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Blackbody radiation and the dimension of space

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Abstract. A thermal radiation cavity is used as a probe to test for a possible deviation of space from exact tri-dimensionality. We consider a D-dimensional cavity assuming that such derivation would be reflected in the observed discrepancy between the standard theoretical prediction and experimental values of the radiance and the spectral distribution. Corrections to the Stefan-Boltzmann radiance and the Planck spectrum are calculated, which turn out to be proportional to (D-3), as expected. For a small cavity there are further corrections to these formulas, coming from its size and shape. Considering then cosmic background radiation, *i.e.*, the largest possible cavity, we get from the reported experimental errors for the radiance and the spectrum, a bound $|D-3| \lesssim 10^{-3}$. This value is considerably larger then the bounds recently obtained using other precisely measured phenomena as probes.

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1. Introduction

In recent years the possibility of space-time having a dimension different from 4 has frequently been considered. This conjecture was originally advanced in Kaluza's famous paper [1] of 1921, which proposed a space-time dimensionality of 5 to unify classical gravitation and electromagnetism, giving rise to the Kaluza-Klein theories, dormant for several decades until their revival in the late 1960's in a quantum field theoretical version that evolved into the modern supersymmetric theories trying to unify gravity with the other interactions [2]. This theoretical road to higher dimensionalities converged with a rather distant one at first, namely, that opened up by the dual models of strong interactions [3], which led first to the string theory of strong interactions and then to the grand unified "string theories of everything" of our days, all of them mathematically consistent only in a space-time with dimension larger than 4.

Other theoretical schemes dealing with space dimensionalities different from 3 even at low energies have arised in the study of critical phenomena [4], and of fractal structures [5]. In fact, even the possibility of a non-continuous space-time has been considered, whose discreteness would become apparent at high energy [6].

At some point the question of space-time dimensionality must be addressed experimentally, looking for residual effects of abnormal dimensions detectable at presently available energies, much lower than the Planck energy 10^{19} Gev. These residual effects would appear as irreducible deviations of experimental results from predictions of the standard three- or four-dimensional theories. This criterion was first used by Zeilinger and Svozil in a 1985 paper [7]. Expecting deviations from four-dimensionality to be quite small at low energies, they defined fractional dimensions in terms of the Hausdorff measure of regions of space-time. Choosing the anomalous magnetic moment g of the electron as a probe, and performing the QED perturbative loop integrations through the Hausdorff measure, instead of the usual Riemann-Stieltjes one, they obtained to first order in the fine structure constant α

$$g(\widetilde{D}) \simeq 2 + \left(\frac{\alpha}{2\pi}\right) \pi^{(\widetilde{D}/2-2)} \Gamma\left(3 - \frac{\widetilde{D}}{2}\right), \tag{1}$$

with $\widetilde{D}$ the space-time dimension, defined through its Hausdorff measure. Identifying the diference Δg between the standard theoretical prediction (using 4-dimensional QED) and the experimental result for g, with the quantity $(\Delta g)_{\text{theor.}} = g(\widetilde{D}) - g(4)$, they obtain

$$\widetilde{D} - 4 \simeq -\left(\frac{2\pi}{\alpha}\right)\left(\frac{2}{C + \ln \pi}\right)\Delta g, \tag{2}$$

where C = Euler's constant. From the best available Δg they get the deviation from 4-dimensionality

$$\widetilde{D} - 4 \simeq -(5.3 \pm 2.5) \times 10^{-7}. \tag{3}$$

Working along this line of thought, Müller and Schäfer [8] chose two very precisely measured phenomena: the advance of Mercury's perihelion and the Lamb shift of the $2p_{1/2}$ and $2s_{1/2}$ states in hydrogen. In each case, instead of introducing fractional dimensions through the Hausdorff measure of sets, they followed a somewhat more delicate procedure: they generalized the theory of the phenomenon under consideration to a space with one time and D space dimensions, with D an integer. Solving the differential equations of the extended theory, they analytically continued the final result to arbitrary values of D. Identifying again the difference between the standard ($D = 3$) result and the experimental value, with a deviation from the standard prediction caused by de assumed dimensionality D, they arrived at bounds to the quantity D (or $(D - 3)$). In the case of the advance of Mercury's perihelion, they wrote down Einstein's equations for general relativity in an arbitrary number of dimensions, considered the sun-planet problem and got for the perihelion shift per revolution

$$\Delta\phi \simeq \frac{6\pi m^2}{L^2} + \pi(D-3) \equiv \Delta\phi_3 + \pi(D-3), \tag{4}$$

