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Constructing integrals of motion by elementary means

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Abstract. To construct constants of the motion for a mechanical system is in general a difficult task; however, for elementary problems it is possible to get some of them by direct elimination of the time variable from the solution. Although the method is of limited value because it is based on a previous knowledge of the solution and on the possibility of getting rid of the time varible, it leads to explicit expressions for constants of the motion even in some unusual situations, such as problems with time-dependent forces, and thus, it may be of interest for introductory courses. A classification of these constants is made according to whether they depend only on one generic point of phase space or they are also functions of the initial conditions; in the first case, they can be true integrals of motion and thus serve to eliminate degrees of freedom from the description.

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1. Introduction

Consider a closed or autonomous mechanical system having N degres of freedom. Associated to it there exist at most $2N$-1 significative constants of the motion, *i.e*, functions of the generalized coordinates and velocities or momenta that do not depend explicitly on time and that evaluate to a constant along the motion (In order to simplify the discussion we shall use indistinctly the sets of variables $(q_i, \dot{q}_i)$ or (q_i, p_i), according to convenience) [1]. The complete set of these constants determines the trajectory of the system in $2N$-dimensional phase space.

An alternative definition of a constant of the motion for a Hamiltonian system can be given in terms of Poisson brackets. Even if accessory to the discussion that follows, it seems convenient to include it here for reference. Consider a dynamical system described by the Hamilton equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{1}$$

where the Hamiltonian H is a function of the coordinates q_i and the corresponding momenta p_i. The total time derivative of the dynamical variable $G(q_i, p_i; t)$

is then given by

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \sum_i \frac{\partial G}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial G}{\partial p_i}\dot{p}_i$$
$$= \frac{\partial G}{\partial t} + \sum_i \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \sum_i \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i}$$

or

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + [G, H], \tag{2}$$

where the Poisson bracket of G and H is defined by

$$[G, H] = \sum_i \left(\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} \right).$$

Eq. (2) shows that G is a constant of the motion if *a)* it does not depend explicitly on time and *b)* its Poisson bracket with the Hamiltonian is zero. This is probably the most usual definition of a constant of the motion; it is even translated into quantum mechanics, where *b)* above is sustituted by *b')*: if its commutator with the Hamiltonian is zero (as follows from the general correspondence of Poisson brackets with commutators). Notice that the rule applies to all Hamiltonian systems, but it serves only to check whether a dynamical variable is a constant of the motion, not to construct it.

The definition given above excludes constants of motion that depend explicitly on time; thus for instance, a simple procedure as the mere rewriting of an equation $p = p(q, t\,;\, q_0, p_0)$ in the form $p_0 = p_0(q, p, t;\, q_0)$, does not yield in general a constant of the motion in the above sense. We should remark that more general definitions that allow the explicit time dependence of the constants of motion, are also frequent in the literature and may be useful for certain purposes; examples of such usage may be found in Refs. [2] and [3]. From this viewpoint, the initial conditions, for instance, are constants of motion and a completely integrable system has $2N$ of them.

Among the possible constants of the motion, there are some that have a particular importance, namely the so-called integrals of motion. Their importance comes from the fact that to each one of them corresponds a conserved quantity, *i.e.*, a quantity that defines the state of motion of the system and thus serves to eliminate a degree of freedom from the description. The integrals of motion are (continuous, single-valued, differentiable) independent functions of q_i and p_i that are defined over the whole (accesible) q, p-space, have a constant value along the trajectory and are in involution. We recall that two functions F, G are said to be in involution when their Poisson bracket is zero (or, translated into quantum mechanics, when their commutator is zero). A system with N degrees of freedom can have at most N integrals of motion; in such a case, the system is called completely integrable [4]. A totally integrable Hamiltonian system has regular trajectories only, whereas non-integrable systms usually give rise to highly irregular and chaotic motions.

