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The Suppes-Zanotti theorem and the Bell inequalities

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Note on the paper *The Suppes-Zanotti theorem and the Bell inequalities*, **by T.A. Brody.**

Shortly after the tragic death of T.A Brody, the editors of Revista Mexicana de Física received a referee's comment on the accompanying paper, recommending its publication in RMF and asking the author to add some clarifying remarks to his text. Undoubtedly T.A Brody would have willingly taken into account the referee's valuable suggestions. Upon request of the director of RMF we have carefully read the manuscript and studied the reviewer's comments, and we concluded that all of them would contribute to improve the presentation, but wouldn't alter the fundamental ideas and conclusions of the paper. We have therefore recommended to leave the paper untouched instead of running the risk of falsifying Brody's view. We hope that the referee will agree with us, in view of the circumstances and the nature of his comments. This paper became unfortunately the last contribution of T.A Brody to the Revista Mexicana de Física.

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Abstract. A detailed proof is given for the Suppes-Zanotti theorem, which states that the existence of certain trivariate probability distribution is both a necessary and a sufficient condition for the validity of the Bell inequality. This condition is not satisfied in the usually considered experimental situations (correlations of spin projections from pairs of particles with total spin zero, or of pairs of cascade photons). It is shown that the three commonly adduced locality criteria bear no relation to this condition and are not even very plausible. Hence the rather extreme conclusions often drawn from locality considerations are not acceptable. Lastly, the Bell situation is formalised in a more natural way, in which the problem of a seeming contradiction with quantum theory cannot arise.

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1. Introduction

The violation of the Bell inequality [1,4] by both quantum theory and experiment has commonly been explained along the lines suggested by Bell in the cited paper: it is taken to mean that no local hidden-variable models can offer an adequate explanation of quantum mechanics [5,6]. From this it is concluded that in the debate between Einstein and Bohr [7,8] realism is the loser [9], or that the ensemble interpretation of quantum mechanics is untenable [10].

Such conclusions stem from Bell's original argument; in deriving the inequality he required that the spin projection of one particle issuing from the dissociation of a spin-0 pair could be measured in complete independence of what happens to the other particle —a stipulation that he called locality. If the resulting inequality is not satisfied by quantum systems, it is argued, they must be non-local.

This conclusion is not in fact valid, since it has been shown [11–17] that the Bell inequality holds between four dichotomic variables (*i.e.* variables that take only the values ± 1) if and only if any three possess a trivariate joint probability distribution (jpd), a condition which is trivially fulfilled if the four variables have a quadrivariate jpd. Thus locality or non-locality is irrelevant, for the existence or non-existence of a jpd depends on quite other considerations.

The present paper gives a general proof of the theorem that underlies this argument, and attempts to clarify all the steps. This appears to be necessary, partly because these details have not so far been brought together, partly because the argument, in spite of its simplicity, is often misunderstood [18] or ignored [19]. For the sake of completeness, the next section briefly discusses the three principal criteria of locality that have been proposed. Then section 3 exhibits the relation between the Bell inequality and the existence of a jpd, by giving a simple proof of the Suppes-Zanotti theorem; section 4 discusses the conditions under which a jpd can or cannot exist; in section 5 an experimentally more satisfactory formalism is given, in which a jpd for those correlations which are involved in the Bell inequality but which are not measurable cannot exist.

2. Non-locality

Intuitively, the concept of locality used by Bell [1] is of course sound: events that have a sufficiently great spatial separation should not, in principle, affect each other. If this sort of locality could be shown not to hold at all (or even not very often), the consequences for the whole of the scientific effort would be nothing short of disastrous, for it would no longer be possible to consider any system as sufficiently isolated for a manageably simple theoretical model to apply to it. What gives this notion its strong appeal is not only its agreement with common experience, but also —at least in classical physics— the fact that the known forces fall off with the inverse square of distance or even faster. Nevertheless, exceptions are of course documentable: gravitational effects are significant over extremely long distances, and we are close to the technical ability to generate intense and well focussed light beams that could provide causal links right across the entire universe. It is therefore necessary to consider formulations of the locality concept which do not admit such exceptions.

Locality (which we shall not distinguish from separability) is a clear-cut concept in the framework of special relativity, but is less easily formulated within non-relativistic theories. Therefore the locality criteria employed in connection with the Bell inequality will be evaluated by comparison with the relativistic concept. Here two points representing events are non-separable (or non-local with respect to each other) if their four-distance is timelike, so that the earlier one could — but need not— be causally linked to the other; if it is spacelike, on the other hand, there cannot exist any causal link between them, and their relation is termed local. They may nevertheless be correlated, through a common cause within the (necessarily existing) intersection of their past light cones. The concept of non-separability can immediately be extended to events that are represented by finite spatio-temporal regions; for systems considered to possess an indefinite life time it is less easy to formulate. The impossibility of a causal link between events with a space-like separation has no known exception, and therefore is a satisfactory basis for locality, which we shall define as the corresponding property of a theoretical model that satisfies the relativistic separability condition, or a suitable equivalent of it. Note that causal links can have a well-defined direction, since the identity of the forward and backward light cones are conserved separately under Lorentz transformations; this will be relevant below (section 3).

