Dirac equation in orthogonal curvilinear coordinates

E. Ley-Koo and R.C. Wang*

Instituto de Física, Universidad Nacional Autónoma de México, apartado postal 20-364, 01000 México, D.F.

Abstract. We discuss the covariant form of the Dirac equation in orthogonal curvilinear coordinates and exhibit it explicitly. We present explicit forms of the Dirac equation in some of the common orthogonal curvilinear coordinates.

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1. Introduction

The construction of the Dirac equation in Minkowski spacetime is guided by the requirement of covariance under Lorentz transformations, which ensures consistency with the relativity principle [1]. The Lorentz transformations are linear transformations between cartesian spacetime coordinates, involving fixed bases of unit vectors; the corresponding Dirac γ matrices are also fixed once a given representation is chosen. In the study of some problems it may be convenient or necessary to use curvilinear instead of cartesian coordinates. In such cases, the coordinate transformations are no longer linear and the unit vector bases change from point to point, but the covariance requirement must still be satisfied. In the solution of such problems it is common to use the curvilinear coordinates keeping the reference to the original cartesian unit vectors and fixed Dirac γ matrices [2]. The question of covariance is not usually discussed, so its analysis is taken up in this paper. V. Bargmann [3] studied the problem originally.

As the background for the discussion, Section 2 contains a brief review of the covariance of the Dirac equation in cartesian coordinates, with emphasis on the effect of Lorentz transformations on Dirac spinors and γ matrices. In Section 3, we obtain explicitly the covariant form of the Dirac equation in Minkowski spacetime and orthogonal curvilinear coordinates, including the general expressions for the difference between the covariant and partial derivatives, and for the coordinate dependent Dirac γ' matrices. In Section 3, the explicit forms of the Dirac equation in some of the common orthogonal curvilinear coordinates are presented, concluding with some didactical remarks.

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2. Dirac spinors and γ matrices under Lorentz transformations

The analysis of the covariance of the Dirac equation in cartesian coordinates can be found in the appropriate textbooks [2]. Nevertheless, our own version of such an analysis is presented in this section for the sake of completeness, and as a point of reference for the corresponding discussion in orthogonal curvilinear coordinates. We introduce the Lorentz transformations, first, emphasizing their interpretation as rotations in the four-dimensional spacetime. Then, we analyze the covariance of the Dirac equation to show the corresponding transformation properties of the spinors and γ matrices, constructing the spinor representation of the rotations along the way.

The four-vector giving the spacetime positions of events in Minkowski space can be written as

$$\mathbf{x} = \hat{\mathbf{e}}_\mu \, x_\mu = \hat{\mathbf{e}}'_\mu \, x'_\mu, \tag{1}$$

in terms of its cartesian components $x_\mu(x, y, z, x_4 = ict)$, in two inertial frames of reference with orthogonal unit vector bases $\hat{\mathbf{e}}_\mu$ and $\hat{\mathbf{e}}'_\mu$, respectively. We use the Einstein summation convention over repeated dummy indices $\mu = 1, 2, 3, 4$. The Lorentz transformations give the relation between the primed and unprimed components, and follow by projecting equation (1) along the desired direction:

$$x'_\mu = \hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}_\nu \, x_\nu \equiv O_{\mu\nu} x_\nu. \tag{2}$$

The elements of the transformation matrix $O_{\mu\nu} \equiv \hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}_\nu$ are simply the projections of the unit vectors of one basis along the directions of the unit vectors of the other basis. The inverse transformation can be written in the alternative forms

$$x_\mu = \hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}'_\nu \, x'_\nu \equiv (O^{-1})_{\mu\nu} x'_\nu = O_{\nu\mu} x'_\nu = (\tilde{O})_{\mu\nu} x'_\nu. \tag{2a}$$

This allows us to conclude that the transformation matrix is orthogonal

$$O^{-1} = \tilde{O} \quad \text{or} \quad O\tilde{O} = \tilde{O}O = 1,$$

i.e.

$$O_{\mu\lambda} \tilde{O}_{\lambda\nu} = O_{\mu\lambda} O_{\nu\lambda} = \delta_{\mu\nu}$$

and

$$\tilde{O}_{\mu\lambda} O_{\lambda\nu} = O_{\lambda\mu} O_{\lambda\nu} = \delta_{\mu\nu}, \tag{3}$$

The orthogonality of the transformation is a natural consequence of the orthogonal character of the unit vector bases. This orthogonality makes it possible to interpret the Lorentz transformations as rotations in Minkowski space.

The invariance of the square of the spacetime interval between two events

$$\Delta\mathbf{x} \cdot \Delta\mathbf{x} = \Delta x_\mu \Delta x_\mu = \Delta x'_\mu \Delta x'_\mu, \tag{4}$$

can be established directly using equations (1) or (2–3). In some textbooks [2], this invariance is taken as the starting point to establish the orthogonal character of the Lorentz transformations.

The proper Lorentz transformations include the identity transformations $O_{\mu\nu} = \delta_{\mu\nu}$, which can be used to establish that $\det \mathbf{O} = 1$.