As has been established in the last decades [4], for most dynamical systems only a few constants exist, and even then, no general method is known for their construction nor for finding their total number. It is not unfrequent, for example, that the only integral of motion is the Hamiltonian (in the case of autonomous Hamiltonian systems), whose value gives the energy of the system. There are certain situations, nevertheless, in which it is possible to construct integrals of motion by reasonably simple means; we briefly recall here several different procedures of practical and theoretical value, without entering into details.

i) The most important example is given by an external force that depends on the coordinates only and is derivable from a potential, $F = -\nabla V$: this is the so-called conservative problem. In this case, by taking the scalar product of the equation of motion $m\ddot{\mathbf{x}} = \mathbf{F}$, observing that $\dot{\mathbf{x}} \cdot \mathbf{F} = -dV/dt$ and integrating over time one gets:

$$\mathcal{E} = \frac{1}{2}m\dot{x}^2 + V = constant,$$

which is the law of conservation of energy. Further, by taking the cross product of the equation of motion with $\mathbf{r}$ it is possible to demonstrate that the angular-momentum vector $\mathbf{r} \times \mathbf{p}$ is conserved if the force is radial, *i.e.*, if F is of the form $\mathbf{r}f(r, \vartheta, \phi)$.

ii) When the system is described by means of the Hamilton equations of motion (1) and H does not depend on a specific coordinate q_s, then the corresponding momentum p_s is conserved, as follows from the second equation in (1).

iii) When the Hamiltonian is (completely) separable or integrable [4], *i.e.*, when there exists a set of N (action-angle) pairs of variables (J_i, ϑ_i) such that H can be written as a sum of functions, each one depending only on one of the J_i, then these action variables are conserved, as follows from the argument just given. The other N constants are obtained by derivation of the Hamiltonian: $\omega_i = \frac{\partial H}{\partial J_i}$, as follows from Eq. (1), with the frequencies given by $\omega_i = \dot{\vartheta}_i$. Even though this is a particular case of (ii), this method is so important—for dealing with multiply periodic systems, for example—that it deserves explicit consideration.

iv) A constructive method for Lagrangian (or Hamiltonian) systems is provided by Noether's theorem, which gives an explicit rule for finding the dynamical variable that is conserved as a result of a given symmetry. The formulation of this theorem goes beyond elementary textbooks and we therefore refrain from any discussion of it (see *e.g.* Ref. [2]). Other, so-called non-Noetherian constants of the motion have been shown to exist, even for more general systems, and methods for their construction have been suggested (see, *e.g.*, Ref. [3]).

The problem of finding the integrals of motion is in general a difficult one. However, if the solutions to the equations of motion are known, it is possible in several cases of interest to use them for the construction of constant functions that do not involve the time. The procedure is obvious and of limited theoretical interest, since often the point in knowing the constants of the motion is just to avoid the need of finding the explicit solution of the problem, or to help to get it by reducing the effective number of degrees of freedom. The method has however a pedagogical value, since it allows to construct constants of the motion from scratch in several elementary but interesting cases, including time-dependent problems.

2. Direct procedure to construct some constants of the motion

In what follows we use a classification of the constants of the motion G into two types: we say that G is *global* (or g-type) if it can be put in the form $G = G(q_i, \dot{q}_i; \alpha_s)$ where α_s represent the set of parameters of the system. Alternatively, we say that G is *specific* (or s-type) if it is of the form $G = G(q_i, \dot{q}_i; \alpha_s; \mathbf{q}_0, \dot{\mathbf{q}}_0)$, *i.e.*, if it depends (non trivially, of course) also on the initial conditions; in particular, this is the case for time-dependent forces, because such forces fix an origin for time.

The procedure to construct the G's is as follows. Assume you know the explicit solution of a problem in the form

$$q_i = q_i(\mathbf{q}_0, \dot{\mathbf{q}}_0; \alpha_s; t), \qquad \dot{q}_i = \dot{q}_i(\mathbf{q}_0, \dot{\mathbf{q}}_0; \alpha_s; t),$$