The measurements of a spin component along direction α, giving a result a, for one of a pair of particles and along direction β, with result b, for the other, form an event pair which can be local to each other if their separation is space-like. The relevance to the Bell case is less evident: not only is the theoretical description formulated in a non-relativistic framework, where no such restrictions on causal links exist, but the parts containing any space and time dependence of the wave functions are factored out since they are irrelevant to the expectation values for the spin correlations; it is not clear how the latter could be affected by a non-separability expressible only in the space-time coordinates they no longer contain. In fact the derivations of the Bell inequality that have so far been given imply that it should hold whether the relevant measurement events have a spacelike separation or not; this is not compatible with the relevance of a locality condition of the relativistic type.

The difficulty is not merely formal. The existence of a correlation between two events is not by itself evidence of non-separability, for such a correlation may be due to a common cause in their past. But if space and time variables cannot be used to establish a discrimination, then only conditions on the correlations or the quantities involved in their computation could appear as criteria of separability. Three forms for such a criterion have appeared in the literature:

(i) Bell's criterion [1]. Bell stipulates that a, the spin projection of the first particle, should depend only on α, the measurement angle, and λ, the hidden variable(s), and similarly for b:

$$a = A(\alpha, \lambda) \qquad b = B(\beta, \lambda). \tag{1}$$

But this criterion may be satisfied even when the measurement events are not space-wise separated, *e.g.* for the Bell inequality between the orbital and spin components in different directions of a single electron [20]; it may be violated even when the relativistic criterion holds, if the hidden variables λ connect a and b as in (1): whenever the correlation between a and b differs from zero, then, given b and β , we can determine λ and hence a from Eqs. (1), or at least find that the probabilities with which its two values appear are altered. It may be concluded that Bell's criterion, while it is presumably satisfied for the spin and cascade-photon cases, cannot exactly be termed a locality criterion; it is, rather, a criterion of functional independence. It is then not surprising to find that it is neither necessary nor sufficient for the validity of the Bell inequality, as is discussed below; and Bell's derivation of the inequalities requires a further condition, as we shall see below.

(ii) A related criterion is due to Stapp [21–23] and Eberhard [24,25] and has been used by Peres [26] and, somewhat differently, by Santos [27]; it may be called the counterfactual criterion, since it stipulates that the value of a would have been the same if instead of measuring b at an angle β we had done so at another angle β'. Of course, the weakness of this criterion [28,29] is that it is not susceptible to experimental verification. In general, a counterfactual argument is acceptable only if the counterfactual situation envisaged does not run counter to the theory; but in quantum mechanics, the specific details (the "hidden variables", if they can be defined) which yielded a particular value of a are not describable, and only the corresponding expectation values may be compared, theoretically as well as experimentally; thus the counterfactual criterion yields (in an obvious notation) the prediction

$$\langle A(\alpha,\beta)\rangle = \langle A(\alpha,\beta')\rangle.$$

In this statistical sense the criterion is compatible with quantum mechanics, and the two measurements of a and b are separable; but this is not, of course, sufficient to deduce the Bell inequality.

(iii) A third form of separability criterion is the factorisability criterion [30], that it should be possible to write the jpd of a and b in the form

$$p(a,b) = \int_\Lambda p_1(a|\lambda)p_2(b|\lambda)d\mu(\lambda) \tag{2}$$

where p_1 and p_2 are conditional probability densities and $d\mu$ is the probability density of λ. Here p can depend on α and β as parameters, p_1 on α, and p_2 on β. The condition (2) is unsatisfactory: it can be shown [31] that the existence of $p(a,b)$ is a sufficient condition for (2) to be always satisfied; the Bell inequality would then follow, provided that μ is the same for all instrument angles; and this remaining criterion, as will be seen below, is entirely equivalent to the criterion that a jpd should exist, —a criterion that has nothing to do with locality. The Clauser-Horne condition is also subject to the criticism made above, namely that it is unrelated to any space-time coordinates.

Thus none of the three criteria agree in their physical meaning with the relativistic criterion. It could be argued that quantum theory requires a differently formulated locality criterion; but even if this had been achieved, such a criterion would likewise be irrelevant, as will be shown in the next section. It could also be argued that since the particles' propagation can be represented by plane waves, which do not suffer inverse-square attenuation, we have here the exceptional case mentioned above of a parallel light beam, for which locality is not valid, or at least not valid for the distances realisable in the laboratory; but such a violation of locality does not conflict with any presently known physical law, and none of the extreme conclusions mentioned above would follow.

