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Gravitational magnetism and general relativity

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Abstract. The relationship between the General Theory of Relativity and Blackett's gravi-magnetic hypothesis is discussed. The gravi-magnetic hypothesis proposes that the magnetic moment and the internal angular momentum of a rotating body are proportional, and that the proportionality constant is $\beta G^{1/2}/c$ where β is a dimensionless empirical constant, G is the gravitational constant and c is the speed of light. Basing this work on this hypothesis it is shown that in the Post- Newtonian approximation the equations of motion of the internal angular momentum of a spinning test body lead to the equation of motion of a magnetic moment acted upon by a magnetic field.

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In a letter [1] expressing his views on Kaluza's idea for the unification of gravitation and electromagnetism, Einstein made the following comment:

"It often appears to me that the magnetic field of the Earth is based upon an yet unknown connection between gravitation and electromagnetism..."

This remark shows how difficult Einstin considered the problem of the origin of the earth's magnetic field and at the same time his suggested solution: a unified theory of gravity and electromagnetism.

An interesting proposal explaining astrophysical magnetic fields (perhaps related to the idea expressed by Einstein) was made by Blackett in 1947 [2]. His proposal, known as the gravi-magnetc hypothesis, postulate that the magnetic moment m and the internal angular momentum S of a rotating body satisfy the following basic relation

$$m = \beta \frac{G^{1/2}}{c} \mathrm{S}, \tag{1}$$

where β is a dimensionless empirical constant, G is the gravitational constant, and c is the speed of light. This relation is of interest in physics for at least two reasons. First, it contains the two basic constants G and c which appear naturally in the general theory of relativity [3]. Second, it allows for the conception of a rotating body without charge as the source of a magnetic field B, since

$$B = \frac{2m}{r^3}, \tag{2}$$

and if one substitutes Eq. (1) into Eq. (2) the result is:

$$B = 2\beta \frac{G^{1/2}}{c} \frac{\mathrm{S}}{r^3}. \tag{3}$$

Although the gravi-magnetic hypothesis is one of the most interesting attempts at explaining astrophysical magnetic fields, it has been largely ignored because of the lack of definitive laboratory proof [4]. Here it is important to remember that no completely convincing theory has been proposed to account for astrophysical magnetic fields [2,4,5]. The Dynamo Theory [6], one of the most promising, is a very complicated theory involving Maxwell's electrodynamics and hydrodynamics, in contrast to the simplicity of the gravi-magnetic hypothesis theory. Furthermore, the dynamo theory presents some practical difficulties when applied to a specific astrophysical body. For isntance, Ingersoll [5] has pointed out recently that no planetary dynamo, not even the Earth, is well understood, primarily because of lack of obserational data from the interior of the planet.

In 1979, Sirag [4] reconsidered Blackett's gravi-magnetic hypothesis. He discovered that not only the Earth, the Sun and the star 78 Virgis, fit into relation (1) (as had been previously proven by Blackett [2]), but that the moon and the planets Mercury, Jupiter, Venus and Sturn do as well. Moreover, Sirag added Pulsar Hercules $X-1$ to the list.

The fact that a pulsar fits into relation (1) is of great importance in the reconsideration of the gravi-magnetic hypothesis. As Sirag [4] pointed out, the Hercules data is both fairly accurate and represents an extremely high magnetic field (5×10^{12} Gauss) and angular velocity (5 rad. s^{-1}).

The main purpose of this paper it to relate Blackett's gravi- magnetic hypothesis to the general theory of relativity [3]. The motivation for this work came from the observation that in General Relativity a rotating body without charge behaves as the source of a gravitatioal field. To be specific, there is an exact solution of the gravitational field equations of general relativity (known as the Kerr Solution [3]) which represents the field as being exterior to a rotating axially symmetric body. It is worthwhile to mention that in Newtonian Gràvitational Theory the field of an axially symmetric body is independent of its rotational motion, unlike the situation in general relativity. Thus, if the gravi-magnetic hypothesis is considered seriously, it follows that a rotating body without charge is the source of both the magnetic field and the gravitational field. This conclusion appears to be important as well as interesting. It may eventually shed some light on the connection between electromagnetic and gravitational phenomena.

