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Continuous orthogonality property for some classical polynomials of a discrete variable

N.M. Atakishiyev*

Physics Institute, Baku 370143, USSR

S.K. Suslov

Kurchatov Institute of Atomic Energy, Moscow 123182, USSR

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Abstract. The continuous orthogonality property for some classical polynomials of a discrete variable is studied. An application to a relativistic quasipotential model of the N-dimensional harmonic oscillator is discussed.

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1. Introduction

In the paper [1] the general theory of the classical orthogonal polynomials of a discrete variable on quadratic lattices was developed on the basis of a difference equation of the hypergeometric type. For some other classes of non-uniform lattices this theory was extended in Ref. [2]. The classification of polynomials arising in these cases was carried out in Refs. [3,4], these results have been further developed in Refs. [5–7]. As a result it became possible to generalize a simple approach to constructing a theory of the classical orthogonal polynomials, suggested in Ref. [5], for solutions of the corresponding hypergeometric-type difference equation on non-uniform lattices.

For further considerations it is convenient to write the hypergeometric-type difference equation in the self-adjoint form

$$\frac{\Delta}{\Delta x(z-\frac{1}{2})}\left[\sigma(z)\rho(z)\frac{\nabla y(z)}{\nabla x(z)}\right]+\lambda\rho(z)y(z)=0 \tag{1.1}$$

As was shown in Ref. [1] polynomial solutions $y=y_n(x), x=x(z)$ of the equation (1.1) for lattices of the form $x(z)=c_0z^2+c_1z+c_2$ do exist for definite values of $\lambda=\lambda_n$, and they can be represented as the following analogue of Rodrigues' formula:

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*Visitor at Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México-Cuernavaca.

$$y_n(x) = \frac{B_n}{\rho(z)} \nabla^{(n)} [\rho_n(z)]. \tag{1.2}$$

Here B_n are constants and $\rho(z)$ is a solution of the equation

$$\frac{\Delta}{\Delta x(z - \frac{1}{2})} [\sigma(z)\rho(z)] = \tau(z)\rho(z), \tag{1.3}$$

where $\sigma(z) = \tilde{\sigma}[x(z)] - \frac{1}{2}\tilde{\tau}[x(z)]\Delta x(z - \frac{1}{2})$, $\tau(z) = \tilde{\tau}[x(z)]$ ($\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in the variable x of degrees at most two and one, respectively), the function $\rho_n(z)$ is defined by the equality

$$\rho_n(z) = \rho(z+n) \prod_{k=1}^{n} \sigma(z+k), \tag{1.4}$$

and the symbol $\nabla^{(n)}$ denotes the following difference operator:

$$\nabla^{(n)} = \left(\frac{\nabla}{\nabla x_1}\right)\left(\frac{\nabla}{\nabla x_2}\right) \cdots \left(\frac{\nabla}{\nabla x_n}\right), \quad x_k = x(z + \frac{k}{2}).$$

Under definite conditions (see Refs. [5, 7]) polynomial solutions $y_m(x)$ and $y_n(x)$ of the equation (1.1) will be orthogonal for $m \neq n$, *i.e.*

$$\sum_{z_i=a}^{b-1} y_m(x_i) y_n(x_i) \rho_i \Delta x_{i-\frac{1}{2}} = 0, \tag{1.5}$$

where $x_i = x(z_i)$, $\rho_i = \rho(z_i)$, $\Delta x_{i-\frac{1}{2}} = \Delta x(z_i - \frac{1}{2})$, $z_{i+1} = z_i + 1$.

Those polynomial solutions of the equation (1.1), for which the discrete orthogonality relation (1.5) with a constant sign weight function ρ_i and a monotonic function x_i holds, following to Refs. [1,6,7], we shall call the *classical orthogonal polynomials of a discrete variable on a quadratic lattice.* The consistent description of the theory and the classification arising here of *Hahn, Meixner, Kravchuk* and *Charlier polynomials* (lattice $x(z) = z$) and also *Racah* and *dual Hahn polynomials* (lattice $x(z) = z(z+1)$) are given in Refs. [1, 5, 7].

As follows from a number of works (see, for example, Refs. [8, 9]), for some of the just mentioned polynomials the continuous orthogonality property, well-known in the theory of differential equations, also holds. In particular, for the example of the Meixner polynomials it was demonstrated [10] that if we represent the sum in (1.5) as a sum of the residues of some contour integral and subsequently open up the contour in the complex plane, then as a result of this after analytic continuation in the parameter we come to the continuous orthogonality relation for the Meixner-Pollaczek polynomials [8,11,12]. The continuous orthogonality property was also proved by direct calculation of the integral for the Hahn polynomials (linear lattice) [13,10,14], for the Racah and the dual Hahn polynomials (quadratic lattice) [9] and for some other systems of polynomials (see, for example, Refs. [15,16]).

Since all mentioned polynomials are particular solutions to the equation (1.1), it would naturally be desirable to study the continuous orthogonality property on a more general basis and to attempt the classification of the resulting orthogonal families.

In the present work * the continuous orthogonality property of polynomial solutions to the equation (1) is proved for the case of quadratic lattices. For the lattices of the form $x(z) = c_0z^2 + c_1z + c_2$ from the unified point of view the systems of polynomials are considered, which are orthogonal in the continuous sense. In this way it becomes possible to include some other important orthogonal families in the simple scheme developed earlier for constructing the theory of the classical orthogonal polynomials of a discrete variable. Some of the polynomials thus arising, for instance, the Meixner-Pollaczek polynomials, though have been known for a long time, but so far were considered separately.

As a simple example of the further generalization of this approach to the other classes of non-uniform lattices [2] the system of the Askey-Wilson polynomials is discussed.

Besides, in this work some other questions are also considered: representations through hypergeometric functions for the Racah polynomials and the dual Hahn polynomials are derived from Rodrigues' formula; an application in a relativistic quasipotential [18,19] model of the N-dimensional harmonic oscillator in the configurational representation [20] is discussed.