for $i = 1, 2, ..., N$. Take one of the q_i or $\dot{q}_i$, say $\dot{q}_1$, and invert it (assuming you can do it) to get the time t as a function of $\dot{q}_1$, the initial values $\mathbf{q}_0$, $\dot{q}_0$ and the parameters of the problem, α_s. Insert this expression for t into the remaining equations for q_i and $\dot{q}_i$ and you get $2N - 1$ relations that hold for all t but do not contain t explicitly. In each one of these relations, transfer all variables $q_i, \dot{q}_i$ to one side of the equation so as to express the results in the form $G_j(q_i, \dot{q}_i)$=const; then you have got $2N - 1$ constants of the motion. It should be clear that the method can be applied only to those $q_i, \dot{q}_i$ that are known as functions of time, and once it has been possible to invert one of these functions to obtain t. Although you cannot be too optimistic, you will always get al least one constant of the motion, when the method is applicable. Needless to say, to test if a given G is indeed constant it suffices to show, by substituting the equations of motion, that the time derivative $\dot{G}$ is zero; of course if the Hamiltonian is at hand, one can show that the Poisson bracket $[G, H]$ gives zero.

3. Time-dependent forces

Let us first study some non-conservative (nonautonomous) one-dimensional problem, of the type $m\ddot{x} = F(t)$; in these cases the Hamiltonian is obviously not conserved, but still a constant of the motion exists, as will come out from the examples.

1. $F(t) = F_0 e^{-\alpha t}$, with $\alpha > 0$ (and $F_0 > 0$). The explicit solution is

$$\dot{x} = \dot{x}_0 + \frac{F_0\left(1 - e^{-\alpha t}\right)}{m\alpha}, \tag{3a}$$

$$x = x_0 + \left(\dot{x}_0 + \frac{F_0}{m\alpha}\right) t - \frac{F_0\left(1 - e^{-\alpha t}\right)}{m\alpha^2}. \tag{3b}$$

From $(3a)$ we obtain

$$t = -\alpha^{-1} \ln \left[1 + \alpha m \frac{(\dot{x}_0 - \dot{x})}{F_0}\right],$$

which introduced into Eq. $(3b)$ gives

$$G = m(\dot{x} - \dot{x}_0) + \alpha m x + \left(m\dot{x}_0 + \frac{F_0}{\alpha}\right) \ln \left[1 + \alpha m \frac{(\dot{x}_0 - \dot{x})}{F_0}\right]. \tag{4}$$

The value of this constant is $\alpha m x_0$ for a given trajectory. This function G is of type s, since it depends non-trivially on $\dot{x}_0$. Notice that it is defined for $\dot{x} < \dot{x}_0 + \frac{F_0}{m\alpha}$ only; it is actually evident from Eq. $(3a)$ that $\dot{x}$ attains its maximum allowed value $\dot{x}_0 + \frac{F_0}{m\alpha}$ asymptotically with time.

In the limit as α goes to zero, and the external force becomes time independent, an expansion of Eq. (4) in powers of α to first order gives

$$G' \equiv G - \frac{\alpha m^2 \dot{x}_0^2}{2F_0} = \alpha m \left[x - \frac{m\dot{x}^2}{2F_0}\right].$$

Hence, for a constant force F_0, G is of the global type and becomes a function of the Hamiltonian $H = \frac{1}{2}m\dot{x}^2 - F_0 x$ (as it should, since the Hamiltonian is constant in this case).

2. $F = A\cos\omega t$, with A and ω constant. The solution is

$$\dot{x} = \dot{x}_0 + \frac{A}{m\omega} \sin\omega t, \tag{5a}$$

$$x = x_0 + \dot{x}_0 t + \frac{A}{m\omega^2}(1 - \cos\omega t). \tag{5b}$$

From Eq. $(5a)$ we obtain, with $p = m\dot{x}$,

$$t = \omega^{-1} \sin^{-1} \left[\frac{\omega(\dot{p} - \dot{p}_0)}{A}\right],$$

which introduced into $(5b)$ gives

$$G = mx + \frac{A}{\omega^2}\left[1 - \frac{\omega^2(p - p_0)^2}{A^2}\right]^{\frac{1}{2}} - \frac{A}{\omega^2} - \frac{p_0}{\omega} \sin^{-1} \left[\frac{\omega(p - p_0)}{A}\right]. \tag{6}$$

This result is once more of type s, and it is defined only for

$$-A \leq \omega(p - p_0) \leq A$$

as follows also directly from inspection of Eq. (5a). In the limit as $\omega \to 0$, the restrictions on the value of p are lifted and Eq. (6) reduces to the Hamiltonian for the constant-force problem.