3. Locality is irrelevant

Bell's derivation [1] uses (1) to write the correlation between measurements when the angles are α and β as

$$\rho_{\alpha\beta} = \int_\Lambda A(\alpha,\lambda)B(\beta,\lambda)d\mu(\lambda).$$

Then, considering a further direction γ and its corresponding function $C(\gamma,\lambda)$, we have

$$\begin{aligned}\rho_{\alpha\beta} - \rho_{\alpha\gamma} &= \int_\Lambda A(\alpha\lambda)B(\beta,\lambda)d\mu(\lambda) - \int_\Lambda A(\alpha,\lambda)C(\gamma,\lambda)d\mu(\lambda) \\ &= \int_\Lambda A(\alpha\lambda)B(\beta,\lambda)\left[1 - B(\beta,\lambda)C(\gamma,\lambda)\right]d\mu(\lambda),\end{aligned}$$

where we have used the fact that $B^2(\beta,\lambda) = 1$, and so obtain the trivariate Bell inequality

$$|\rho_{\alpha\beta} - \rho_{\alpha\gamma}| \le 1 \pm \rho_{\beta\gamma}. \qquad (3)$$

In a similar way, using only the functions A and B, but each with two directions, α, α' and β, β' [32], we find the quadrivariate Bell inequality

$$\begin{aligned}|\rho_{\alpha\beta} - \rho_{\alpha\beta'}| + |\rho_{\alpha'\beta} + \rho_{\alpha'\beta'}| &= \int_\Lambda \left[|A(\alpha,\lambda)\left\{B(\beta,\lambda) - B(\beta',\lambda)\right\}|\right. \\ &\quad \left. + |A(\alpha',\lambda)\left\{B(\beta,\lambda) + B(\beta'\lambda)\right\}|\right] d\mu(\lambda) \le 2. \qquad (4)\end{aligned}$$

The inequality in (4) follows because, of $(B(\beta,\lambda) - B(\beta',\lambda))$ and $(B(\beta,\lambda) + B(\beta',\lambda))$, one is necessarily 0 and the other ± 2, while $A(\alpha,\lambda)$ and $A(\alpha',\lambda)$ at most effect sign changes. Also, $\mu(\lambda)$ is a normalised distribution function.

Many other forms of the Bell inequality are known [33–41] but the arguments presented here can be adapted to them without difficulty, and they will not be further discussed. As is well known [4], the inequalities (3) and (4) are violated for a wide range of angles of measurement, both theoretically and experimentally, a situation which has given rise to the discussions alluded to above.

The "locality" assumption made by Bell is needed in the argument leading to (3) or (4) in order to enable the second members to be written as single integrals, with common factors taken outside the parentheses in the integrands. But a further assumption is required, since one of these common factors is $\mu(\lambda)$; to make it a common factor we require the second part of the Clauser-Horne locality criterion, namely that the distribution function $\mu(\lambda)$ be the same in all the applications of (2); if this is the case, the quadrivariate distribution $p(a, a', b, b')$ for the set of possible outcomes exists, for in terms of the functions A and B of Eq. (1) it may be written as

$$p(a, a', b, b') =$$

$$\frac{1}{16} \int_{\Lambda} [1 + aA(\alpha, \lambda)] [1 + a'A(\alpha', \lambda)] [1 + bB(\beta, \lambda)] [1 + b'B(\beta', \lambda)] \, d\mu(\lambda) \qquad (5)$$

It is trivial to show that (5) satisfies all the conditions of a probability density. Inversely, the existence of the distribution (5) ensures that $\mu(\lambda)$ is the same in the four applications of (2), which all correspond to marginals derived from (5). Thus the Clauser-Horne locality condition is equivalent to the existence of the jpd (5), and should not be interpreted as a locality condition. Nor are locality conditions needed in the other derivations of the inequality, as has been discussed in detail elsewhere [42,43,16]. For instance, Wigner [44] considers (adapting somewhat his notation) probabilities such as $p(+ + --)$, the probability that $a = a' = +1$, $b = b' = -1$; the four correlations on the left of (4) can then be written as sums of the form

$$\rho_{\alpha\beta} = \sum_{a,a',b,b'} abp(a, a', b, b') \qquad (6)$$

and so on, and the inequalities then follow by essentially the same argument as led to (3) and (4). But now no locality assumption has been made, either explicitly or implicitly; indeed, in the marginal distribution of a and b we might have the "non-local" case

$$q(a, +) = \sum_{a',b'} p(a, a', +, b') \neq q(a, -) = \sum_{a',b'} p(a, a', -, b'),$$

yet (6) remains a valid definition and the derivation of (3) and (4) goes through as before; the only necessary assumption is that the jpd (5) should exist. Thus the Bell inequality may be derived without assuming either locality or non-locality (or, for that matter, the existence of hidden variables). Nothing but the sixteen elementary probabilities have been assumed.