2. Continuous orthogonality property

For polynomial solutions of the equation (1.1) the following orthogonality property is valid. *If the conditions*

$$\int_C \nabla[\rho_1(z)x_1^k(z)]\,dz = 0 \qquad (k = 0,1,2,\ldots), \tag{2.1}$$

are satisfied then polynomial solutions of (1.2) will be orthogonal on the contour C in the complex plane of the variable z:

*Which is the extended version of our preprint [17]

$$\int_C y_m[x(z)]\, y_n[x(z)]\, \rho(z)\nabla x_1(z)\, dz = 0 \qquad (m \neq n). \tag{2.2}$$

Proof. As in the case of the discrete orthogonality relation (1.4) (see, for example, Refs. [5,7]), the easiest way to prove (2.2) is on the basis of difference equation similarly to the proof for the well-known property of the Sturm-Liouville problem. Let us write the equations for the polynomials $\bar{y}_k(z) = y_k[x(z)](k = m, n)$ in the self-adjoint form (1.1), multiply the equation for $\bar{y}_m$ by $\bar{y}_n$ and the equation for $\bar{y}_n$ by $\bar{y}_m$ and then subtract the second equation from the first one. As a result we obtain

$$(\lambda_m - \lambda_m)\bar{y}_m(z)\, \bar{y}_n(z)\, \rho(z)\nabla x_1(z) = \Delta\left\{\sigma(z)\, \rho(z)\, W\left[\bar{y}_m(z), \bar{y}_n(z)\right]\right\}, \tag{2.3}$$

where

$$W\left[\bar{y}_m(z), \bar{y}_n(z)\right] = \bar{y}_m(z)\frac{\nabla y_n(z)}{\nabla x(z)} - \bar{y}_n(z)\frac{\nabla \bar{y}_m(z)}{\nabla x(z)} = \begin{vmatrix} \bar{y}_m(z) & \bar{y}_n(z) \\ \frac{\nabla \bar{y}_m(z)}{\nabla x(z)} & \frac{\nabla \bar{y}_n(z)}{\nabla x(z)} \end{vmatrix}$$

is an analogue of the Wronskian and, as it can be shown, for the quadratic lattices is a polynomial in $x(z - \frac{1}{2})$:

$$W\left[\bar{y}_m(z), \bar{y}_n(z)\right] = \sum_k C_k x^k(z - \tfrac{1}{2}).$$

Therefore the formula (2.3) for this case has the form

$$(\lambda_m - \lambda_n) y_m[x(z)]\, y_n[x(z)]\, \rho(z)\, \nabla x_1(z) = \sum_k C_k \Delta[\sigma(z)\rho(z) x^k(z - \tfrac{1}{2}). \tag{2.4}$$

Since

$$\Delta[\sigma(z)\rho(z)x^k(z - \tfrac{1}{2})] = \nabla[\sigma(z+1)\rho(z+1)x^k(z + \tfrac{1}{2})] = \nabla[\rho_1(z)x_1^k(z)],$$

then integrating both sides of the equality (2.4) over such a contour C, for which (2.1) is valid, we come to the relation (2.2).

Remark. Our reasoning remains valid for other classes of nonuniform lattices considered in Ref. [2].

We have proved the orthogonality property of polynomial solutions to the equation (1.1) on a contour in the complex plane. In some cases (see below) it becomes possible to choose the contour in the complex plane. In some cases (see below) it becomes possible to choose the contour C in (2.2) in such a way as to obtain the continuous orthogonality relation

$$\int_a^b P_m(t)P_n(t)\rho(t)\,dt = 0 \qquad (m \neq n). \tag{2.5}$$

This arises for the real system of polynomials $p_n(t)$, obtained from the polynomials $y_n(x)$ by a change of variable and appropriate choice of parameters.

It is also natural to call the polynomials $p_n(t)$ the classical orthogonal polynomials of a discrete variable on the quadratic lattices.

3. Continuous orthogonality for the Hahn, Meixner, Racah and dual Hahn polynomials

On the basis of our approach we shall single out polynomial solutions of the equation (1.1) on the quadratic lattices, which are orthogonal in the continuous sense. It is necessary for this to find such a solutions $\rho(z)$ to the equation (1.3) and to choose such a contour C that the following requirements are satisfied:

1) The integrals in (2.1) and (2.2) exist;

2) the conditions (2.1) holds;

3) the transition from (2.2) to the real orthogonality property (2.5) is possible.

To satisfy these requirements it is convenient to make use of the following circumstances.

1° In the approach under consideration the function $\rho(z)$ isdefined as a solution of the equation (1.3) up to an arbitrary periodic factor (its period being equal to unity), which does not influence the form of polynomials obtained by the formula (1.2). This arbitrariness may be exploited when choosing a function $\rho(z)$, satisfying the conditions (2.1).

The conditions (2.1) may be rewritten as

$$\int_C f(z)dz = \int_{C'} f(z')dz', \tag{3.1}$$

where $f(z) = \rho_1(z)x_1(z)$ and C' is the contour obtained from a contour C by the shift $z' = z - 1$. The validity of this equality follows from Cauchy's theorem provided that there are no singularities of the function $\rho_1(z)$ between the contours C and C'

2° When choosing a contour C in (2.2) and for a subsequent proof of the reality of entering in (2.5) polynomials, it is convenient to make use of the symmetry relations for polynomial solutions to the equation (1.1), which follow from Rodrigues' formula (1.2).

Taking these features into consideration, we shall treat the systems of the classical orthogonal polynomials of a discrete variable on lattices of the form $x(z) = c_0 z^2 + c_1 z + c_2$, for which the property (2.5) is satisfied. By a linear change of variable these lattices can always be reduced to either the linear $x(z) = z$ or the quadratic $x(z) = z(z+1)$ ones.

1. *Lattice* $x(z) = z$

The real continuous orthogonality property holds for the Hahn and Meixner-Pollaczek polynomials.

1° In the case of the Hahn polynomials $h_n^{(\alpha,\beta)}(x,N)$ *, using the arbitrariness in the choice of the periodical factor, we select the following solution of the equation (1.3):

$$\rho(z) = \Gamma(\beta + z + 1)\Gamma(z - N + 1)\Gamma(\alpha + N - z)\Gamma(-z). \tag{3.2}$$

As a contour C we consider one which comes from infinity along the imaginary axis separating the poles of the expressions $\Gamma(\beta + z + 1)\Gamma(z + -N + 1)$ and $\Gamma(\alpha + N - z)\Gamma(-z)$. The choice of such contour is possible when $\alpha > -1$, $\beta > -1$ and N is any complex number.

The conditions (2.1) in the form (3.1) will be satisfied by virtue of Cauchy's theorem, since between the contours C and C' (obtained from C by the shift by unity to the left) there are no singularities of the integrand and, besides, $\rho_1(z)x_1^k(z) \to 0$ as $|\operatorname{Im} z| \to \infty$. Consequently, for the Hahn polynomials $h_n^{(\alpha,\beta)}(z,N)$ the orthogonality property (2.2) with the complex weight (3.2) is valid.

We now proceed to the real orthogonality property. Since $|\Gamma(3)|^2 = \Gamma(\zeta)\Gamma(\zeta^*)$, the function (3.2) will take real values in the case when the arguments of two pairs of Γ-functions are complex conjugates, i.e. when

$$\beta + z + 1 = -z^*, \qquad z - N + 1 = \alpha + N^* - z^*, \tag{3.3}$$

or when

$$\beta + z + 1 = \alpha + N^* - z^*, \qquad z - N + 1 = -z^*. \tag{3.4}$$

1) In the case (3.3) the weight $\rho(z)$ in (3.2) becomes real if as a contour C to choose the line $\operatorname{Re} z = -\beta + 1/2$ provided that $\operatorname{Re} N = -\alpha + \beta/2$. Four further consideration it is convenient to put $\operatorname{Im} z = 1/2(t + \gamma)$ and $\operatorname{Im} N = \gamma$, where t is a real variable.