3. $F = At^n$, with $A > 0$ and $n \neq -1, -2$. The solution is

$$p = p_0 + \frac{At^{n+1}}{(n+1)}, \tag{7a}$$

$$mx = mx_0 + p_0 t + \frac{At^{n+2}}{(n+1)(n+2)}, \tag{7b}$$

with $p = m\dot{x}$. From Eq. (7a),

$$t = \left[\frac{(n+1)(p - p_0)}{A}\right]^{\frac{1}{(n+1)}}.$$

Introducing this into Eq. (7b) we obtain

$$G = mx - p_0 \left[\frac{(n+1)(p - p_0)}{A}\right]^{\frac{1}{(n+1)}} - \frac{A\left[\frac{(n+1)(p-p_0)}{A}\right]^{\frac{(n+2)}{(n+1)}}}{(n+1)(n+2)}. \tag{8}$$

This result is in general of type s, and restricted to $(n+1)(p - p_0) \geq 0$ if $n \neq 0$. For $n = 0$, one gets once more a constant force and G becomes of type g.

In the above examples, a time-dependent force gives rise to an s-type G; more specifically, to a function G that depends non-trivially on the initial value of the momentum. This may be understood as a consequence of the fact that the time-dependent force breaks the symmetry of the system with respect to time translations, *i.e.*, it introduces an origin for t, and the constant of the motion—which is not the Hamiltonian any more— depends explicitly on the conditions of the system at this origin. One might suspect that a time-dependent force always gives rise to an s-type G, although this is not proved in general, as far as we know.

4. The damped oscillator

We study now the one-dimensional, linear oscillator acted on by a damping force and described by the equation

$$m\ddot{x} = -m\omega^2 x - 2m\gamma\dot{x}. \tag{9}$$

This includes the physically interesting case of the radiating dipole: ($\tau = \frac{2e^2}{3mc^3}$)

$$m\ddot{x} = -m\omega^2 x + m\tau\dddot{x},$$

in the approximate description to first order in τ, in which the radiation term $m\tau\dddot{x}$ is replaced by its first-order approximation $-m\tau\omega^2\dot{x}$ and hence $\gamma = \frac{1}{2}\tau\omega^2$.

Let us start with the underdamped case, $\omega > \gamma$. The solution of Eq. (9) is

$$x = a\exp(i\sigma t) + a^* \exp(-i\sigma^* t), \tag{10a}$$

$$\dot{x} = i\sigma a\exp(i\sigma t) - i\sigma^* a^* \exp(-i\sigma^* t), \tag{10b}$$

with

$$a = \frac{\sigma^* x_0 - i\dot{x}_0}{\sigma + \sigma^*} \tag{11}$$

and

$$\sigma = \omega\left(1 - \frac{\gamma^2}{\omega^2}\right)^{\frac{1}{2}} + i\gamma. \tag{12}$$

Upon elimination of $\exp(i\sigma t)$ from Eqs. (10) we obtain, using (11),

$$t = \frac{i}{\sigma^*}\ln\left(\frac{\sigma x + i\dot{x}}{\sigma x_0 + i\dot{x}_0}\right).$$

Introducing this result into Eq. (10a) we get

$$x = a\left(\frac{\sigma x + i\dot{x}}{\sigma x_0 + i\dot{x}_0}\right)^{-\frac{\sigma}{\sigma^*}} + a^* \frac{\sigma x + i\dot{x}}{\sigma x_0 + i\dot{x}_0},$$

which with the help of Eq. (11) leads after some minor algebra to

$$G = \frac{1}{2}m\left|(\sigma x + i\dot{x})^{\frac{2\sigma}{\sigma+\sigma^*}}\right|^2. \tag{13}$$

We have arrived at a global G, that reduces to the Hamiltonian of the harmonic oscillator $H^0 = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2)$ as the damping factor γ goes to zero and hence σ and σ^* go to ω.