Infinitesimal Lorentz transformations differ slightly from the identity transformation

$$O_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}, \qquad \tilde{O}_{\mu\nu} = \delta_{\nu\mu} + \epsilon_{\nu\mu}, \tag{5}$$

where $|\epsilon_{\mu\nu}| \ll 1$. From the orthogonality property, equation (3), it follows that the infinitesimal rotational parameters are antisymmetric

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \tag{6}$$

Then we can rewrite

$$O_{\mu\nu} = \delta_{\mu\nu} + \tfrac{1}{2}(\epsilon_{\mu\nu} - \epsilon_{\nu\mu}) = \delta_{\mu\nu} + \tfrac{1}{2}(\epsilon_{\alpha\beta}\mathbf{L}^{\alpha\beta})_{\mu\nu}, \tag{5a}$$

in terms of the generators of the rotation $\mathbf{L}^{\alpha\beta} = \mathbf{E}^{\alpha\beta} - \mathbf{E}^{\beta\alpha}$, where $\mathbf{E}^{\alpha\beta}$ is a 4×4 matrix whose μ-th row and ν-th column element is $(\mathbf{E}^{\alpha\beta})_{\mu\nu} = \delta_\mu^\alpha \delta_\nu^\beta$.

Any finite Lorentz transformation can be constructed as a succession of infinitesimal transformations of the type of equation (5a), taking the exponential form

$$O = e^{\epsilon_{\alpha\beta}\mathbf{L}^{\alpha\beta}/2}. \tag{7}$$

The summations in the exponent imply a multiplication of exponentials, and the order of such factors must be taken appropriately because, in general, rotations do not commute with each other. It is straightforward to establish that the matrix elements of each one of the exponential factors can be written as

$$\left(e^{\epsilon_{\alpha\beta}\mathbf{L}^{\alpha\beta}/2}\right)_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu} + \frac{1}{2!}\epsilon_{\mu\lambda}\epsilon_{\lambda\nu} + \frac{1}{3!}\epsilon_{\mu\lambda}\epsilon_{\lambda\sigma}\epsilon_{\sigma\nu} + \cdots. \tag{8}$$

The Dirac equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + k_0\right)\psi(x_\nu) = 0, \tag{9}$$

is written in terms of the inverse of the Compton wavelength of the particle, $k_0 = mc/\hbar$, the four-component Dirac spinor wavefunction, ψ, and the 4×4 Dirac γ_μ matrices that satisfy the anticommutation rules

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \mathbf{I}. \tag{10}$$

The Dirac matrices are not uniquely defined. According to Pauli's fundamental theorem [2], other representations of these matrices, γ'_μ, satisfying the corresponding commutation rules, can be constructed from the starting representation

$$\gamma'_\mu = \mathbf{S}\gamma_\mu \mathbf{S}^{-1}, \tag{11}$$

involving a nonsingular 4×4 matrix $\mathbf{S}$. The same matrix relates the new spinor wavefunction with the original one,

$$\psi'(x_\nu) = \mathbf{S}\psi(x_\nu). \tag{12}$$

We are interested, now, in analyzing the changes in the quantities appearing in the Dirac equation when the Lorentz transformation of equation (2) takes place. From the covariance requirement on the Dirac equation, we expect to be able to write it as

$$\left(\gamma'_\mu \frac{\partial}{\partial x'_\mu} + k_0\right)\psi'(x'_\nu) = 0. \tag{9a}$$

The four-gradient operators are connected by the Lorentz transformation of equation (2),

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = O_{\mu\nu}\frac{\partial}{\partial x_\nu}, \tag{13}$$

as it follows from the application of the chain rule and of equation (2a). The new spinor wavefunction is expected to be related to the original one through a linear rearrangement of their components

$$\psi'(x'_\mu) = \mathbf{\Lambda}\psi(x_\nu), \tag{14}$$

where $\mathbf{\Lambda}$ is a 4×4 matrix.

By substituting equations (13) and (14) in equation (9a), multiplying from the left by $\mathbf{\Lambda}^{-1}$, and rearranging the factors in the first term we obtain

$$\left(\mathbf{\Lambda}^{-1}\gamma'_\mu O_{\mu\nu}\mathbf{\Lambda}\frac{\partial}{\partial x_\nu} + k_0\right)\psi(x_\lambda) = 0, \tag{9b}$$

which is just the original equation (9), if we identify

$$\gamma_\nu = \mathbf{\Lambda}^{-1}\gamma'_\mu O_{\mu\nu}\mathbf{\Lambda}. \tag{15}$$

This equation (15) gives the connection between the Dirac γ matrices, reflecting their behavior as components of a four-vector, equation $(2a)$, and as spinor operators; it can also be rewritten as

$$\gamma'_\mu O_{\mu\nu} = \mathbf{\Lambda}\gamma_\nu\mathbf{\Lambda}^{-1}. \tag{15a}$$

