Electric and magnetic four-vectors in classical electrodynamics

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(recibido el 4 de febrero de 1988; aceptado el 30 de julio de 1988)

Abstract. Following the procedure customarily used to define the four-spin, we define electric and magnetic field four-vectors and use them to write Maxwell equations in a manifestly Lorentz covariant form. We prove that these equations are physically equivalent to the standard covariant formulation of Maxwell equations. We then show how to develop covariant electrodynamics from our set of equations.

PACS: 03.50.De

1. Introduction

Why it is not possible to formulate the equations of electromagnetism in manifestly covariant form using electric and magnetic 4-vectors? A possible answer to this question comes immediately to mind: the electric and magnetic fields are not the spatial components of any 4-vectors. It is only a very particular arrangement of their components which form a fully covariant object, the electromagnetic field tensor $F^{\mu\nu}$. Only using this tensor the manifestly covariant form of Maxwell equations can be achieved.

However, think of the covariant generalization of the equation of motion of a particle with spin in external electromagnetic fields. Here, it is not apparent that the spin of the particle $\mathbf{s}$ should be the spatial part of any 4-vector. But this does not prevent us from *defining* a spin 4-vector s^μ whose spatial part reduces to $\mathbf{s}$ in the proper frame of the particle and so subject to the constraint $s^\mu u_\mu = 0$, with u_μ the 4-velocity of it. The 4-spin so defined can be used to obtain a covariant expression for the equation of motion of the particle [1-3].

At this point one may wonder again. Why it is not possible to *define* 4-vectors E^μ and B^μ exactly as we have defined the 4-spin, and use them to cast Maxwell equations in a manifestly Lorentz covariant form? The purpose of this paper is to show that this is indeed possible and that by using them a perfectly consistent description of electromagnetic interactions can be achieved.

We must point out that 4-vectors E^μ and B^μ have been used previously for describing electromagnetic interactions in charged fluids [4], and employed as "interesting secondary objects" in the study of the algebraic properties of the electromagnetic field [5]. But in both cases they are introduced only as a convenient way of splitting the electromagnetic field tensor in its electric and magnetic "components" from the point of view of an observer in a certain comoving frame.

In this work we shall consider E^μ and B^μ as primary objects, defined in order to put Maxwell equations in an explicitly covariant form; in other words we shall use them as a starting point for an alternative introduction to covariant electrodynamics. The plan of the paper is as follows: In section 2 we define the 4-vectors E^μ and B^μ and obtain the covariant equations of motion they satisfy. In section 3 we show the equivalence between those equations and Maxwell's; we recover some standard results of classical electrodynamics, and we discuss the covariant description of the electrodynamics of continuous media. Section 4 contains our conclusions and some additional comments.

2. Defining E^μ and B^μ

To define the 4-vectors E^μ and B^μ, we must select a particular —but otherwise arbitrary— inertial reference frame $\mathcal{R}$, to which we shall sometimes refer as the frame of our fiducial observer, and set:

$$E^\mu = (0, \mathbf{E}) \qquad \text{and} \qquad B^\mu = (0, \mathbf{B}) \tag{1}$$

where $\mathbf{E}$ is the electric and $\mathbf{B}$ is the magnetic field in that frame. In all other frames E^μ and B^μ are obtained by Lorentz transformations of Eq. (1). Obviously, E^μ and B^μ must satisfy

$$E^\mu E_\mu = -|\mathbf{E}|^2 , \quad B^\mu B_\mu = -|\mathbf{B}|^2 \tag{2}$$

and

$$E^\mu u_\mu = B^\mu u_\mu = 0 , \tag{3}$$

where u_μ is a time-like unit 4-vector, the 4-velocity of our fiducial observer. To be precise, E^μ is a 4-vector but B^μ must be called a pseudo 4-vector.

Until this point all appears to be completely analogous to the definition of the 4-spin. However, there exist an important difference, s^μ is defined as we have said only in the case of massive particles; and, as we represent the spin degrees of freedom as a three-vector in the rest frame of the particle, this becomes the only frame in which we can define s^μ. There is no similar way of selecting a frame[6] for the definition of E^μ and B^μ, every frame can be selected and therefore there are many unequivalent ways of defining them.