For small values of γ ($\gamma \ll \omega$) Eq. (13) gives to first order in γ

$$G = [H^0 + m\gamma x\dot{x}]\left[1 + \frac{i\gamma}{\omega}\ln\frac{\omega x + i\dot{x}}{\omega x - i\dot{x}}\right]$$
$$= H^0\left[1 + \frac{\gamma}{\omega}\left(\frac{m\omega x\dot{x}}{H^0} - 1\right)\sin^{-1}\frac{m\omega x\dot{x}}{H^0}\right],$$

for the quantity that is conserved along the motion. Notice that even though the energy of the damped oscillator decreases exponentially with time, there exists a function G associated to its motion that remains constant. However, this constant is not an integral of motion, since it is not single-valued. This kind of constants has been discussed in the literature [3].

In the overdamped case ($\gamma > \omega$) one obtains from the solution of Eq. (9)

$$G = (\dot{x} + \beta x)^{\beta}(\dot{x} + \varepsilon x)^{-\varepsilon}, \tag{14}$$

instead of Eq. (13), with

$$\beta = \gamma\left[1 + \left(1 - \frac{\omega^2}{\gamma^2}\right)^{\frac{1}{2}}\right], \qquad \varepsilon = \gamma\left[1 - \left(1 - \frac{\omega^2}{\gamma^2}\right)^{\frac{1}{2}}\right].$$

As $\frac{\gamma}{\omega} \to \infty$, Eq. (14) reduces to $G = (\dot{x} + 2\gamma x)^{2\gamma}$, corresponding to an $x(t)$ that decreases exponentially with time.

5. Systems with two degrees of freedom

1. Let us consider two coupled one-dimensional oscillators, described by the Hamiltonian (we are taking the masses and the natural frequencies equal to 1, for simplicity, and we assume $0 \leq C \leq 1$)

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + Cxy. \tag{15}$$

With $x = 2^{-\frac{1}{2}}(Z + z)$, $\quad y = 2^{-\frac{1}{2}}(Z - z)$, we get two uncoupled oscillators described by

$$Z = A_1\exp(i\Omega t) + B_1\exp(-i\Omega t), \tag{16a}$$

$$z = A_2\exp(i\omega t) + B_2\exp(-i\omega t), \tag{16b}$$

where

$$\Omega = (1+C)^{\frac{1}{2}}, \qquad \omega = (1-C)^{\frac{1}{2}}.$$

The problem has thus been reduced to that of two independent oscillators of different frequencies. We therefore know that not only the total Hamiltonian (15) is constant, but also the individual Hamiltonians are, and these can be taken as two independent constants of the motion. Now we can use the explicit solutions to construct a different combination that is also a constant, as follows. From Eq. (16*a*) and its time derivative we get

$$\exp(i\Omega t) = \frac{(\Omega Z - i\dot{Z})}{2A_1\Omega}.$$

Similarly, from (16*b*) and its time derivative we get

$$\exp(-i\omega t) = \frac{(\omega z + i\dot{z})}{2B_2\omega}.$$

The above equations can be combined in the form

$$\left(\frac{\Omega Z - i\dot{Z}}{2A_1\Omega}\right)^{\omega} = \left(\frac{\omega z + i\dot{z}}{2B_2\omega}\right)^{-\Omega}.$$

In an analogous way one gets

$$\left(\frac{\Omega Z + i\dot{Z}}{2B_1\Omega}\right)^{\omega} = \left(\frac{\omega z - i\dot{z}}{2A_1\omega}\right)^{-\Omega}.$$

These two equations can be combined to give

$$G = (\Omega Z - i\dot{Z})^{\eta}(\omega z + i\dot{z}) - (\Omega Z + i\dot{Z})^{\eta}(\omega z - i\dot{z}), \tag{17}$$

where $\eta = \frac{\omega}{\Omega} = \left[\frac{(1-C)}{(1+C)}\right]^{\frac{1}{2}}$. This constant is of type g and it is defined for all values of Z, $\dot{Z}$, z and $\dot{z}$. If the oscillators are very weakly coupled, then $\eta \cong 1$ and G can be approximated by the expression

$$G \cong 2i\omega(Z\dot{z} - z\dot{Z}) = 2i\omega(y\dot{x} - x\dot{y}) = 2i\omega l,$$

where l is the angular momentum of the motion in the xy-plane. On the other hand, if the coupling is strong ($C \cong 1$) the system behaves still as an oscillator in the z direction but as a free particle in the z direction: $G \cong 2i\dot{z}$. Eq. (17) defines a constant function that is a generalization of the angular momentum to arbitrary values of η, $0 \leq \eta \leq 1$.