Next, we will construct the explicit form of the spinor transformation matrix $\mathbf{\Lambda}$ appearing in equations (14) and (15). Using the freedom allowed by Pauli's theorem, equation (11), we also choose the same fixed representation of the Dirac γ matrices in equations (9) and $(9a)$, *i.e.* we can take $\gamma'_\mu = \gamma_\mu$. First, we consider infinitesimal Lorentz transformations, equation (5), writing the corresponding spinor matrices

$$\mathbf{\Lambda} = \mathbf{I} + \tfrac{1}{4}\epsilon_{\mu\nu}\mathbf{T}^{\mu\nu}, \qquad \mathbf{\Lambda}^{-1} = \mathbf{I} - \tfrac{1}{4}\epsilon_{\mu\nu}\mathbf{T}^{\mu\nu}, \tag{16}$$

where $\mathbf{T}^{\mu\nu}$ are 4×4 matrices that share the antisymmetric character $T^{\mu\nu} = -T^{\nu\mu}$ of the rotational parameters $\epsilon_{\mu\nu}$. Substitution in equation $(15a)$ leads to the relation

$$\gamma_\mu\mathbf{T}^{\lambda\nu} - \mathbf{T}^{\lambda\nu}\gamma_\mu = 2\delta_{\lambda\mu}\gamma_\nu - 2\gamma_\lambda\delta_{\mu\nu}, \tag{17}$$

which is satisfied by the explicit spinor representation of the generators

$$\mathbf{T}^{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{18}$$

Then the spinor representation of the rotation corresponding to the finite Lorentz transformation of equation (7) takes the exponential form

$$\mathbf{\Lambda} = e^{\epsilon_{\mu\nu}T^{\mu\nu}/4} = e^{\epsilon_{\mu\nu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/8}, \tag{19}$$

constructed with a succession of infinitesimal transformations of the type of equation (16).

3. Covariant form of the Dirac equations

Let $q_a(q_1, q_2, q_3, q_4)$ be the orthogonal curvilinear coordinates of points in Minkowski space. The Jacobian matrix for the transformation from the cartesian to the curvilinear coordinates can be written as the product of a rotation matrix

and a scale matrix,

$$\frac{\partial x_\mu}{\partial q_a} = H_{ab}O_{b\mu}. \tag{20}$$

The matrix $\mathbf{O}$ describes the rotation to go from the unit vector basis associated with the cartesian coordinates to the corresponding basis associated with the curvilinear coordinates. Its properties and representations are the same as in equations (3), (5) and (7), the difference in Sections 2 and 3 being that the rotational parameters are the same everywhere cartesian coordinates are used, because the unit vector bases are fixed, while those parameters change from point to point when curvilinear coordinates are used, because the unit vectors change from point to point. The matrix $\mathbf{H}$ describes the change of scales associated with the curvilinear coordinates; it is a diagonal matrix if the coordinates are orthogonal,

$$H_{ab} = h_a \delta_{ab}, \tag{21}$$

where h_a is the scale factor associated with the coordinate q_a. The Jacobian matrix for the inverse transformation is the inverse of equation (20),

$$\frac{\partial q_a}{\partial x_\mu} = (O^{-1})_{\mu b}(H^{-1})_{ba}. \tag{22}$$

The Dirac equation in cartesian coordinates, equation (9), can be rewritten in terms of curvilinear coordinates as

$$\left(\gamma_\mu \frac{\partial q_a}{\partial x_\mu}\frac{\partial}{\partial q_a} + k_0\right)\psi[x_\nu(q_b)] = 0. \tag{23}$$

This form, as pointed out in the introduction, keeps the reference to the original cartesian unit vectors and Dirac matrices, as witnessed by the continued use of the index μ and of the original wavefunction ψ.

If we want to write the Dirac equation in terms of the curvilinear coordinates and the associated vectors, which may change their orientation from point to point, it is necessary to recognize that the wavefunction is changed under the corresponding rotations. The transformation of the spinor wave-function has the same form as equation (14),

$$\psi'(q_a) = \mathbf{A}(q_a)\psi[x_\nu(q_a)], \tag{24}$$

with the important difference that the rotational parameters, and consequently the transformation matrix, equation (19), depend on the location.

Multiplication of equation (23) from the left by the matrix $\mathbf{\Lambda}$, insertion of the unit 4×4 matrix $\mathbf{\Lambda}^{-1}\mathbf{\Lambda}$ in the first term of the same equation, and straightforward calculation allow us to write the Dirac equation for the wavefunction in curvilinear coordinates

$$\left[\mathbf{\Lambda}\gamma_\mu \frac{\partial q_a}{\partial x_\mu}\mathbf{\Lambda}^{-1}\left(\frac{\partial}{\partial q_a} + \mathbf{\Lambda}\frac{\partial \mathbf{\Lambda}^{-1}}{\partial q_a}\right) + k_0\right]\psi'(q_b) = 0. \tag{23a}$$

Here we can identify the coordinate dependent Dirac γ matrices

$$\gamma'_a = \mathbf{\Lambda}\gamma_\mu \frac{\partial q_a}{\partial x_\mu}\mathbf{\Lambda}^{-1}, \tag{25}$$

and the compensating field

$$B_a = -\mathbf{\Lambda}\frac{\partial \mathbf{\Lambda}^{-1}}{\partial q_a} = \frac{\partial \mathbf{\Lambda}}{\partial q_a}\mathbf{\Lambda}^{-1}, \tag{26}$$

in terms of which the general form of the Dirac equation in orthogonal curvilinear coordinates becomes

$$\left[\gamma'_a\left(\frac{\partial}{\partial q_a} - B_a\right) + k_0\right]\psi'(q_b) = 0. \tag{23b}$$

The expression for the Dirac γ'_a matrices, equation (25), can be simplified through the following considerations. It involves matrix factors of the form