Using the symmetry relation

*The Hahn polynomials with the discrete orthogonality property were introduced by P.L. Chebyshev in 1875 (see Ref. [21]).

$$h_n^{\alpha,\beta}(z,N) = (-1)^n h_n^{(\beta,\alpha)}(-\beta - z - 1, -\alpha - \beta - N), \tag{3.5}$$

which follows from Rodrigues' formula (1.2), it is easy to show that the polynomials [10]

$$P_n(t) = P_n^{(\alpha,\beta)}(t,\gamma) = i^{-n} h_n^{(\alpha,\beta)}(z,N), \tag{3.6}$$

where $z = \frac{i}{2}(t+\gamma) - \frac{1}{2}(\beta+1)$ and $N = i\gamma - \frac{1}{2}(\alpha+\beta)$, are real for the real values of the variable t and the parameters α, β and γ. As a result when $\alpha > -1$ and $\beta > -1$ the orthogonality property (2.3) with the weight given in table 1. In our normalization the estimate

$$\rho(t) \sim \frac{\pi}{2^{\alpha+\beta}} |t|^{\alpha+\beta} e^{-\pi|t|},$$

holds as $|t| \to \infty$. Here the symbol $f(t) \sim g(t)$ means that $\lim[f(t)g^{-1}(t)] = 1$ when $|t| \to \infty$.

As $\gamma \to \infty$ the limiting relation [10, 7].

$$P_n^{(\alpha,\beta)}(\gamma s, \gamma) = \gamma^n [P_n^{(\alpha,\beta)}(s) + O(\gamma^{-2})],$$

is valid, where $P_n^{(\alpha\beta)}(s)$ are the Jacobi polynomials.

2) In their turn, the conditions (3.4) are satisfied when $\alpha = \beta$ and $N = N^*$. Using the relation

$$h_n^{(\alpha,\beta)}(z,N) = (-1)^n h_n^{(\beta,\alpha)}(N - z - 1, N),$$

in a similar way one can prove the property (2.5) for the polynomials [13,10]

$$q_n^{(\alpha)}(t,\delta) = i^{-n} h_n^{(\alpha,\alpha)}(z,N),$$

where $z = \frac{i}{2}t - \frac{1}{2}(\alpha - \delta + 1)$ and $N = \delta - \alpha$ (see table 1). The equalities

$$q_n^{(\alpha)}(t,\delta) = P_n^{(\alpha,\alpha)}(t, -i\delta) = P_n^{(\alpha-\delta,\alpha+\delta)}(t,0) \tag{3.7}$$

holds. We note that the polynomials $q_n^{(0)}(t,\delta)$ with

$$\rho(t) = \frac{\pi}{2}[\cosh \pi t + \cos \pi\delta]^{-1},$$

have also been considered in Refs. [22–25] for $\delta = 0$ and in Refs. [26–31, 8] for $-1 < \delta < 1$. We also notice that the polynomials $q_n^{(\alpha)}(t, 1/2)$ have the weight function

$$\rho(t) = 2^{-(2\alpha+1)}|\Gamma(\alpha + \tfrac{1}{2} + it)|^2.$$

The polynomials $p_n^{(\alpha,\beta)}(t,\gamma)$ and $q_n^{(\alpha)}(t,\delta)$ are closely related to the unitary irreducible representations of the Lorentz group $SO(3,1)$ for the principal and complementary series, respectively [10]. We used to call them the *Hahn polynomials of an imaginary argument.* *

2° The conditions (2.1) are also satisfied for the Meixner polynomials $m_n^{(\gamma\mu)}(z)$ if $\rho(z) = (-\mu)^z\Gamma(\gamma+z)\Gamma(-z)$ and the contour C separates the poles of the functions $\Gamma(\gamma + z)$ and $\Gamma(-z)$. The product of these Γ-functions will be real if $\gamma + z = -z^*$, *i.e.*, when $\operatorname{Re} z = -\gamma/2$ and $\operatorname{Im}\gamma = 0$. In this connection we may choose the line $z = it - \gamma/2$ as a contour C. Then

$$(-\mu)^z = e^{-i(\arg\mu\pm\pi)\gamma/2}|\mu|^{it-\gamma/2}e^{-(\arg\mu\pm\pi)t}.$$

Since the first factor does not depend on t, then the real weight arises when $|\mu| = 1$.

By the symmetry relation

$$m_n^{(\gamma,\mu)}(z) = \mu^{-n}m_n^{(\gamma,\mu^{-1})}(-\gamma - z),$$

it is not difficult to show that the polynomials

$$P_n^{\lambda}(t,\varphi) = \frac{1}{n!}e^{-in\varphi}m_n^{(2\lambda,\mu)}(it - \lambda), \qquad \mu = e^{-2i\varphi},$$

*The complex orthogonality property for the Hahn polynomials was also proved by Askey [14] by the direct evaluation of the integral (2.2) after he had become acquainted with our paper [10]. In fact, the real form of the polynomials $p_n(x; a, b, c, d)$ treated by Askey, differs from (3.6) only by an unessential shift of the argument

$$P_n(x; a, b, c, d) = \text{const.}P_n^{(2\operatorname{Re}a-1,2\operatorname{Re}b-1)}(\operatorname{Im}a + \operatorname{Im}b - 2x,\ \operatorname{Im}b - \operatorname{Im}a).$$

The orthogonality of these polynomials for the case corresponding to the principal series of the unitary irreducible representations of the Lorentz group is also discussed in Ref. [32].

are real for the real value of t, λ and φ. As a result we come to the orthogonality* for the *Meixner-Pollaczek polynomials* $P_n^{\lambda}(t,\varphi)$ [33,34,11,12,8] (see table 1).