2. Let us now consider a system of two interacting particles (with coordinates x, y) described by the Hamiltonian (notice that this is a two-dimensional Toda lattice [4])

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \lambda \exp(x - y). \tag{18}$$

Upon the change of variables

$$r = y - x, \qquad z = \frac{1}{2}(x + y), \tag{19}$$

the Hamiltonian takes the form

$$H = \dot{z}^2 + \frac{1}{4}\dot{r}^2 - \lambda \exp(-r),$$

which allows us to immediately identify two integrals of motion, namely the total linear momentum (associated to the ignorable z variable), and the Hamiltonian associated to the relative motion; we therefore write

$$\dot{z} = \dot{z}_0, \tag{20a}$$

$$\frac{1}{4}\dot{r}^2 - \lambda \exp(-r) = H_0. \tag{20b}$$

To construct a third constant of the motion we solve Eqs. (20). The solution of (20a) is immediate:

$$z = z_0 + \dot{z}_0 t \tag{21a}$$

and to solve (20b) we introduce the change of variable $r = \ln f^2$, leading to $\dot{f}^2 - H_0 f^2 = \lambda$, whose solution is $f = \left(\frac{\lambda}{H_0}\right)^{\frac{1}{2}} \sinh \sqrt{H_0}(t - t_1)$, with t_1 an integration constant; hence

$$r = \ln\left[\frac{\lambda}{H_0} \sinh^2 \sqrt{H_0}(t - t_1)\right]. \tag{21b}$$

Using Eq. (21a) to eliminate the time variable from (21b) we obtain after some minor algebra

$$G = z - \frac{\dot{z}_0}{\sqrt{H_0}} \sinh^{-1}\left(\frac{H_0 e^r}{\lambda}\right)^{\frac{1}{2}}. \tag{22}$$

The value of this constant is $z_0 + \dot{z}_0 t_1$. Since $\dot{z}_0$ and H_0 are of type g (see Eqs. (20)), Eq. (22) gives a third g-type constant of the motion.

6. Final comments

When applied to elementary problems, the elimination of time from the explicit solutions of the equations of motion can lead to interesting examples of constants of the motion, and thus it provides a natural approach to the subject. The method, despite its elementary character, allows to construct constants of the motion even for unusual circumstances, as shown in the examples.

When the constant G is a single-valued function of type g, it defines a surface on phase space on which the trajectory lies; the knowledge of its specific value amounts to the elimination of a degree of freedom of the motion. However, if the constant happens to be of type s, then *in addition* to G one needs to know some initial values (see, *e.g.*, Eqs. (4) or (6)) to define the specific trajectory of the motion and only with this extra information can the process of reduction be performed. We thus see the meaning of the distinction, since only g-type constants may lead to genuine integrals of motion in the conventional sense of being conserved quantities defined in the whole phase space and thus allowing by themselves a reduction of the number of degrees of freedom.

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Resumen. La construcción de constantes de movimiento para un sistema mecánico es una tarea difícil en general; sin embargo, para problemas elementales resulta posible obtener algunas de ellas mediante la eliminación directa del tiempo en la solución. Pese a que el método es de aplicación limitada, debido a que se basa en el conocimiento previo de la solución y en la posibilidad de invertir ésta para despejar la variable temporal, puede conducir a expresiones explícitas para integrales de movimiento aun en situaciones poco usuales —como en problemas con fuerzas dependientes del tiempo— y, por lo tanto, resulta de interés para cursos introductorios. Hay dos clases de constantes de movimiento: las que dependen de un solo punto genérico del espacio fase, y las que además son funciones de las condiciones iniciales; las primeras pueden ser integrales de movimiento que sirven para eliminar grados de libertad de la descripción.