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{27}$$

where $A = \epsilon_{\mu\nu}\mathbf{T}^{\mu\nu}/4$, equations (18) and (19), and $B = \gamma_\mu$. The corresponding commutators can be evaluated to be

$$[\tfrac{1}{4}\epsilon_{\alpha\beta}T^{\alpha\beta}, \gamma_\mu] = \gamma_\lambda\epsilon_{\lambda\mu}. \tag{28}$$

Therefore for each exponential factor in $\mathbf{\Lambda}$, equation (19),

$$e^{\epsilon_{\alpha\beta}T^{\alpha\beta}/4}\gamma_\mu e^{-\epsilon_{\alpha\beta}T^{\alpha\beta}/4}$$

$$= \gamma_\mu + \gamma_\lambda\epsilon_{\lambda\mu} + \frac{1}{2!}\gamma_\lambda\epsilon_{\lambda\sigma}\epsilon_{\sigma\mu} + \frac{1}{3!}\gamma_\lambda\epsilon_{\lambda\sigma}\epsilon_{\sigma\rho}\epsilon_{\rho\mu} + \cdots \tag{29}$$

$$= \gamma_\lambda(\delta_{\lambda\mu} + \epsilon_{\lambda\mu} + \frac{1}{2!}\epsilon_{\lambda\sigma}\epsilon_{\sigma\rho}\epsilon_{\rho\mu} + \cdots).$$

It can be recognized that the expression in the last parenthesis is the $(\lambda\mu)$ matrix element of the corresponding factor in the rotation matrix, equations (8) and (7). Consequently, by successive application of this result to the exponential factors involved in the rotation operators it is established that

$$\mathbf{\Lambda}\gamma_\mu\mathbf{\Lambda}^{-1} = \gamma_\tau O_{\tau\mu}, \tag{30}$$

which expresses the effect of the rotation on the γ matrices, just like equation (15b). By substituting the result of equation (30) and equation (22) in equation (25), we finally obtain

$$\gamma'_a = \gamma_\tau O_{\tau\mu}(O^{-1})_{\mu b}(H^{-1})_{ba} = \frac{\gamma_a}{h_a}. \tag{25b}$$

In conclusion, the covariant form of the Dirac equation in orthogonal curvilinear coordinates is

$$\left[\gamma'_a\frac{D}{Dq_a} + k_0\right]\psi'(q_b) = 0, \tag{23c}$$

where the coordinate dependent γ' matrices differ from the ordinary ones by the scale factor, equation (25b), and the covariant and partial derivatives in equations (23c) and (23b) differ by the compensating field B_a, equation (26).

This general form of the Dirac equation naturally includes the case in which cartesian coordinates are used. In such a case, the scale matrix is the unit matrix, all the scale factors being equal to one, and the rotational parameters have the same values at all points, the new unit vector basis being fixed. Correspondingly, the γ' matrices have the same form as the γ matrices, equation (25b), and the compensating field is null, equation (26).

4. Explicit forms and discussion

The explicit forms of the Dirac equation in circular cylindrical, elliptic cylindrical, parabolic cylindrical, spherical, prolate spheroidal, oblate spheroidal and parabolic coordinates are presented in this section. The equations are numbered as in Section 3 after the initials of each type of orthogonal curvilinear coordinates.

Circular cylindrical coordinates

$$x_1 = \rho\cos\varphi, \qquad x_2 = \rho\sin\varphi, \qquad x_3 = z, \qquad x_4 = ict;$$

$$q_1 = \rho, \qquad q_2 = \varphi, \qquad q_3 = z, \qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = H e^{\varphi L^{12}}, \qquad (CC20)$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{-\varphi L^{12}} H^{-1}, \quad (CC22)$$

$$\Lambda = e^{\varphi T^{12}/2}, \qquad \Lambda^{-1} = e^{-\varphi T^{12}/2}, \qquad (CC19)$$

$$B_\varphi = \tfrac{1}{2}T^{12} = \tfrac{1}{2}\gamma_1\gamma_2, \qquad \gamma'_\varphi B_\varphi = -\frac{1}{2\rho}\gamma_1, \qquad (CC26.2)$$

$$\left[\gamma_1\left(\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right) + \gamma_2\frac{\partial}{\rho\partial\varphi} + \gamma_3\frac{\partial}{\partial z} + \gamma_4\frac{\partial}{ic\partial t} + k_0\right]\psi' = 0. \qquad (CC23b)$$

Elliptic cylindrical coordinates

$$x_1 = f\cosh u\cos v, \qquad x_2 = f\sinh u\sin v, \qquad x_3 = z, \qquad x_4 = ict;$$

$$q_1 = u, \qquad\qquad q_2 = v, \qquad\qquad q_3 = z, \qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix} f\sqrt{\cosh^2 u - \cos^2 v} & 0 & 0 & 0 \\ 0 & f\sqrt{\cosh^2 u - \cos^2 v} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \dfrac{\sinh u\cos v}{\sqrt{\cosh^2 u-\cos^2 v}} & \dfrac{\cosh u\sin v}{\sqrt{\cosh^2 u-\cos^2 v}} & 0 & 0 \\ -\dfrac{\cosh u\sin v}{\sqrt{\cosh^2 u-\cos^2 v}} & \dfrac{\sinh u\cos v}{\sqrt{\cosh^2 u-\cos^2 v}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = H e^{\alpha L^{12}},$$