The polynomials $q_n^{(0)}(t,1/2)$ and $P_n^{(1/2)}(t,\pi/2)$ are orthogonal on the interval $(-\infty,\infty)$ with the weight $\rho(t) = \text{const.} \cosh^{-1} \pi t$. Therefore the equality

$$q_n^{(0)}(t,\tfrac{1}{2}) = (\tfrac{1}{2})_n P_n^{(1/2)}(t,\tfrac{\pi}{2})$$

is valid [10]. The more general expression has the form [35]

$$q^{(\alpha)}(t,\tfrac{1}{2}) = \frac{(2\alpha+n+1)_n}{2^{2n}} P_n^{(\alpha+1/2)}(t,\tfrac{\pi}{2})$$

2. *Lattice* $x(z) = z(z+1)$

1° In the case of the Racah polynomials $U_n^{(\alpha,\beta)}(x,a,b)$ the conditions (2.1) will be satisfied for a function

$$\begin{aligned}\rho(z) =&\Gamma(a+z+1)\Gamma(a-z)\Gamma(z-a+\beta+1)\Gamma(\beta-a-z)\Gamma(z-b+1)\\ &\Gamma(-b-z)\Gamma(b+\alpha-z)\Gamma(b+\alpha+z+1)\sin\pi(2z+1),\end{aligned}$$

if the contour C separates the poles of Γ-functions, going out to the left and to the right, respectively. By analogy with the Jacobi polynomials the parameters α and β are considered to be real, while the parameters a and b may take complex values. The weight will be real if

$$\left.\begin{aligned} a+z+1 &= \beta - a^* - z^* \\ \beta - a + z + 1 &= a^* - z^* \end{aligned}\right\}, \qquad \left.\begin{aligned} z-b+1 &= b^* + \alpha - z^* \\ b+\alpha+z+1 &= -b^* - z^* \end{aligned}\right\}.$$

i.e. when $\operatorname{Re} z = -1/2$, $\operatorname{Re} a = \beta/2$, $\operatorname{Re} b = -\alpha/2$. For further consideration it is convenient to put $\operatorname{Im} z = s$, $\operatorname{Im} a = \delta$ and $\operatorname{Im} b = \gamma$.

Using the relation

$$u_n^{(\alpha,\beta)}[x(z),a,b] = u_n^{(\alpha,\beta)}[x(-z-1),\beta-a,-\alpha-b],$$

*As is known [10, 14], there exists a close relationship between the Meixner-Pollaczek polynomials and the Meixner polynomials, though the former satisfy a continuous orthogonality relation and the later, a discrete one. In particular, the orthogonality for the Meixner-Pollaczek polynomials can be obtained by the analytic continuation of the discrete orthogonality for the Meixner polynomials with the aid of the Sommerfeld-Watson transformation [10].

$P_n(t)$	$P_n^{(\alpha,\beta)}(t,\gamma)$	$q_n^{(\alpha)}(t,\delta)$	$P_n^{\lambda}(t,\phi)$
(a,b)	$(-\infty,\infty)$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\rho(t)$	$\frac{1}{4\pi}\left\vert\Gamma\left(i\frac{t-\gamma}{2}+\frac{\alpha+1}{2}\right)\Gamma\left(i\frac{t+\gamma}{2}+\frac{\beta+1}{2}\right)\right\vert^2$	$\frac{1}{4\pi}\left\vert\Gamma\left(i\frac{t}{2}+\frac{\alpha+\delta+1}{2}\right)\Gamma\left(i\frac{t}{2}+\frac{\alpha-\delta+1}{2}\right)\right\vert^2$	$\frac{(2\sin\phi)^{2\lambda}}{2\pi}\vert\Gamma(\lambda+it)\vert^2 e^{(2\phi-\pi)t}$
	$(\alpha>-1,\beta>-1)$	$(\vert\delta\vert<\alpha+1)$	$(\lambda>0, 0<\phi<\pi)$
a_n	$\frac{1}{2^n n!}(\alpha+\beta+n+1)_n$	$\frac{1}{2^n n!}(2\alpha+n+1)_n$	$\frac{(2\sin\phi)^n}{n!}$
d_n^2	$\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\vert\Gamma\left(\frac{\alpha+\beta}{2}+i\gamma+n+1\right)\vert^2}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}$	$\frac{\Gamma^2(\alpha+n-1)\Gamma(\alpha+\delta+n+1)\Gamma(\alpha-\delta+n+1)}{n!(2\alpha+2n+1)\Gamma(2\alpha+n+1)}$	$\frac{\Gamma(2\lambda+n)}{n!}$
α_n	$\frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}$	$\frac{(n+1)(2\alpha+n+1)}{(2\alpha+2n+1)(\alpha+n+1)}$	$\frac{n+1}{2\sin\phi}$
β_n	$\frac{\gamma(\beta^2-\alpha^2)}{(\alpha+\beta+n)(\alpha+\beta+2n+2)}$	0	$-(\lambda+n)\frac{\cos\phi}{\sin\phi}$
γ_n	$\frac{2(\alpha+n)(\beta+n)\left[\gamma^2+\left(\frac{\alpha+\beta}{2}+n\right)^2\right]}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}$	$(\alpha+n)\frac{\left[(\alpha+n)^2-\delta^2\right]}{2\alpha+2n+1}$	$\frac{2\lambda+n-1}{2\sin\phi}$
	$i^{-n}h_n^{(\alpha,\beta)}(Z,N)$,	$P_n^{(\alpha,\alpha)}(t,-i\delta)=$	$\frac{e^{-in\phi}}{n!}m_n^{(2\lambda,\mu)}(it-\lambda)$,
	$Z=i\frac{t+\gamma}{2}-\frac{\beta+1}{2}$,	$=P_n^{(\alpha-\delta,\alpha+\delta)}(t,0)$	$\mu=e^{-2i\phi}$
	$N=i\gamma-\frac{\alpha+\beta}{2}$		

TABLE I. Continuous orthogonality for the Hahn and Meixner polynomials.

