text{constant } q^{-(\nu+1)z} \frac{g\left(t, q^{1+z+\nu/2}\right)}{g\left(t, q^{z-\nu/2}\right)}.
 \end{aligned}$$

As a result from the formulas (8) and (10) we obtain the following particular

solution of the equation (2):

$$y_\nu(z) = \frac{Aq^{(\nu+1)z}}{\rho(z, a, b, c, d)} \int_{C'} \frac{\rho(t, aq^{\nu/2}, bq^{\nu/2}, cq^{\nu/2}, dq^{\nu/2})}{g(t, q^{1+z+\nu/2})g^{-1}(t, q^{z-\nu/2})} x'(t) dt. \quad (38)$$

The contour  $C'$  is located on the imaginary axis:  $t = it'$ ,  $0 \leq t' \leq \pi \ln^{-1} q$ ; while  $-1 \leq x(t) \leq 1$ .

According to Ref. 40 the integral representation of the (38) type defines a very well-poised  ${}_8\phi_7$  [42]. This permits one to draw an analogy with the known Euler integral representation for the hypergeometric function. We have obtained that this function is a solution of the equation (2) and it has all properties considered in Section 6. With the appropriate choice of the parameters (see Ref. 41) in the limit  $q \rightarrow 1$  the formula (38) goes over to (37).

We have discussed here only the simplest solutions of the equation (2). Similarly using the main theorem one can construct difference functions of hypergeometric type in other cases. Further development of such class of special functions should certainly be made. In particular, it is worthwhile to study as explicitly as possible the deep analogy with classical hypergeometric functions including new interpretations and forthcoming generalizations of many well-known results. It is also interesting to develop group theoretical approach to the theory of these functions.

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**Resumen.** Se demuestra un teorema que permite construir en forma explícita algunas soluciones particulares para ecuaciones diferenciales del tipo hipergeométrico sobre redes no-uniformes. Se derivan las propiedades principales de estas soluciones. Discutimos como ejemplos los polinomios ortogonales clásicos de una variable discreta sobre redes no-uniformes, las funciones de segunda clase de una variable discreta, así como las analogías diferenciales sobre redes lineales para las funciones especiales clásicas de la física matemática.