$$(EC20)$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = e^{-\alpha L^{12}} \begin{pmatrix} \dfrac{1}{f\sqrt{\cosh^2 u - \cos^2 v}} & 0 & 0 & 0 \\ 0 & \dfrac{1}{f\sqrt{\cosh^2 u - \cos^2 v}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (EC22)$$

where

$$\cos\alpha = \frac{\sinh u \cos v}{\sqrt{\cosh^2 u - \cos^2 v}}, \qquad \sin\alpha = \frac{\cosh u \sin v}{\sqrt{\cosh^2 u - \cos^2 v}}.$$

$$\Lambda = e^{\alpha T^{12}/2}, \Lambda^{-1} = e^{-\alpha T^{12}/2} \qquad (EC19)$$

$$B_u = \frac{1}{2}\frac{\partial\alpha}{\partial u}T^{12} = -\frac{\sin v \cos v}{2(\cosh^2 u\ \cos^2 v)}\gamma_1\gamma_2, \qquad (EC26.1)$$

$$\gamma_u' B_u = -\frac{\sin v \cos v}{2f(\cosh^2 u\ \cos^2 v)^{3/2}}\gamma_2,$$

$$B_v = \frac{1}{2}\frac{\partial\alpha}{\partial v}T^{12} = -\frac{\sinh u \cosh u}{2(\cosh^2 u\ \cos^2 v)}\gamma_1\gamma_2, \qquad (EC26.2)$$

$$\gamma_v' B_v = -\frac{\sinh u \cosh u}{2f(\cosh^2 u\ \cos^2 v)^{3/2}}\gamma_1,$$

$$\left\{\frac{1}{f\sqrt{\cosh^2 u - \cos^2 v}}\left[\gamma_1\left(\frac{\partial}{\partial u} - \frac{\sinh u \cosh u}{2(\cosh^2 u - \cos^2 v)}\right)\right.\right.$$
$$\left.\left. + \gamma_2\left(\frac{\partial}{\partial v} + \frac{\sin v \cos v}{2(\cosh^2 u - \cos^2 v)}\right)\right] + \gamma_3\frac{\partial}{\partial z} + \gamma_4\frac{\partial}{ic\partial t} + k_0\right\}\psi' = 0.$$
$$(EC23'')$$

Parabolic cylindrical coordinates

$$x_1 = \tfrac{1}{2}(\xi^2 - \eta^2), \qquad x_2 = \xi\eta, \qquad x_3 = z, \qquad x_4 = ict;$$
$$q_1 = \xi, \qquad\qquad q_2 = \eta, \qquad q_3 = z, \qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix} \sqrt{\xi^2 + \eta^2} & 0 & 0 & 0 \\ 0 & \sqrt{\xi^2 + \eta^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{\xi}{\sqrt{\xi^2+\eta^2}} & \frac{\eta}{\sqrt{\xi^2+\eta^2}} & 0 & 0 \\ -\frac{\eta}{\sqrt{\xi^2+\eta^2}} & \frac{\xi}{\sqrt{\xi^2+\eta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = He^{\alpha L^{12}}, \qquad (PC20)$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = e^{-\alpha L^{12}} \begin{pmatrix} \frac{1}{\sqrt{\xi^2+\eta^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\xi^2+\eta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (PC22)$$

where

$$\cos\alpha = \frac{\xi}{\sqrt{\xi^2 + \eta^2}}, \qquad \sin\alpha = \frac{\eta}{\sqrt{\xi^2 + \eta^2}},$$

$$\Lambda = e^{\alpha T^{12}/2}, \qquad \Lambda^{-1} = e^{-\alpha T^{12}/2}, \qquad (PC19)$$

$$B_\xi = \frac{1}{2}\frac{\partial\alpha}{\partial\xi}T^{12} = -\frac{\eta}{2(\xi^2+\eta^2)}\gamma_1\gamma_2, \qquad \gamma'_\xi B_\xi = -\frac{\eta}{2(\xi^2+\eta^2)^{3/2}}\gamma_2, \quad (PC26.1)$$

$$B_\eta = \frac{1}{2}\frac{\partial\alpha}{\partial\eta}T^{12} = \frac{\xi}{2(\xi^2+\eta^2)}\gamma_1\gamma_2, \qquad \gamma'_\eta B_\eta = -\frac{\xi}{2(\xi^2+\eta^2)^{3/2}}\gamma_1, \quad (PC26.2)$$

$$\left\{ \frac{1}{\sqrt{\xi^2+\eta^2}}\left[\gamma_1\left(\frac{\partial}{\partial\xi} + \frac{\xi}{2(\xi^2+\eta^2)}\right) + \gamma_2\left(\frac{\partial}{\partial\eta} + \frac{\eta}{2(\xi^2+\eta^2)}\right)\right] \right.$$

$$\left. + \gamma_3\frac{\partial}{\partial z} + \gamma_4\frac{\partial}{ic\partial t} + k_0 \right\}\psi' = 0. \qquad (PC23b)$$