where m is the sun mass, L the system's angular momentum, $\Delta\phi_3 = 6\pi m^2/L^2$ the standard result form $(3 + 1)$-dimensional general relativity, and $\pi(D - 3)$ represents the effect of the assumed deviation of space from tri-dimensionality. Using the experimental result fort $\Delta\phi$ they obtained the bound

$$|D-3| \lesssim 10^{-9}. \tag{5}$$

For the Lamb shift problem they wrote down the Schrödinger equation with relativistic and spin-orbit terms in D spatial dimensions, and calculated the effect $(\Delta E)_{LS}$ on the lamb shift,

$$(\Delta E)_{LS} = E_{D,2p_{1/2}} - E_{D,2s_{1/2}} \simeq (D-3)\frac{(Z\alpha)^2 m}{12}, \tag{6}$$

where Z is the nuclear charge, m the electron mass and α the fine structure constant. This term is present even in the non-relativistic limit because the SO(4) symmetry of the normal quantum mechanical Kepler problem is broken for $D \neq 3$. Equating $(\Delta E)_{LS}$ with the difference between the standard ($D = 3$) QED prediction for the Lamb shift and its experimental value, they get

$$|D-3| \lesssim 3.6 \times 10^{-11}, \tag{7}$$

this being the most stringent bound to date on possible deviations from tridimensionality at low energies.

Following the same line of argument, we propose using thermal radiation in a cavity as a probe to test for possible deviations from tridimensionality. Measurement of the radiance and the spectrum are reasonably accurate [9], and discrepancies with respect to the Stefan-Boltzmann law and the Planck frequency distribution, respectively, should provide bounds for the quantity $(D-3)$.

Blakbody radiation has been employed in the past to get bounds on a possible photon mass [10], assuming that the disagreement found between experimental values and theoretical predictions could be associated with the effect of a tiny mass for light, and using Proca's instead of Maxwell's equations to describe electromagnetic radiation. In this article we will take a strictly zero photon mass, and assume that the observed differences come exclusively from a dimensional effect.

2. A D-dimensional radiation cavity

For a tridimensional cavity the partition function Z is given by

$$\ln Z = -2 \sum_{\substack{\text{normal}\\ \text{modes}}} \ln\left(1 - e^{-\beta\varepsilon}\right) \simeq -2 \int_0^\infty d\nu(\ldots)\ln(1 - e^{-\beta h\nu}) \tag{8}$$

where $\beta = \frac{1}{kT}$, $\varepsilon = h\nu$ (neglecting as usual the zero-point energy); the factor 2 is the number of photon polarizations, and $(\ldots)$ is a function of the frequency ν of geometrical origin, whose exact value is not written down to focus attention on the quantities of interest. The nature of the approximation involved when going from the sum to the integral will be considered latter.

This formula generalizes in D dimensions to

$$\ln Z_D = -\lambda_D \sum_{\substack{\text{normal}\\ \text{modes}}} \ln\left(1 - e^{-\beta h\nu}\right) \simeq -\lambda_D \int_0^\infty d\nu(\ldots)\ln(1 - e^{-\beta h\nu}) \tag{9}$$

where λ_D is the number of photon polarizations in D dimensions and the new factor $(\ldots)$ in the integral now comes from adding over all normal modes in a D-cube. This equation is valid assuming $\varepsilon_n = nh\nu$, *i.e.* the same spectrum as in three dimensions (the zero point energy is simply neglected from the beginning, because, being a constant, it affects neither the radiance nor the spectrum, just as in the ordinary case).

The value of λ_D and these energy eigenvalues are readily obtained from a generalization of the Maxwell free Lagrangian to D-dimensional space [11]:

$$\mathcal{L}_D = -\frac{1}{4}\int d^D x F^{\mu\nu} F_{\mu\nu}, \tag{10}$$

with $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu \quad (\mu, \nu = 0, 1, \ldots, D)$. Writing down for A^μ an expansion in normal modes analogous to that in (3+1) space-time, the Hamiltonian from Eq. (10) immediately gives the spectrum $\varepsilon_n = (n + \frac{1}{2})h\nu \to nh\nu$ if the zero-point energy is neglected. Just as in the ordinary case, the longitudinal photon is eliminated through the equations of motion, *i.e.*, the Maxwell equations generalized to D-space; the time-like photon is absent due to gauge invariance. Hence the remaining number of independent A^μ components, or photon polarizations, is $\lambda_D = D - 1$.