Let us display the explicit connection between the components of E^μ and B^μ in a frame $\mathcal{R}'$ with those in frame $\mathcal{R}$. If $E^{\mu'} = (\mathcal{E}'_0, \vec{\mathcal{E}'})$ and frame $\mathcal{R}'$ is moving with a velocity $\mathbf{v}$ as seen from $\mathcal{R}$, we must have ($c = 1$):

$$\mathcal{E}'_0 = -\gamma\, \mathbf{v} \cdot \mathbf{E}, \tag{4}$$

$$\vec{\mathcal{E}'} = \mathbf{E} - \frac{\gamma}{\gamma + 1}\,(\mathbf{v} \cdot \mathbf{E})\,\mathbf{v}, \tag{5}$$

where $\gamma = (1 - v^2)^{-1/2}$, and analogous expressions for the components of $B^{\mu'}$. As equation (5) shows $\vec{\mathcal{E}'}$ is a three-vector which has little to do with the electric field in frame $\mathcal{R}'$. Therefore E^μ and B^μcannot be considered as merely "covariant generalizations" of the usual notions of electric and magnetic fields. The fields E^μ and B^μmust be regarded as "new" fields with similar but not the same physical interpretation as $\mathbf{E}$ and $\mathbf{B}$. In spite of this, we will call them the electric and the magnetic field 4-vectors, since they become essentially the $\mathbf{E}$ and $\mathbf{B}$ vectors in frame $\mathcal{R}$.

Let us proceed now to obtain the equations of motion for E^μ and B^μ. Apart from fields, charges and currents the equations may also include the 4-vector u^μ, and they must reduce to the standard Maxwell equations in frame $\mathcal{R}$. The simplest set of equations that satisfy these requirements (in Gaussian units and with $c = 1$) is:

$$\partial_\mu E^\mu = 4\pi\, \rho_c, \tag{6a}$$

$$\partial_\mu B^\mu = 0, \tag{6b}$$

$$\varepsilon^{\alpha\beta\gamma}\, \partial_\gamma E_\beta + \partial_t B^\alpha = 0, \tag{6c}$$

$$\varepsilon^{\alpha\beta\gamma}\, \partial_\gamma B_\beta - \partial_t E^\alpha = 4\pi\, J_c^\alpha, \tag{6d}$$

where ρ_c is the density of proper charge, J_c^λ the conduction current 4-vector (*i.e.* the charge and current densities as measured by the fiducial observer), $\varepsilon^{\alpha\beta\gamma} = u_\lambda \varepsilon^{\lambda\alpha\beta\gamma}$ with $\varepsilon^{\lambda\alpha\beta\gamma}$ the four dimensional Levi-Civita symbol ($\varepsilon^{0123} = 1$), and ∂_t ($= u^\lambda \partial_\lambda$) stands for the invariant directional derivative in the "time direction". The relationships $J^\lambda = J_c^\lambda + u^\lambda\, \rho_c$ and $\rho_c = J_\lambda u^\lambda$ are the links between J_c^λ and ρ_c with the usual current 4-vector J^λ. We can also express the force (Lorentz force density) exerted on charges ρ_c and currents J_c^α in terms of E^μ and B^μas:

$$f^{\alpha} = \rho_c E^{\alpha} + \varepsilon^{\alpha}{}_{\sigma\lambda} J_c^{\sigma} B^{\lambda} - u^{\alpha} J_c^{\lambda} E_{\lambda}. \tag{7}$$

Equations (6) and (7) are explicitly covariant and describe completely the dynamics of the electric and magnetic 4-fields. Notice the dependence on the unit 4-vector u^{μ}, this dependence reflects the arbitrariness in the selection of frame $\mathcal{R}$ and at the same time makes the form of equations (6) and (7) independent of that choice.