$P_n(t)$	$P_n^{\alpha,\beta}(t,\gamma,\delta)$	$P_n(t,a,b,c)$
(a,b)	$(0,\infty)$	$(0,\infty)$
$\rho(t)$	$\frac{1}{2\pi^2}\lvert\Gamma\left(it^{\frac{1}{2}} - i\gamma + \frac{\alpha+1}{2}\right)\Gamma\left(it^{\frac{1}{2}} + i\gamma + \frac{\alpha+1}{2}\right)\rvert^2$ $\times\lvert\Gamma\left(it^{\frac{1}{2}} - i\delta + \frac{\beta+1}{2}\right)\Gamma\left(it^{\frac{1}{2}} + i\delta + \frac{\beta+1}{2}\right)\rvert^2 \sinh 2\pi t^{\frac{1}{2}}$ $(\alpha > -1, \beta > -1)$	$\frac{1}{2\pi^2}\lvert\Gamma\left(it^{\frac{1}{2}} + a\right)\Gamma\left(it^{\frac{1}{2}} + b\right)\Gamma\left(it^{\frac{1}{2}} + c\right)\rvert^2 \sinh 2\pi t^{\frac{1}{2}}$ $(a > 0, b > 0, c > 0;$ $a = b^*, \quad \mathrm{Re}\, a > 0, \quad c > 0)$
a_n	$\frac{1}{n!}(\alpha+\beta+n+1)_n$	$\frac{1}{n!}$
d_n^2	$\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\left\lvert\Gamma(\frac{\alpha+\beta}{2}+i\gamma+i\delta+n+1)\right\rvert^2}{n!(\alpha+\beta+n+1)\Gamma(\alpha+\beta+2n+1)}$ $\times\lvert\Gamma(\frac{\alpha+\beta}{2} - i\gamma + i\delta + n + 1)\rvert^2$	$\frac{1}{n!}\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n)$
α_n	$\frac{(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}$	$n+1$
β_n	$\frac{1}{4}(\alpha+1)(\beta+1) + \frac{1}{2}(\gamma^2+\delta^2) + \frac{n}{2}(\alpha+\beta+n+1) +$ $\frac{(\beta^2-\alpha^2)(\gamma^2-\delta^2)}{2(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$	$ab + ac + bc + 2n^2 +$ $+n(2a+2b+2c-1)$
γ_n	$\frac{(\alpha+n)(\beta+n)\left[\left(\frac{\alpha+\beta)}{2}+n\right)^2+(\gamma-\delta^2)\right]\left[\left(\frac{\alpha+\beta)}{2}+n\right)^2+(\gamma+\delta^2)\right]}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$	$(a+b+n+1)(a+c+n-1)(b+c+n-1)$
	$(-1)^n u_n^{(\alpha,\beta)}(x,a,b), \quad x = -t - \frac{1}{4},$ $a = \frac{\beta}{2} + i\delta, b = -\frac{\alpha}{2} + i\gamma$	$(-1)^n W_n^{(c-\frac{1}{2})}(x, a - \frac{1}{2}, \frac{1}{2} - b),$ $x = -t - \frac{1}{4}$

TABLE II. Continuous orthogonality for the Racah and dual Hahn polynomials.

which follows from (1.2), it is not difficult to show that the polynomials

$$P_n^{(\alpha,\beta)}(t,\gamma,\delta) = (-1)^n u_n^{(\alpha,\beta)}(x,a,b), \tag{3.7}$$

where $x = -t - 1/4$, $z = is - 1/2$, $a = i\delta + \beta/2$ and $b = i\gamma - 1/2$, are real for the real values of t, α, β, γ and δ. As a result, we come to the property (2.5) (see table 2). The asymptotic behaviour

$$\rho(t) \sim (2\pi)^2 t^{\alpha+\beta} e^{-2\pi t^{1/2}}$$

holds as $t \to \infty$. The parameters of the polynomials $p_n^{(\alpha,\beta)}(t,\gamma,\delta)$ are chosen in (3.7) in such a way that the relation

$$P_n^{(\alpha,\beta)}[\frac{\gamma^2+\delta^2}{2} + \frac{\gamma^2-\delta^2}{2}s, \gamma, \delta] = (\gamma^2-\delta^2)^n [P_n^{(\alpha,\beta)}(s) + O(\frac{1}{\gamma^2-\delta^2})]$$

is valid as $\gamma \to \infty$.

The continuous orthogonality for the Racah polynomials was established earlier by the direct evaluation of the integral [9]. Here this property is proved on the basis of our approach. Besides, for the polynomials (3.7) a choice of the parameters is used which emphasizes their analogy with the Jacobi polynomials. The continuous orthogonality for the Racah polynomials was also proved by Miller [36] with the use of symmetry techniques, which contains points of similarity with our method.

2° In the case of the dual Hahn polynomials $W_n^{(c)}(x,a,b)$ in a similar manner it is possible to come to the continuous orthogonality for the polynomials $p_n(t,a,b,c)$ (table 2). For the weight $\rho(t)$ the estimate

$$\rho(t) \sim 2\pi t^{2\,\mathrm{Re}\,a+c-3/2} e^{-\pi t^{1/2}}$$

holds as $t \to \infty$. The equality analogous to (2.5) was established by Wilson [9]. In connection with a relativistic quasipotential model of the three-dimensional harmonic oscillator this property was also treated in Ref. [37].

In accordance with their definition the basic properties of the polynomials, exhibited in tables 1 and 2 —the difference equation, an analogue of Rodrigues' formula, formulas of difference differentiation and so on— can be obtained through the properties of the classical orthogonal polynomials of a discrete variable on linear and quadratic lattices, which have been studied earlier. (See, for example, Refs. [38,7]).

To calculate the square of the norm

$$d_n^2 = \int_a^b P_n^2(t)\rho(t)\,dt \tag{3.8}$$

we shall modify in the following way the considerations for the discrete case carried out in Refs. [5,7]. Let in (1.1) $y = y_n(x)$ and $\lambda = \lambda_n$. Under these assumptions for the functions

$$v_k(z) = \frac{\Delta v_{k-1}(z)}{\Delta x_{k-1}(z)},$$

where $x_0 = x(z)$, $v_0 = y(z)$, the equation [1]

$$\frac{\Delta}{\Delta x_k(z-\frac{1}{2})}\left[\sigma(z)\rho_k(z)\frac{\nabla v_k(z)}{\nabla x_k(z)}\right] + \mu_k\rho_k(z)v_k(z) = 0$$

is valid. Let us multiply both sides of this equation for the function $v_k(z) = v_{kn}(z)$ by $v_{kn}(z)\nabla x_{k+1}(z)$ and integrate over a contour C, for which the conditions (2.1) are satisfied. Then for the quantities

$$\bar{d}_{kn}^2 = \int_C v_{kn}^2(z)\rho_k(z)\nabla x_{k+1}(z)\,dz$$

we obtain

$$\mu_{kn}\bar{d}_{kn}^2 = -\int_C v_{kn}(z)\Delta\left[\sigma(z)\rho_k(z)\frac{\nabla v_{kn}(z)}{\nabla x_k(z)}\right]dz.$$

Hence, using the identity

$$\Delta[f(z)g(z)] = f(z)\Delta g(z) + g(z+1)\Delta f(z)$$

for the functions $f(z) = v_{kn}(z)$ and $g(z) = \sigma(z)\rho_k(z)\frac{\nabla v_{kn}(z)}{\nabla x_k(z)}$, we come to the relation

$$\mu_{kn}\bar{d}_{kn}^2 = \bar{d}_{k+1,n}^2, \tag{3.9}$$

provided that the conditions

$$\int_C \Delta \left[\sigma(z)\rho_k(z)v_{kn}(z)\frac{\nabla v_{kn}(z)}{\nabla x_k(z)}\right] dz = 0$$

are satisfied. From (3.9) we successively find

$$d_n^2 = d_{0n}^2 = \mu_{0n}^{-1} d_{1n}^2 = \mu_{0n}^{-1}\mu_{1n}^{-1} d_{2n}^2 = \ldots = d_{nn}^2 \prod_{k=0}^{n-1} \mu_{kn}^{-1}.$$

Since $v_{nn} = A_n B_n = \text{const.}$ and $A_n = (-1)^n \prod_{k=0}^{n-1} \mu_{kn}^{-1}$ (see Ref. [7]), then as a result we obtain

$$d_n^2 = \int_C y_n^2[x(z)]\rho(z)\nabla x_1(z)\, dz = (-1)^n A_n B_n^2 \int_C \rho_n(z)\nabla x_{n+1}(z)\, dz.$$

The squares of the norms (3.8), calculated on the basis of this equality, are given in tables 1 and 2. The values of integrals are taken from Refs. [39,40,9]. For all considered polynomials tables 1 and 2 contain also the leading terms of the expansion $p_n(t) = a_n t^n + \ldots$ and the coefficients of the recurrence relation $t\, p_n(t) = \alpha_n p_{n+1}(t) + \beta_n p_n(t) + \gamma_n p_{n-1}(t)$.