Using spherical coordinates in D-space the partition function becomes (see Appendix)

$$\ln Z_D = -\frac{2(D-1)L^D \pi^{D/2}}{\Gamma(D/2)c^D}\int_0^\infty d\nu \nu^{D-1}\ln(1 - e^{-\beta h\nu}), \tag{11}$$

with L the linear size of the cavity (assumed a D-cube), and Γ the gamma function. All the thermodynamics of the system is obtained from this expression in the usual way: the internal energy density $u_D = -\frac{1}{V}\left(\frac{\partial \ln Z_D}{\partial \beta}\right)_V$, with $V = L^D$ the cavity volume, is given by

$$u_D = \left[\frac{2(D-1)\pi^{D/2}\Gamma(D+1)\zeta(D+1)k^{D+1}}{\Gamma(\frac{D}{2})h^D c^D}\right] T^{D+1}, \tag{12}$$

where ζ is Riemann's z-function. This expression reduces to the usual one in 3 dimensions when $D = 3$, as it must. To write down the radiance $\mathcal{R}_D$, the geometrical argument leading to $\mathcal{R} = (\frac{c}{4})u$ in three dimensions must be extended to D-space (see Appendix):

$$\mathcal{R}_D = \left[\frac{c\Gamma(\frac{D}{2})}{\sqrt{\pi}(D-1)\Gamma(\frac{D-1}{2})} \right] u_D. \tag{13}$$

This formula reduces to $\mathcal{R} = \frac{c}{4}u$ for $D = 3$.

The spectral radiance $\mathcal{P}_D(\nu, T)$ is obtained from $\mathcal{R}_D = \int_0^\infty d\nu \mathcal{P}_D(\nu, T)$. Using Eq. (11)

$$\mathcal{P}_D(\nu, T) = \frac{2\pi^{(D-1)/2}}{c^{D-1}\Gamma(\frac{D-1}{2})} \frac{h\nu^D}{e^{\beta h\nu} - 1}, \tag{14}$$

which reduces to the Planck density $\mathcal{P}_P = \frac{2\pi}{c^2}\frac{h\nu^3}{e^{\beta h\nu}-1}$ when $D = 3$.

3. Bounds for the dimension of space

To extract bounds for D from Eqs. (12–14), we must first analytically continue these formulas, proved for integer D, to arbitrary values of D. Then we must define quantities $\widetilde{\mathcal{R}}_D$ and $\widetilde{\mathcal{P}}_D$ with the same units as $\mathcal{R}_{SB}$ and $\mathcal{P}_P$, the corresponding Stefan-Boltzmann radiance and Planck spectrum in three dimensions

$$\widetilde{\mathcal{R}}_D \equiv L^{D-3}\mathcal{R}_D, \tag{15}$$

$$\widetilde{\mathcal{P}}_D \equiv L^{D-3}\mathcal{P}_D. \tag{16}$$

To first order in $(D - 3)$ one then gets from Eqs. (12–14),

$$\widetilde{\mathcal{R}}_D = \mathcal{R}_{SB}\left[1 + \alpha_1(D-3)\ln\left(\frac{LkT}{hc}\right)\right], \tag{17}$$

$$\widetilde{\mathcal{P}}_D = \mathcal{P}_P\left[1 + \alpha_2(D-3)\ln\left(\frac{L\nu}{c}\right)\right], \tag{18}$$

with α_1 and α_2 constants of order unity, whose exact values will not be required, as the final result for the bound on $(D - 3)$ will show. The terms proportional to $(D - 3)$ in Eqs. (17) and (18) then represent the first order corrections to the Stefan-Boltzmann radiance and the Planck spectrum when one assumes a space dimensionality different from 3.

To get a bound for D we then compare the term proportional to $(D-3)$ in either of the above two equations with the corresponding experimental error: identifying the central value in each case with $\mathcal{R}_{SB}(T)$ or $\mathcal{P}_P(\nu, T)$, $(D - 3)$ must be small enough for the dimensional correction to be compatible with the reported error.

To carry out this procedure one must choose the appropriate cavity size L, temperature T and frequency ν. The choice is determined by the fact that the $(D-3)$ correction term in Eqs. (17) and (18) has to compete with another theoretical correction, which depends on the form and size of the cavity. Even in three dimensions, the Stefan-Boltzmann value for the radiance, and the Planck spectrum, are rigorously valid only for infinitely large cavities. For finite ones we must change these expressions in the following way [9]:

$$\mathcal{R}_{f,3} = \mathcal{R}_{SB}\left[1 - \frac{5\Lambda}{16\pi^3 V}\left(\frac{hc}{kT}\right)^2\right], \tag{19}$$

$$\mathcal{P}_{f,3} = \mathcal{P}_{P}\left[1 - \frac{\Lambda}{8\pi V}\left(\frac{c}{\nu}\right)^2\right], \tag{20}$$