3. Covariant electrodynamics

3.1 Covariant Maxwell equations

All standard results of classical electrodynamics can be obtained from equations (6) and (7), to show this we will exhibit the complete equivalence between these and the standard form of Maxwell equations. First, we need to uncouple the equations for E^{μ} and B^{μ}, to this end we must take the "curl" (*i.e.* apply the operator $\varepsilon^{\alpha\mu\nu}\partial_{\nu}$) to equations (6c) and (6d) and simplify the results with the help of equations (6a) and (6b). We end with the pair of equations

$$\begin{aligned} \partial^{\mu}\partial_{\mu} E^{\alpha} &= 4\pi\, J^{\alpha\beta} u_{\beta} \\ \partial^{\mu}\partial_{\mu} B^{\alpha} &= -4\pi\, \breve{J}^{\alpha\beta} u_{\beta}, \end{aligned} \tag{8}$$

to write these equations we have defined the antisymmetric tensor $J^{\alpha\beta} = \partial^{\beta} J^{\alpha} - \partial^{\alpha} J^{\beta}$ and its dual $\breve{J}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} J_{\mu\nu}$.

The form of these equations strongly suggests that both of them can be written as a single equation if instead of E^{μ} and B^{μ}we use an antisymmetric tensor $F^{\alpha\beta}$ which, accordingly, must be defined through the conditions

$$E^{\alpha} = F^{\alpha\beta} u_{\beta} \qquad \text{and} \qquad B^{\alpha} = -\breve{F}^{\alpha\beta} u_{\beta}. \tag{9}$$

It is easy to invert these equations to express $F^{\alpha\beta}$ and its dual in terms of E^{μ} and B^{μ}:

$$F^{\alpha\beta} = E^{\alpha} u^{\beta} - E^{\beta} u^{\alpha} + \varepsilon^{\alpha\beta}{}_{\gamma} B^{\gamma}, \tag{10}$$

$$\breve{F}^{\alpha\beta} = B^{\beta} u^{\alpha} - B^{\alpha} u^{\beta} - \varepsilon^{\alpha\beta}{}_{\gamma} E^{\gamma}. \tag{11}$$

it is obvious now that $F^{\alpha\beta}$ must be identified with the electromagnetic field tensor; using it Eqs. (8) take the form

$$\partial^\mu \partial_\mu F^{\alpha\beta} = 4\pi J^{\alpha\beta}, \tag{12}$$

or, equivalently,

$$\partial^\mu \partial_\mu \breve{F}^{\alpha\beta} = 4\pi \breve{J}^{\alpha\beta}. \tag{13}$$

The first-order equations that $F^{\alpha\beta}$ satisfy follow at once if we take the divergence of Eqs. (10) and (11) and use Eqs. (6):

$$\partial_\beta F^{\alpha\beta} = -4\pi J^\alpha, \tag{14}$$

$$\partial_\beta \breve{F}^{\alpha\beta} = 0. \tag{15}$$

This result shows the equivalence between our equations (6) and the standard covariant form of Maxwell equations.

Notice that we can solve Eq. (12) to get

$$F^{\beta\alpha} = \int d^4x'\, G(x - x')\Big(\partial^\alpha J^\beta(x') - \partial^\beta J^\alpha(x')\Big), \tag{16}$$

where $G(x - x')$ is a D'Alambert Green's function. This equation shows very clearly the role of "superpotential"[7] played by $F^{\alpha\beta}$ in classical electrodynamics.

3.2 Energy-Momentum conservation

An important result of Maxwell equations is the theorem of energy-momentum conservation. We are going to show how this result follows from our equations. If in Eq. (7) we use Eqs. (6a) and (6b) to eliminate ρ_c and J_c^μ in favor of E^μ and B^μ, and using the identity

$$\begin{aligned}\varepsilon_{\alpha\beta\rho}\varepsilon_{\mu\nu}{}^{\rho} = {} & (\eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\mu}\eta_{\beta\nu}) + (\eta_{\beta\nu}u_\alpha u_\mu - \eta_{\beta\mu}u_\alpha u_\nu) + \\ & (\eta_{\alpha\mu}u_\beta u_\nu - \eta_{\alpha\nu}u_\beta u_\mu),\end{aligned} \tag{17}$$