Spherical coordinates

$$x_1 = r \sin \theta \cos \varphi, \qquad x_2 = r \sin \theta \sin \varphi, \qquad x_3 = r \cos \theta, \qquad x_4 = ict;$$

$$q_1 = \theta, \qquad q_2 = \varphi, \qquad q_3 = r, \qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a} \right) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r \sin \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \cos \varphi & \sin \varphi & 0 & 0 \\ -\sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = H e^{\theta L^{31}} e^{\varphi L^{12}}, \qquad (S20)$$

$$\left(\frac{\partial q_a}{\partial x_\mu} \right) = e^{-\varphi L^{12}} e^{-\theta L^{31}} \begin{pmatrix} 1/r & 0 & 0 & 0 \\ 0 & \frac{1}{r \sin \theta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (S22)$$

$$\Lambda = e^{\theta T^{31}/2} e^{\varphi T^{12}/2}, \qquad \Lambda^{-1} = e^{-\varphi T^{12}/2} e^{-\theta T^{31}/2}, \qquad (S19)$$

$$B_\theta = \tfrac{1}{2} T^{31} = \frac{1}{2} \gamma_3 \gamma_1, \qquad \gamma_\theta' B_\theta = -\frac{1}{2r} \gamma_3, \qquad (S26.1)$$

$$B_\varphi = e^{\theta T^{31}/2} \tfrac{1}{2} T^{12} e^{-\theta T^{31}/2} = \tfrac{1}{2} (\gamma_1 \gamma_2 \cos \theta - \gamma_2 \gamma_3 \sin \theta). \qquad (S26.2)$$

The last result is evaluated by using equation (27) and the commutator $[\gamma_\mu \gamma_\nu, \gamma_\nu \gamma_\lambda] = 2 \gamma_\mu \gamma_\lambda$.

$$\gamma_\varphi' B_\varphi = \frac{1}{2r \sin \theta} (-\gamma_1 \cos \theta - \gamma_3 \sin \theta),$$

$$\left[\gamma_1 \frac{1}{r} \left(\frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) + \gamma_2 \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} + \gamma_3 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) + \gamma_4 \frac{\partial}{ic \partial t} + k_0 \right] \psi' = 0. \ (S23b)$$

Prolate spheroidal coordinates

$$x_1 = f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \quad x_2 = f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi, \quad x_3 = f\xi\eta, \quad x_4 = ict;$$

$$q_1 = \xi, \qquad\qquad\qquad q_2 = \varphi, \qquad\qquad\qquad q_3 = \eta, \quad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix} f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}} & 0 & 0 & 0 \\ 0 & \sqrt{(\xi^2-1)(1-\eta^2)} & 0 & 0 \\ 0 & 0 & f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi & 0 & \sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta & 0 & \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = He^{\beta L^{31}}e^{\varphi L^{12}},$$

$$(PS20)$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = e^{-\varphi L^{12}}e^{-\beta L^{31}} \begin{pmatrix} \frac{1}{f}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{f\sqrt{(\xi^2-1)(1-\eta^2)}} & 0 & 0 \\ 0 & 0 & \frac{1}{f}\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(PS22)$$

where

$$\cos\beta = \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\xi, \qquad \sin\beta = -\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\eta.$$

$$\Lambda = e^{\beta T^{31}/2}e^{\varphi T^{12}/2}, \qquad \Lambda^{-1} = e^{-\varphi T^{12}/2}e^{-\beta T^{31}/2}, \qquad (PS19)$$

$$B_\xi = \frac{1}{2}\frac{\partial\beta}{\partial\xi}T^{31} = -\frac{1}{2}\sqrt{\frac{1-\eta^2}{\xi^2-1}}\frac{\eta}{(\xi^2-\eta^2)}\gamma_3\gamma_1, \qquad (PS26.1)$$

$$\gamma'_\xi B_\xi = \frac{1}{2f}\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\frac{\eta}{(\xi^2-\eta^2)}\gamma_3,$$

$$B_\varphi = e^{\beta T^{31}/2}\tfrac{1}{2}T^{12}e^{-\beta T^{31}/2} = \tfrac{1}{2}(\gamma_1\gamma_2\cos\beta - \gamma_2\gamma_3\sin\beta), \qquad (PS26.2)$$

$$\gamma'_\varphi B_\varphi = \frac{1}{2f}\left\{-\gamma_1\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\frac{\xi}{(\xi^2-1)} + \gamma_3\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\frac{\eta}{(1-\eta^2)}\right\},$$

$$B_\eta = \frac{1}{2}\frac{\partial\beta}{\partial\eta}T^{31} = -\frac{1}{2}\sqrt{\frac{\xi^2-1}{1-\eta^2}}\frac{\xi}{(\xi^2-\eta^2)}\gamma_3\gamma_1 \qquad (PS26.3)$$

$$\gamma'_\eta B_\eta = -\frac{1}{2f}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\frac{\xi}{\xi^2-\eta^2}\gamma_1,$$

$$\left\{\gamma_1\frac{1}{f}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\left(\frac{\partial}{\partial\xi} + \frac{\xi}{2(\xi^2-1)} + \frac{\xi}{2(\xi^2-\eta^2)}\right) + \gamma_2\frac{1}{f\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial}{\partial\varphi}\right.$$
$$\left. + \gamma_3\frac{1}{f}\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\left(\frac{\partial}{\partial\eta} - \frac{\eta}{2(\xi^2-\eta^2)} - \frac{\eta}{2(1-\eta^2)}\right) + \gamma_4\frac{\partial}{ic\partial t} + k_0\right\}\psi' = 0.$$
$$(PS23b)$$