where Λ is of the form αL, with L a typical length of the cavity, and α a number of order unity (for example, $\lambda = 3L$ for a cube, $\Lambda = L_1 + L_2 + L_3$ for a parallelepiped, $\Lambda \simeq 6R$ for a sphere, etc.). This geometrical correction was calculated summing over the first 10^6 normal modes in Eq. (8), instead of approximating the sum by an integral as indicated there. The corrections due to cavity size and shape in D dimensions must have the same form as those in Eqs. (19) and (20), with perhaps a different numerical factor in Λ, and most probably and exponent $(D-1)$ instead of 2. All we need is that such extra terms vanish too in the limit of very large cavities, and this general feature of the thermodynamic limit cannot be changed by going to dimensionalities different from 3.

Then, to avoid the complication of geometrical corrections competing with dimensional ones we must consider the largest possible cavity, that is, cosmic background radiation (CBR), and this implies $L \simeq 10^{28}$cm (the size of the observed universe), $T \simeq 3$ K and $\nu \simeq 10^{11}$ Hz, to stay close to the maximum in the spectrum.

The experimental errors for both the radiance and the spectrum in CBR measurements are of order ten percent [12], so from Eqs. (17) and (18) the same bound results:

$$|D-3| \lesssim 10^{-3}, \tag{21}$$

This is to be compared with the bounds mentioned in the Introduction, coming from the anomalous magnetic moment of the electron ($|D-3| \lesssim 10^{-7}$), from the advance of Mercury's perihelion ($|D-3| \lesssim 10^{-9}$), and from the Lamb shift ($|D-3| \lesssim 10^{-11}$). Thus we are led to the conclusion that thermal radiation methods cannot improve on existing low energy bounds for the dimension of space: this would imply measuring the cosmic background radiation with a precision greater than 10^{-9}, totally beyond present capabilities.

Appendix

The necesary formulas to go from cartesian to spherical coordinates in D dimensions will be included here, as they turned out to be difficult to find or prove. Let the cartesian coordinates of a D-vector be $(x_1, x_2, \ldots, x_D)$, and its spherical coordinates be $(r, \theta_1, \theta_2, \ldots, \theta_{D-2}, \phi)$, with the ranges $0 \leq \theta_j \leq \pi, \quad 0 \leq \phi \leq 2\pi$ for the angles; they are connected through

$$
\begin{aligned}
x_1 &= r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-2} \sin\phi \\
x_2 &= r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{D-2} \cos\phi \\
&\vdots \\
x_j &= r \sin\theta_1 \ldots \sin\theta_{D-j} \cos\theta_{D-j+1}, \qquad (j > 2) \\
&\vdots \\
x_D &= r \cos\theta_1,
\end{aligned}
\qquad (A-1)
$$

so the metric is

$$g_{ij} = \text{diag } \left(1, r^2, r^2\sin^2\theta_1, r^2\sin^2\theta_1\sin^2\theta_2, \ldots, r^2\sin^2\theta_1 \ldots \sin^2\theta_{D-2}\right), \quad (A-2)$$

and the Jacobian becomes

$$J = \sqrt{g} = r^{D-1}(\sin\theta_1)^{D-2}(\sin\theta_2)^{D-3} \ldots \sin\theta_{D-2}. \qquad (A-3)$$

To calculate the geometrical factor $(\ldots)$ in the radiance, $\mathcal{R}_D = (\ldots)u_D$, the usual construct of a semi-infinite cavity with a little hole at the origin gives the correct result, being careful about the fact that a little hole in D dimensions is $(D-1)$-dimensional.

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Resumen. Se utiliza una cavidad de radiación térmica para explorar la posibilidad de una desviación del espacio respecto a la tridimensionalidad que se le asigna. Analizamos una cavidad de radiación D-dimensional, suponiendo que la desviación sería reflejada en la discrepancia observada entre los valores teóricos y experimentales para la radiancia y la distribución espectral, y calculamos las correspondientes correcciones a la radiancia de Stefan-Boltzman y la distribución espectral de Planck, las cuales resultan proporcionales a $(D-3)$, como era de esperarse. En el caso de una cavidad pequeña existen correcciones adicionales a estas expresiones, provenientes de su forma y tamaño. Considerando entonces la radiación cósmica de 3 K, es decir la cavidad máxima posible, obtenemos, a partir de los errores experimentales reportados para la radiancia y el espectro, la cota $|D-3| \lesssim 10^{-3}$. Este límite es considerablemente mayor que los obtenidos recientemente a partir de otros fenómenos medidos con alta precisión.