Let us begin by considering the Kerr Solution. It has a line element of the following form

$$\begin{aligned} ds^2 = & - c^2dt^2 + d\mathbf{x}^2 + \frac{2MG\rho}{c^2[\rho^4 + (\mathbf{x}\cdot\mathbf{a})^2](\rho^2 + \mathrm{a}^2)^2} \\ & \times \left[\rho^2\mathbf{x}\cdot d\mathbf{x} + \rho d\mathbf{x}\cdot(\mathbf{a}\times\mathbf{x}) + (\mathbf{a}\cdot\mathbf{x})(\mathbf{a}\cdot d\mathbf{x}) + (\rho^2 + \mathbf{a}^2)\rho c dt\right]^2, \end{aligned} \tag{4}$$

where M is the mass of the rotating body, $\mathbf{x}$ is a Quasi Euclidean three-vector; scaler products $x\cdot a, \mathbf{x}^2$ and so on, are defined as in Euclidean Geometry, ρ is defined by

$$\rho^4 - (r^2 - \mathbf{a}^2)\rho^2 - (\mathbf{a}\cdot\mathbf{x})^2 = 0, \tag{5}$$

where $r^2 = \mathbf{x}^2$ and $\mathbf{a}$ is a constant vector related to the internal angular momentum $\mathbf{S}$ of the rotating body by the formula

$$\mathbf{a} = \frac{G}{c^3M}\mathbf{S}. \tag{6}$$

It is important to mention that the total linear momentum $K^\mu(\mu,\nu = 0,1,2,3)$ of the rotating body satisfies, according to the Kerr solution (4) the expressions:

$$\mathbf{K} = 0, \qquad \mathrm{K}^\circ = M. \tag{7}$$

Although it would be interesting to relate the entire Kerr solution (4) to the Gravi-Magnetic hypothesis, we will only work in this paper with its reduced approximate form which is valid for low rates of internal angular momentum and weak fields. We will use the Post-Newtonian approximation [3] in connection with the internal angular momentum precession of a rotating system.

We will write below the equations of motion for a massive test particle with internal angular momentum $\mathcal{S}^\mu$:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{8}$$

$$\frac{d\mathcal{S}^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\mathcal{S}^\beta = 0, \tag{9}$$

where $\Gamma^\mu_{\alpha\beta}$ represents the Christoffel symbols. The internal angular momentum $\mathcal{S}^\mu$ satisfies the additional relation

$$\mathcal{S}^\mu\frac{dx_\mu}{ds} = 0. \tag{10}$$

It is worth mentioning that Equation (8) is only approximate, since normally its right hand side is not zero due to a gravitational force which depends on the Riemmann tensor and $\mathcal{S}^{\mu}$ [7].

We will focus now on Eq. (9) for the internal angular momentum $\mathcal{S}^{\mu}$. It is possible to show that in the post-newtonian approximation this equation reduces to the following expression:

$$\frac{d\vec{\mathcal{S}}}{dt} = c\boldsymbol{\Omega} \times \vec{\mathcal{S}} \tag{11}$$

where

$$\boldsymbol{\Omega} = -\frac{1}{2}\nabla \times g_{oi} = \frac{G}{c^3}\Big[\frac{-\mathbf{S}}{r^3} + \frac{3(\mathbf{S}\cdot\mathbf{x})\mathbf{x}}{r^5}\Big]. \tag{12}$$

Here g_{oi} are the off-diagonal terms of the metric $g_{\mu\nu}$ and $\mathbf{S}$ is the internal angular momentum of the source rotating body. In expression (12) for $\boldsymbol{\Omega}$, we considered only the terms containing $\mathbf{S}$. Indeed, there are two additional terms in (12) which we will not consider here, since they don't contain $\mathbf{S}$. Equation (11) describes at the post-newtonian level precession of the internal angular momentum $\vec{\mathcal{S}}$ relative to a comoving orthonormal frame.