4. Hypergeometric representation for the Racah and dual Hahn polynomials

The Rodrigues-type formula (1.2) for the Racah polynomials was first established in Refs. [41,42]. A representation for these polynomials in the form of the hypergeometric function ${}_4F_3(1)$ was treated by Wilson [9] and then it has become a starting point for a number of investigations. In this connection it is natural to derive the representations for the Racah and dual Hahn polynomials through the hypergeometric functions. The corresponding results for the linear lattice are given in Ref. [7].

With the aid of the notion of divided difference (see, for example, Ref. [43]) Rodrigues' formula (1.2) in the case of the quadratic lattice $x(z) = z(z+1)$ can be brought to the form

$$y_n[x(z)] = \frac{(-1)^n B_n}{(2z-n+2)_n} \sum_{k=0}^{n} \frac{(-n)_k (z - \frac{n-1}{2} + 1)_k (2z+1-n)_k}{k!\,(z - \frac{n-1}{2})_k\,(2z+2)_k} \left[\frac{\rho_n(z-n+k)}{\rho(z)}\right], \tag{4.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)\Gamma^{-1}(a)$. Using (4.1), it is possible to obtain the following hypergeometric representations for the Racah $U_n^{(\alpha,\beta)}(x,a,b)$ and dual Hahn $W_n^{(c)}(x,a,b)$ polynomials:

$$U_n^{(\alpha,\beta)}(x,a,b) = \frac{(-1)^n}{n!}(\beta+1)_n(b-a-n)_n(a+b+\alpha+1)_n$$
$$\times\, {}_4F_3\left(\begin{matrix}-n, & \alpha+\beta+n+1, & a-z, & a+z+1\\ \beta+1, & a-b+1, & a+b+\alpha+1\end{matrix}\,\middle|\,1\right), \qquad (4.2)$$

$$W_n^{(c)}(x,a,b) = \frac{(-1)^n}{n!}(a+c+1)_n(b-a-n)_n$$
$$\times\, {}_3F_2\left(\begin{matrix}-n, & a-z, & a+z+1\\ a+c+1, & a-b+1\end{matrix}\,\middle|\,1\right). \qquad (4.3)$$

In deriving (4.2) and (4.3) from (4.1) we have used the formulas (4) on page 25 and (2) on page 28 from Biley [40] respectively, with the subsequent conventional transition to the polynomial expressions in the arising functions ${}_4F_3(1)$ and ${}_3F_2(1)$.

The hypergeometric representations for the classical orthogonal polynomials of a discrete variable on the quadratic lattice are exhibited in table 3. All expansions in this table can be represented in the unified form:

$$y_n[x(z)] = \sum_{k=0}^{n} C_{kn}[x_{k-1}(a) - x_{k-1}(z)]^{(k)}, \qquad (4.4)$$

where the "generalized power" is defined by

$$[x_{k-1}(a) - x_{k-1}(z)]^{(k)} = \prod_{p=0}^{k-1}[x_{k-1}(a) - x_{k-1}(z-p)]$$
$$= \prod_{p=0}^{k-1}[x(a+p) - x(z)] = (-1)^k \sum_{p=0}^{k} S_k^{(p)}(a)x^p(z).$$

5. On the Askey-Wilson polynomials

The proof of the continuous orthogonality relation (2.2), given for the case of quadratic lattices in the section 2, is in fact of general character and remains valid (with corresponding changes) for all classes of nonuniform lattices, considered in Ref. 2. In this connection, in complete analogy with the case of quadratic lattice,

$Y_n(x)$	${}_pF_q(\xi)$
	Lattice x(z)=z
$h_n^{(\alpha,\beta)}(x,N)$	$\frac{(-1)^n}{n!}(\beta+1)_n(N-n)_n {}_3F_2\left(\begin{matrix} -n, & \alpha+\beta+n+1,-x \\ \beta+1, & 1-N \end{matrix}\middle\vert 1\right)$
$m_n^{(\gamma,\mu)}(x)$	$(\gamma)_n F(-n,-x,\gamma,1-\mu^{-1})$
$K_n^{(p)}(x,N)$	$(-p)^n C_N^n F(-n,-x,-N,p^{-1}), \qquad C_N^n=\frac{N!}{n!(N-n)!}$
$C_n^{\mu}(x)$	${}_2F_0(-n,-x,-\mu^{-1})$
	Lattice x(z) = Z(Z+1)
$U_n^{(\alpha,\beta)}(x,a,b)$	$\frac{(-1)^n}{n!}(\beta+1)_n(b-a-n)_n(a+b+\alpha+1)_n {}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+n+1, & a-z,a+z+1 \\ \beta+1, & a-b+1, & a+b+\alpha+1 \end{matrix}\middle\vert 1\right)$
$W_n^{(c)}(x,a,b)$	$\frac{(-1)^n}{n!}(a+c+1)_n(b-a-n)_n\, {}_3F_2\left(\begin{matrix} -n, & a-z,a+z+1 \\ a+c+1, & a-b+1 \end{matrix}\middle\vert 1\right)$

TABLE III. Hypergeometric representations for some clasical polynomials of a discrete variable.

studied above in detail, it is possible to give a simple derivation of the continuous orthogonality property also for other systems of classical orthogonal polynomials. In particular, this can be done for the polynomials $p_n(t, a, b, c, d, |q)$, introduced by Askey and Wilson [13]. In this case we have $x(z) = \cosh \omega z = \frac{1}{2}(q^z + q^{-z}), q = e^{\omega}$,

$$\sigma(z) = q^{-2z}(q^z - a)(q^z - b)(q^z - c)(q^z - d), \tag{5.1}$$

$$\rho(z) = \rho(z, a, b, c, d) = Af_q(z) \prod_{v=a,b,c,d} g(z, v), \tag{5.2}$$

$$\rho_k(z) = (-1)^k q^{-k(k-2)/2} \rho\left(z + \tfrac{k}{2}, aq^{k/2}, bq^{k/2}, cq^{k/2}, dq^{k/2}\right) \tag{5.3}$$