Oblate spheroidal coordinates

$$x_1 = f\sqrt{(\varsigma^2+1)(1-\omega^2)}\cos\varphi, \quad x_2 = f\sqrt{(\varsigma^2+1)(1-\omega^2)}\sin\varphi, \quad x_3 = f\varsigma\omega, \quad x_4 = ict;$$
$$q_1 = \varsigma, \qquad\qquad q_2 = \varphi, \qquad\qquad q_3 = \omega, \qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix} f\sqrt{\frac{\varsigma^2+\omega^2}{\varsigma^2+1}} & 0 & 0 & 0 \\ 0 & f\sqrt{(\varsigma^2+1)(1-\omega^2)} & 0 & 0 \\ 0 & 0 & f\sqrt{\frac{\varsigma^2+\omega^2}{1-\omega^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\varsigma & 0 & \sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\omega & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\omega & 0 & \sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\varsigma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = He^{\beta L^{31}}e^{\varphi L^{12}}.$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = e^{-\varphi L^{12}}e^{-\beta L^{31}} \begin{pmatrix} \frac{1}{f}\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{f\sqrt{(\varsigma^2+1)(1-\omega^2)}} & 0 & 0 \\ 0 & 0 & \frac{1}{f}\sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(OS22)$$

where

$$\cos\beta = \sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\,\varsigma, \qquad \sin\beta = -\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\,\omega.$$

$$\Lambda = e^{\beta T^{31}/2}e^{\varphi T^{12}/2}, \qquad \Lambda^{-1} = e^{-\varphi T^{12}/2}e^{-\beta T^{31}/2}, \qquad (OS19)$$

$$B_\varsigma = \frac{1}{2}\frac{\partial\beta}{\partial\varsigma}T^{31} = \frac{1}{2}\sqrt{\frac{1-\omega^2}{\varsigma^2+1}}\,\frac{\omega}{(\varsigma^2+\omega^2)}\gamma_3\gamma_1, \qquad (OS26.1)$$

$$\gamma_\varsigma'B_\varsigma = -\frac{1}{2f}\sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\,\frac{\omega}{(\varsigma^2+\omega^2)}\gamma_3,$$

$$B_\varphi = e^{\beta T^{31}/2}\frac{1}{2}T^{12}e^{-\beta T^{31}/2} = \frac{1}{2}(\gamma_1\gamma_2\cos\beta - \gamma_2\gamma_3\sin\beta), \qquad (OS26.2)$$

$$\gamma_\varphi'B_\varphi = \frac{1}{2f}\left\{-\gamma_1\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\,\frac{\varsigma}{(\varsigma^2+1)} + \gamma_3\sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\,\frac{\omega}{(1-\omega^2)}\right\},$$

$$B_\omega = \frac{1}{2}\frac{\partial \beta}{\partial \omega}T^{31} = -\frac{1}{2}\sqrt{\frac{\varsigma^2+1}{1-\omega^2}}\frac{\varsigma}{(\varsigma^2+\omega^2)}\gamma_3\gamma_1, \tag{$OS26.3$}$$

$$\gamma'_\omega B_\omega = -\frac{1}{2f}\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\frac{\varsigma}{(\varsigma^2+\omega^2)}\gamma_1,$$

$$\left\{\gamma_1\frac{1}{f}\sqrt{\frac{\varsigma^2+1}{\varsigma^2+\omega^2}}\left(\frac{\partial}{\partial\varsigma}+\frac{\varsigma}{2(\varsigma^2+1)}+\frac{\varsigma}{2(\varsigma^2+\omega^2)}\right)+\gamma_2\frac{1}{f\sqrt{(\varsigma^2+1)(1-\omega^2)}}\frac{\partial}{\partial\varphi}\right.$$

$$\left.+\gamma_3\frac{1}{f}\sqrt{\frac{1-\omega^2}{\varsigma^2+\omega^2}}\left(\frac{\partial}{\partial\omega}-\frac{\omega}{2(\varsigma^2+\omega^2)}-\frac{\omega}{2(1-\omega^2)}\right)+\gamma_4\frac{\partial}{ic\partial t}+k_0\right\}\psi'=0 \tag{$OS23b$}$$

Parabolic coordinates

$$x_1 = \xi\eta\cos\varphi, \qquad x_2 = \xi\eta\sin\varphi, \qquad x_3 = \tfrac{1}{2}(\xi^2-\eta^2), \qquad x_4 = ict;$$

$$q_1 = \eta, \qquad\qquad q_2 = \varphi, \qquad\qquad q_3 = \xi, \qquad\qquad q_4 = ict.$$

$$\left(\frac{\partial x_\mu}{\partial q_a}\right) = \begin{pmatrix}\sqrt{\xi^2+\eta^2} & 0 & 0 & 0 \\ 0 & \xi\eta & 0 & 0 \\ 0 & 0 & \sqrt{\xi^2+\eta^2} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\xi}{\sqrt{\xi^2+\eta^2}} & 0 & -\frac{\eta}{\sqrt{\xi^2+\eta^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\eta}{\sqrt{\xi^2+\eta^2}} & 0 & \frac{\xi}{\sqrt{\xi^2+\eta^2}} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}\cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} = He^{\beta L^{31}}e^{\varphi L^{12}}, \tag{$P20$}$$

$$\left(\frac{\partial q_a}{\partial x_\mu}\right) = e^{-\varphi L^{12}}e^{-\beta L^{31}}\begin{pmatrix}\frac{1}{\sqrt{\xi^2+\eta^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\xi\eta} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\xi^2+\eta^2}} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{$P22$}$$