The Eq. (12) defines the angular velocity Ω of precession. It is interesting to compare this equation with that of a magnetic field due to a magnetic moment $\mathbf{m}$

$$\mathbf{B} = -\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{x})\mathbf{x}}{r^5}. \tag{13}$$

It is apparent that except for the constant factor G/c^3, Eq. (12) has exactly the same form as Eq. (13). It is precisely the factor G/c^3 which makes it difficult to detect the precessional angular velocity. For instance, the order of magnitude of the precessional angular velocity of earth's rotation is

$$\Omega \sim 0.1 \ \text{ second of arc per year.} \tag{14}$$

This effect has not yet been detected [3].

We will now write Eq. (11) in the following form

$$\frac{d\vec{\mathcal{S}}}{dt} = \left(\frac{G^{\frac{1}{2}}}{c}\right)\left(-\frac{\mathbf{S}}{r^3} + \frac{3(\mathbf{S}\cdot\mathbf{x})\mathbf{x}}{r^5}\right) \times \left(\frac{G^{\frac{1}{2}}}{c}\right)\vec{\mathcal{S}}. \tag{15}$$

This expression suggests that the gravi-magnetic hypothesis takes on the following form

$$\mathbf{m} = \frac{G^{\frac{1}{2}}}{c}\mathbf{S}, \tag{16}$$

$$\vec{\mu} = \frac{G^{\frac{1}{2}}}{c}\vec{\mathcal{S}}. \tag{17}$$

Here $\mathbf{m}$ and $\vec{\mu}$ play the role of magnetic moments associated with the source rotating body and the spinning test particle respectively. It is not difficult to see that (16) and (17) reproduce the gravi-magnetic relation (1) provided that $\beta = 1$.

By sustituting the relations (16) and (17) into formula (15) we learn that

$$\vec{\tau} = \mathbf{B} \times \vec{\mu}, \tag{18}$$

where $\mathbf{B}$ is given by

$$\mathbf{B} = -\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{x})\mathbf{x}}{r^5} \tag{19}$$

and

$$\vec{\tau} = \frac{d\vec{\mathcal{S}}}{dt}. \tag{20}$$

Equation (18) may be recognized as the particular equation of a torque $\vec{\tau}$ exerted on a magnetic moment $\vec{\mu}$ moving in a magnetic field $\mathbf{B}$. Thus, starting with the Post-Newtonian approximation of general relativity, and using the gravi-magnetic hypothesis in the forms (16) and (17) we were able to reproduce a well known formula (18) of electromagnetism.

There are of course a number of problems arising from the preceding results, which need further discussion. First of all, it is known that the condition $\beta = 1$ used in (16) and (17) doesn't agree with the experimental results [2,4]. Furthermore, one finds that for astrophysical bodies such as planets the direction of $\mathbf{m}$ is not parallel to $\mathbf{S}$ [5]. Finally, Blackettt's assumption cannot explain the reversal problem [6].

Although the result (18) is based on the Post-Newtonian approximation, it seems intriguing and merits further research.

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Resumen. Se discute la relación entre la Teoría General de la Relatividad y la hipótesis gravi-magnética de Blackett. La hipótesis gravimagnética propone que el momento magnético y el momento angular interno de un cuerpo en rotación son proporcionales, y que la constante de proporcionalidad es $\beta G^{1/2}/c$, donde β es una constante empírica sin dimensiones, G es la constante gravitacional y c es la velocidad de la luz. Al basar este trabajo en esta hipótesis, se muestra que en la aproximación postnewtoniana las ecuaciones de movimiento del momento angular interno de un cuerpo de prueba con espín conducen a las ecuaciones de movimiento de un momento magnético bajo la influencia de un campo magnético.