Here $f_q^{-1}(z) = \Gamma_q(2z)\Gamma_q(-2z)(q^z - q^{-z}), \Gamma_q(w)$ is the q–gamma function (see, for example, Refs. [44,45],

$$g^{-1}(zv) = \prod_{k=0}^{\infty}(1 - vq^{z+k})(1 - vq^{k-z}) = \prod_{k=0}^{\infty}[1 - v(q^z + q^{-z})q^k + v^2 q^{2k}], \quad |q| < 1,$$

and A is a constant. In deriving (5.2) and (5.3) from (1.3) and (1.4), respectively, the following relations were used:

$$\frac{f_q(z+1)}{f_q(z)} = q^{-4z-2}, \qquad \frac{g(z+1, v)}{g(z, v)} = q^{2z+1}\frac{q^{-z} - v}{q^{z+1} - v},$$

$$q^{2z+2} f_q(z+1) = -q^{1/2} f_q(z + \tfrac{1}{2}), \quad (1 - vq^{-z-1})g(z+1, v) = g(z + \tfrac{1}{2}, vq^{1/2}).$$

We also note the useful equalities

$$f_q(-z) = -f_q(z), \qquad g(-z, v) = g(z, v), \qquad g(z \pm 2\pi i \log^{-1} q, v) = g(z, v).$$

Since

$$\sigma(z)\rho(z) = \rho_1(z - 1) = -q^{\frac{1}{2}}\rho(z - 1/2, aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}),$$

$$\Delta\left[\sigma(z)\rho(z)\frac{\nabla y(z)}{\nabla x(z)}\right]=\Delta\left[\rho_1(z-1)\frac{\delta y(z-1/2)}{\delta x(z-1/2)}\right]$$

$$=-q^{1/2}\Delta\left[\rho(z-\tfrac{1}{2},aq^{\frac{1}{2}},bq^{1/2},cq^{1/2},dq^{1/2})\frac{\delta y(z-1/2)}{\delta x(z-1/2)}\right]$$

$$=-q^{1/2}\delta\left[\rho\left(z,aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}\right)\frac{\delta y(z)}{\delta x(z)}\right],$$

where $\delta f(z)=\Delta f(z-1/2)$, then the equation (1.1) for the case of the Askey-Wilson polynomials takes the form [16]

$$\frac{\delta}{\delta x(z)}\left[\rho(z,aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2})\frac{\delta y(z)}{\delta x(z)}\right]=\lambda q^{-1/2}\rho(z,a,b,c,d)y(z). \qquad (5.4)$$

Correspondingly, since

$$\nabla^{(n)}[\rho_n(z)]=\nabla^{(n-1)}\left[\frac{\delta\rho_n(z-\frac{1}{2})}{\delta x(z+\frac{n-1}{2})}\right]$$

$$=\nabla^{(n-2)}\left[\frac{\delta}{\delta x(z+\frac{n-2}{2})}\left(\frac{\delta\rho_n(z-\frac{2}{2})}{\delta x(z+\frac{n-2}{2)}}\right)\right]=\cdots$$

$$=\left(\frac{\delta}{\delta x(z)}\right)^n[\rho_n(z-\tfrac{n}{2})],$$

then the Rodrigues-type formula (1.2) for the Askey-Wilson polynomials can be rewritten as [16]

$$y_n(x)=\frac{(-1)^n q^{-n(n-2)/2}B_n}{\rho(z,a,b,c,d)}\left(\frac{\delta}{\delta x(z)}\right)^n\left[\rho_n(z,aq^{n/2},bq^{n/2},cq^{n/2},dq^{n/2})\right]. \qquad (5.5)$$

We shall also briefly discuss series expansions for the Askey-Wilson polynomials. For the lattice $x(z)=\frac{1}{2}(q^z+q^{-z})$ an analogue of formula (4.1) has the form

$$y_n[x(z)]=\frac{(-1)^n B_n 2^n q^{nz-n(n-3)/4}}{(1-q)^n(q^{2z-n+1},q)_n}$$

$$\times\sum_{k=0}^{n}q^{kn}\frac{(q^{-n},q)_k(q^{z-\frac{n}{2}+1},q)_k(-q^{z-\frac{n}{2}+1},q)_k(q^{2z-n},q)_k}{(q,q)_k\,(q^{z-n/2},q)_k\,(-q^{z-n/2},q)_k\,(q^{2z+1},q)_k}\left[\frac{\rho_n(z-n+k)}{\rho(z)}\right], \qquad (5.6)$$

where $(a,q)_0 = 1$, $(a,q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$. With the aid of a Pearson-type equation it is not difficult to derive that in the case of the Askey-Wilson polynomials,

$$\frac{\rho_n(z-n+k)}{\rho(z)} = q^{2nz-n(n-1)}(aq^{-z},q)_n(bq^{-z},q)_n(cq^{-z},q)_n(dq^{-z},q)_n$$
$$\times \frac{(aq^z,q)_k(bq^z,q)_k(cq^z,q)_k(dq^z,q)_k}{(a^{-1}q^{1-n+z},q)_k(b^{-1}q^{1-n+z},q)_k(c^{-1}q^{1-n+z},q)_k(d^{-1}q^{1-n+z},q)_k}\left(\frac{q^{-2n+2}}{abcd}\right)^k \tag{5.7}$$

Substituting (5.7) into (5.6) and first using Watson's transformation [40] followed by Sears' transformation [13], we come to the standard representation for the Askey-Wilson polynomials in terms of the basis hypergeometric function ${}_4\varphi_3$:

$$y_n[x(z)] = \frac{2^n B_n q^{-\frac{n(3n-s)}{4}}}{a^n(1-q)^n}(ab,q)_n(ac,q)_n(ad,q)_n$$
$$\times\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, & abcdq^{n-1}, & aq^z, & aq^{-z} \\ & ab, & ac, & ad \end{matrix}\middle|\, q,q\right). \tag{5.8}$$

We note that for these polynomials the representation (4.4) also holds.

The continuous orthogonality relation for the Askey-Wilson polynomials was proved by direct evaluation of the integral [16], for simplifications of the proof see Refs. [46–48]. It is not hard to derive this relation with the use of the approach under consideration. In fact, the solution (5.2) of the equation (1.3) is a periodic function of period $2\pi i \log^{-1} q$. The contour C we define in the following way: $z = is, -\pi\log^{-1} q \le s \le \pi \log^{-1} q$. The conditions (2.1) hold owing to Cauchy's theorem and periodicity of the function $\rho_1(z)x_1^k(z)$. As a result we come to the continuous orthogonality relation (2.5), where $a=-1$, $b=1$, $t=x$ and the weight function is equal to

$$\rho(t) = (1-t^2)^{-1/2}\frac{\prod_{k=0}^{\infty}[1-2(2t^2-1)q^k+q^{2k}]}{\prod_{v=a,b,c,d}\prod_{k=0}^{\infty}(1-vtq^k+v^2q^{2k})},$$

provided that $\max(|a|,|b|,|c|,|d|) < 1$ and $-1 < q < 1$.