where

$$\cos \beta = \frac{\xi}{\sqrt{\xi^2 + \eta^2}}, \qquad \sin \beta = \frac{\eta}{\sqrt{\xi^2 + \eta^2}}.$$

$$\Lambda = e^{\beta T^{31}/2} e^{\varphi T^{12}/2}, \qquad \Lambda^{-1} = e^{-\varphi T^{12}/2} e^{-\beta T^{31}/2}, \qquad (P19)$$

$$B_\eta = \frac{1}{2} \frac{\partial \beta}{\partial \eta} T^{31} = \frac{\xi}{2(\xi^2 + \eta^2)} \gamma_3 \gamma_1, \qquad (P26.1)$$

$$\gamma'_\eta B_\eta = -\frac{1}{2} \frac{1}{\sqrt{\xi^2 + \eta^2}} \frac{\xi}{(\xi^2 + \eta^2)} \gamma_3,$$

$$B_\varphi = e^{\beta T^{31}/2} \tfrac{1}{2} T^{12} e^{-\beta T^{31}/2} = \tfrac{1}{2}(\gamma_1 \gamma_2 \cos \beta - \gamma_2 \gamma_3 \sin \beta), \qquad (P26.2)$$

$$\gamma'_\varphi B_\varphi = \frac{1}{\sqrt{\xi^2 + \eta^2}} \left(-\frac{1}{2n} \gamma_1 - \frac{1}{2\xi} \gamma_3 \right),$$

$$B_\xi = \frac{1}{2} \frac{\partial \beta}{\partial \xi} T^{31} = -\frac{\eta}{2(\xi^2 + \eta^2)} \gamma_3 \gamma_1, \qquad (P26.3)$$

$$\gamma'_\xi B_\xi = -\frac{1}{2} \frac{1}{\sqrt{\xi^2 + \eta^2}} \frac{\eta}{(\xi^2 + \eta^2)} \gamma_1,$$

$$\left\{ \frac{1}{\sqrt{\xi^2 + \eta^2}} \left[\gamma_1 \left(\frac{\partial}{\partial \eta} + \frac{1}{2\eta} + \frac{\eta}{2(\xi^2 + \eta^2)} \right) + \gamma_3 \left(\frac{\partial}{\partial \xi} + \frac{\xi}{2(\xi^2 + \eta^2)} + \frac{1}{2\xi} \right) \right] \right.$$

$$\left. + \frac{1}{\xi \eta} \gamma_2 \frac{\partial}{\partial \varphi} + \gamma_4 \frac{\partial}{ic \partial t} + k_0 \right\} \psi' = 0.$$

$$(P23b)$$

As shown in general in Section 3, and as illustrated with specific examples in this section, the differences of the Dirac equation in curvilinear and cartesian coordinates arise from the existence of scale factors and the change of orientation of the unit vector basis associated with the curvilinear coordinates. The scale factors make the difference between the Dirac γ' and γ matrices as indicated by equation (25b). The change of orientation of the unit vector basis from point to point is the source of the compensating field, equation (26).

In the specific examples, we can distinguish between the cases of cylindrical coordinates and of the other coordinates base on surfaces generated by the rotation of conics around their symmetry axes. For the cases of cylindrical coordinates, only the scale factors h_1 and h_2 may be different from one, and only the rotational parameter $\epsilon_{12} = -\epsilon_{21} = \varphi$ or α may be different from zero. For the coordinates based on conoidal surfaces of revolution, the three scale factors with $\mu = 1, 2, 3$ may be different from one, and the non-zero rotational parameters are $\epsilon_{12} = -\epsilon_{21} = \varphi$ and $\epsilon_{13} = -\epsilon_{31} = \theta$ or β. It is the values of these scale factors and rotational parameters which determine the structure of the γ' matrices and compensating fields entering in equations (23b).

The solutions of the Dirac equation in the forms of equations (23) and (23b) are related to each other through the rotation transformation of equation (24). While equation (23b) emphasizes the covariance of the Dirac equation, its solution may be more complicated than the solution of the hybrid form of equation (23) due to the presence of the compensating field. Of course, the solution of equation (23b) can also be constructed from that of equation (23) through the use of equation (24).

The anticommutation rules for the γ' matrices can be obtained from equations (25b) and (10):

$$\gamma'_a \gamma'_b + \gamma'_b \gamma'_a = 2\frac{\delta_{ab}}{h_a h_b}\mathbf{I} \equiv 2g_{ab}\mathbf{I} \tag{10'}$$

where $g_{ab} = \delta_{ab}(h_a h_b)$ is recognized to be the metric tensor.

The analysis presented in this paper can be used by the interested reader as a preliminary step to study the Dirac equation in Riemmann curved spacetime [4]. In fact, the validity of the equations of Section 3 is extended to Riemmann spacetime when the appropriate metric is used. This can be understood by identifying the type of flat spacetime assumed in our analysis as a locally flat spacetime in a small neighborhood of each point in the curved spacetime.

Reference

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Resumen. Se discute y exhibe explícitamente la forma covariante de la ecuación de Dirac en coordenadas curvilíneas ortogonales. Se presentan las formas explícitas de la ecuación de Dirac en algunas de las coordenadas curvilíneas ortogonales más comunes.