($\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowsky metric), we can rewrite[8] equation (7) in the form

$$f^\alpha + \partial_\beta T^{\alpha\beta} = 0, \tag{18}$$

where we have defined the 4-tensor $T^{\alpha\beta}$ as

$$\begin{aligned} T^{\alpha\beta} = -\frac{1}{4\pi}\Big(& E^\alpha E^\beta + B^\alpha B^\beta - (\varepsilon^{\alpha\lambda\gamma}u^\beta + u^\alpha\varepsilon^{\beta\lambda\gamma})E_\gamma B_\lambda + \\ & (u^\alpha u^\beta - \frac{1}{2}\eta^{\alpha\beta})(E_\lambda E^\lambda + B_\lambda B^\lambda)\Big). \end{aligned} \tag{19}$$

This is the symmetrical energy-momentum tensor of the electromagnetic field and Eq. (18) is the balance equation for the energy-momentum flow into a system of fields and charges on interaction. Using this tensor we can introduce various useful "secondary objects", for example:

$$U = T_{\alpha\beta}u^\alpha u^\beta = -\frac{1}{8\pi}(E_\mu E^\mu + B_\mu B^\mu) \qquad \text{(invariant energy density)}, \tag{20}$$

$$S^\alpha = T_{\mu\nu}(\eta^{\alpha\mu} - u^\alpha u^\mu)u^\nu = \frac{1}{4\pi}\varepsilon^\alpha{}_{\beta\gamma}E^\beta B^\gamma \qquad \text{(Poynting 4-vector)}. \tag{21}$$

These objects appear in the study of the algebraic properties of the electromagnetic field [5].

3.3 Electrodynamics of continuous media

Our approach is appropriate for the covariant description of electromagnetic phenomena in continuous media. In this case the arbitrariness in the choice of a frame for making the basic definitions is removed, the natural choice becomes the rest frame of the medium. For describing the electromagnetic response of the medium we only need to introduce, in an analogous way to what we used in section 2 for defining E^μ and B^μ, the space-like induction field 4-vectors D^μ and H^μ. That is, they must reduce to the macroscopic vectors **D** and **H** measured by our fiducial observer in the medium's proper frame. The macroscopic electromagnetic properties of the medium are included in two tensors $\epsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$, 4-dimensional generalizations of the permitivity and permeability tensors, respectively. The constitutive equations can be written in the form

$$D_\alpha = \epsilon_{\alpha\beta}E^\beta, \qquad H_\alpha = \mu^{-1}_{\alpha\beta}B^\beta \tag{22}$$

In a conducting medium we must have also

$$J_c^\alpha = \sigma^{\alpha\beta} E_\beta, \tag{23}$$

with $\sigma^{\alpha\beta}$ the 4-tensor of electrical conductivity, this is a covariant expression for Ohm's law. The permitivity, permeability and conductivity 4-tensors are symmetric and must fulfill $\epsilon_{\alpha\beta}u^\beta = 0$, $\mu^{-1}_{\alpha\beta}u^\beta = 0$ and $\sigma_{\alpha\beta}u^\beta = 0$; u^β can be interpreted now as the 4-velocity of the medium.

In the medium, the macroscopic Maxwell equations are

$$\partial_\mu D^\mu = 4\pi\rho_c, \tag{24a}$$

$$\partial_\mu B^\mu = 0, \tag{24b}$$

$$\varepsilon^{\alpha\beta\gamma}\partial_\gamma E_\beta + \partial_t B^\alpha = 0, \tag{24c}$$

$$\varepsilon^{\alpha\beta\gamma}\partial_\gamma H_\beta - \partial_t D^\alpha = 4\pi J_c^\alpha, \tag{24d}$$

where ρ_c and J_c^α now stand for the macroscopic proper charge and conduction current densities.