The discussion of various particular and limiting cases of the polynomials denoted $P_n(t,a,b,c,d|q)$ can be found in Ref. [16]. The values of squared norms may be evaluated by direct analogy with the cases, considered in section 3. For another way of the evaluation of squared norms see Ref. [36]. Some other examples of the continuous orthogonality relations for classical polynomials of a discrete variable are given in Ref. [49].

6. A model of the harmonic oscillator in the relativistic configurational $\mathbf{r}_N$-space

As an example of the utilization of the polynomials of a discrete variable we discuss an $O(N)$-symmetric model in the configurational $\mathbf{r}_N$-space [20,50], which is a simple extension of the relativistic three-dimensional harmonic oscillator, studied in detail in Refs. [51,37], to the N-dimensional case. Firstly, we note that in view of the $O(N)$ symmetry the dependence of the wave function ($\hbar = m = c = 1$)

$$\Psi_{Nn\ell}(r, \theta_1, \dots, \theta_{n-1}) = C_{Nn\ell} \left\{ r^{\left(\frac{N-1}{2}\right)} \right\}^{-1} \mathcal{X}_{Nn\ell}(r) Y_N^{\ell}(\theta_1, \dots, \theta_{N-1}), \qquad (6.1)$$

on the angles $\theta_1, \dots, \theta_{N-1}$ is described by the spherical harmonics $Y_N^{\ell}(\theta_1, \dots, \theta_{N-1})$ (indices, corresponding to the azimuthal quantum numbers, are omitted because they are irrelevant for further consideration). Since in the requirement that wave-functions be square-integrable in the $\mathbf{r}_N$-space, enters the measure (for details see Ref. [50])

$$\rho_N(r) d^N x = \rho_N(r) r^{N-1} dr\, d\Omega_{N-1} = \left| r^{\left(\frac{N-1}{2}\right)} \right|^2 dr\, d\omega_{N-1}, \qquad (6.2)$$

then the extraction of the factor

$$\left\{ r^{\left(\frac{N-1}{2}\right)} \right\}^{-1} = i^{\frac{1-N}{2}} \Gamma(-ir) \Gamma^{-1}\left(\tfrac{N-1}{2} - ir\right) = i^{\frac{1-N}{2}} (-ir)^{-1}_{\frac{N-1}{2}},$$

in (6.1) reduces an N-dimensional problem to finding the eigenfunctions of the radial part of a hamiltonian, i.e.

$$\widetilde{H}(r) \mathcal{X}_{Nn\ell}(r) = E_{Nn\ell} \mathcal{X}_{Nn\ell}(r), \qquad (6.3)$$

where $n = 0, 1, 2, \dots$ is the radial quantum number. A model of the oscillator under consideration is specified by the differential-difference operator $H(\mathbf{r}_N)$, the radial part of which has the form

$$\widetilde{H}(r) = \frac{1}{2} \left\{ \exp\left(-i \frac{d}{dr}\right) + [1 + \omega^2 r^{(2)}] \left[1 + \frac{L(L+1)}{r^{(2)}}\right] \exp\left(i \frac{d}{dr}\right) \right\}, \qquad (6.4)$$

where $L = 1 + \frac{1}{2}(N-3)$.

We represent the eigenfunction of $\widetilde{H}(r)$ as

$$\mathcal{X}_{Nn\ell}(r) = (r)^{(L+1)} M_\nu(r) \mathcal{P}(r), \tag{6.5}$$

having thus extracted the functions $(r)^{(L+1)}$ and $M_\nu(r) = [\nu(\nu-1)]^{-ir/2}$ $\Gamma(\nu + ir)$, $2\nu = 1 + (1 + 4\omega^{-2})^{1/2}$,which determine the asymptotic behaviour of $\mathcal{X}_{Nn\ell}(r)$ at the points $r = 0$ and $r = \infty$, respectively. Then for its polynomial part $\mathcal{P}(r)$ we obtain the following difference equation (cf. with Ref. [37])

$$\left[(\nu + ir)(L + 1 + ir)e^{-i\frac{d}{dr}} - (\nu - ir)(L + 1 - ir)e^{i\frac{d}{dr}}\right] \mathcal{P}(r) = 2i \frac{E_{Nnl}}{\omega} r \mathcal{P}(r). \tag{6.6}$$

The solutions of (6.6) corresponding to the eigenvalues $E_{Nn\ell} = \omega(\nu + 2n + L + 1)$ are the dual Hahn polynomials

$$\mathcal{P}_n^{\nu,L}(r^2) = W_n^{(0)}\left(-r^2 - \tfrac{1}{4}, L + \tfrac{1}{2}, \tfrac{1}{2} - \nu\right) = (-1)^n P_n\left(r^2, L + 1, \nu, \tfrac{1}{2}\right). \tag{6.7}$$

The normalization constant $C_{Nn\ell}$ entering in (6.1) is equal to

$$C_{Nnl} = \left[\tfrac{1}{2}(n+1)_{\nu+L}\Gamma(n + L + \tfrac{3}{2})\,\Gamma\left(n + \nu + \tfrac{1}{2}\right)\right]^{-1/2}, \tag{6.8}$$

and the orthogonality of the wavefunctions (6.1) is the consequence of the continuous orthogonality relation for the dual Hahn polynomials $W_n^{(c)}(x, a, b)$ (see § 3, section 2). Thus formulas (6.1), (6.5), (6.7) and (6.8) completely define the explicit forms of the wavefunctions $\Psi_{Nnl}(r, \theta_1, \ldots, \theta_{N-1})$ for the present relativistic model of the oscillator. In the nonrelativistic limit (i.e. when the velocity of light c tends to infinity) they coincide with the wavefunctions of the nonrelativistic N-dimensional harmonic oscillator in the coordinate representation.

In complete analogy to the three-dimensional case [37] it is easy to construct by the Infeld-Hull factorization method both the spectrum-generating algebra $SU(1,1)$ and the radial n and orbital 1 quantum numbers raising and lowering operators. It suffices for this to replace 1 by $L = 1 + \frac{1}{2}(N - 3)$ in the corresponding formulas of the paper just mentioned.

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Resumen. Se estudian las propiedades de ortogonalidad en el continuo de ciertos polinomios básicos de variable discreta. Se discute una aplicación al modelo de cuasipotencial relativista del oscilador armónico en N dimensiones.