It is convenient [10], in analogy with Eq. (10), to introduce the macroscopic induction field tensor $H^{\alpha\beta} = D^\alpha u^\beta - D^\beta u^\alpha + \varepsilon^{\alpha\beta}{}_\gamma H^\gamma$. The relationship between $H^{\alpha\beta}$and $F^{\mu\nu}$can be written as $H^{\alpha\beta} = \chi^{\alpha\beta}{}_{\mu\nu}F^{\mu\nu}$; using Eqs. (10) and (24), it is easy to see that the material tensor $\chi^{\alpha\beta}{}_{\mu\nu}$ can be written as

$$\chi^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}(u^\alpha u_\mu \epsilon^\beta_\nu - u^\beta u_\mu \epsilon^\alpha_\nu - u^\alpha u_\nu \epsilon^\beta_\mu + u^\beta u_\nu \epsilon^\alpha_\mu - \varepsilon^{\alpha\beta\gamma}\mu^{-1}_{\gamma\lambda}\varepsilon^\lambda{}_{\mu\nu}). \tag{25}$$

The formulas given in this section can be used as the starting point for a covariant description of the electrodynamics of moving media [10,12].

Interestingly, as can be seen from the above equations, in an isotropic, non-magnetic ($\mu = 1$) and non-dispersive medium the electromagnetic effects of matter can be thought of as a change in the space-time metric:

$$\eta^{\mu\nu} \rightarrow g^{\mu\nu} = \eta^{\mu\nu} - (\epsilon - 1)u^\mu u^\nu, \tag{26}$$

with ϵ the dielectric constant of the medium. This metric is similar to that used by Synge for describing ligth propagation in transparent media in general relativity

[9] and with a recently proposed metric for describing geodesically the path of a charged particle in electromagnetic fields [11].

4. Conclusions

In this work we have shown how we can formulate the basic equations of classical electrodynamics using properly defined electric and magnetic 4-vectors. This formulation is physically equivalent to the standard one, although its basic objects have different meanings. Our description of electromagnetism can be useful for formulating covariantly the electrodynamics of moving matter, a point that we hardly begin to touch in this work.

The 4-fields we have introduced do not have the same physical interpretation as the standard electric and magnetic fields, except in the fiducial observer frame $\mathcal{R}$, but we have proved that all electromagnetic *phenomena* can be described correctly using whatever set of fields. This, we think, is an evidence of the somewhat arbitrary character of the theoretical constructs we must invent for describing physical reality. This is an important point, since many times such constructs are presented as "real objects" as if they were imposed on us by the physical world.

On the other hand, it should be clear that, following the method used in this paper for defining E^{μ} and B^{μ} and to generalize Maxwell equations to four dimensions, every set of equations at our disposal can be put in a manifestly Lorentz covariant form. However, this is not sufficient to guarantee that they must give a correct description of the phenomena in a relativistic framework. In our case we found a correct description only because Maxwell equations were covariant (although not manifestly so) from the start. But this is an experimental fact. Covariance alone cannot guarantee the correctness of any mathematical formula as a physical law [12] as sometimes may be implied from certain presentations.

What we have said must not be interpreted as demeriting the *heuristic* value of covariance in the search of new laws, but this should not be confused with a fool-proof method of inventing them.

Acknowledgements

We wish to thank J.L. Fernández Chapou for some helpful discussions, and also, F.D. Micha for her help with the final corrections of the manuscript.

Addenda

We have found two recent papers which discuss ideas akin to the ideas presented in this paper: M.T. Teli, *Pramana*, **24**, (1985), 485; and J. Stachel, in *J.C. Maxwell, the Sesquicentennial Symposium* edited by M.S. Berger, Elsevier Scientific B.V. Amsterdam, 1984.

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Resumen. Introducimos cuadrivectores de campo eléctrico y magnético definiéndolos según el procedimiento acostumbrado para definir el cuadriespín y los empleamos para expresar las ecuaciones de Maxwell en forma covariante. Probamos que las ecuaciones así obtenidas son físicamente equivalentes a la forma covariante usual de las ecuaciones de Maxwell. También mostramos cómo puede desarrollarse la electrodinámica covariante a partir de nuestras ecuaciones.