This is no more than a particular case of the general conclusion to be derived from two theorems to be established here:

Theorem I

This theorem was given, with slight restrictions, by Suppes and Zanotti [11]. Three random variates x, y, z, which are dichotomic will satisfy the trivariate Bell inequality of type (3),

$$\langle xy \rangle + \langle xz \rangle - \langle yz \rangle \leq 1 \tag{7}$$

and its cyclic permutations [1], together with the condition

$$-1 \leq \langle xy \rangle + \langle xz \rangle + \langle yx \rangle \tag{8}$$

if and only if the joint probability distribution $p(x, y, z)$ exists.

Here $\langle xy \rangle$ etc. are the covariances, *i.e.* the expectation values of the corresponding products. If the means of the three variates are zero, the covariances will be equal to the correlation coefficients.

We begin by defining compatibility among probability distributions. A set of joint probability distributions in subsets of a given set of variates are said to be (pairwise) compatible if the marginal distributions of the maximal common subset of variates for any pair of distributions coincide. Thus the bivariate distributions of x and y and of x and z are compatible if the two marginals for x coincide.

For the case of three dichotomic variates, compatibility is immediately seen to require, firstly, that only 7 of the 12 probabilities in the three distributions are independent; calling these p_i, $i = 1 \ldots 7$ and using an obvious notation for writing all 12, we find the following table:

$$\begin{array}{lll}
(++\,.) = p_1 & (+.+) = p_5 & (.++) = p_7 \\
(+-\,.) = p_2 & (+.-) = p_1 + p_2 - p_5 & (.+-) = p_1 + p_3 - p_7 \\
(-+\,.) = p_3 & (-.+) = p_6 & (.-+) = p_5 + p_6 - p_7 \\
(--\,.) = p_4 & (-.-) = p_3 + p_4 - p_6 & (.--) = p_2 + p_4 - p_5 - p_6 + p_7.
\end{array} \tag{9}$$

The second requirement for compatibility is of course that the 5 probabilities defined in terms of p_1 to p_7 in this table be non-negative; this yields the inequalities

$$\begin{array}{ll}
p_1 + p_2 \leq p_5 & p_1 + p_3 \leq p_7 \\
p_3 + p_4 \leq p_6 & p_5 + p_6 \leq p_7
\end{array} \tag{10}$$

$$p_2 + p_4 + p_7 \leq p_5 + p_6$$

It is not necessary to require normalisation of the second and third distributions; this follows automatically.

If now these three compatible bivariate distributions are to be the marginals of a trivariate distribution $p(x, y, z)$, then each of the 12 probabilities is the sum of two trivariate probabilities; thus,

$$(++.) = (+++) + (++-)$$

and so on. Since there are only 7 independent quantities, one of the trivariate probabilities, say $\alpha = (+++)$, is not determined; if the others are expressed in terms of α, there are 8 conditions to be simultaneously satisfied if all trivariate probabilities are to be non-negative (their normalisation is automatic). These conditions may be resumed as

$$\begin{aligned} m \equiv \max(0, p_5 - p_2, p_7 - p_3, p_7 - p_6) \\ \leq \min(p_1, p_5, p_7, p_4 - p_6 + p_7) \equiv M, \end{aligned} \tag{11a}$$

$$m \leq \alpha \leq M. \tag{11b}$$

Eq. (11a) is equivalent to 16 independent inequalities. Of these 7 are trivial, in the sense that they are valid if for each a certain p_i is non-negative; five others are the inequalities of the compatibility conditions for the three bivariate distributions. The remaining four inequalities are

$$p_5 \leq p_2 + p_7, \tag{12}$$

$$p_7 \leq p_3 + p_5, \tag{13}$$

$$p_7 \leq p_1 + p_6, \tag{14}$$

$$p_6 \leq p_4 + p_7. \tag{15}$$

Now the three covariances in Eq. (7) are, in terms of the p_i,

$$\begin{aligned} \langle xy \rangle &= p_1 - p_2 - p_3 + p_4, \\ \langle xz \rangle &= -p_1 - p_2 + p_3 + p_4 + 2p_5 - 2p_6, \\ \langle yz \rangle &= -p_1 + p_2 - p_3 + p_4 - 2p_5 - 2p_6 + 4p_7, \end{aligned} \tag{16}$$

so that the four inequalities (12) to (15) are precisely equivalent to the four inequalities

$$\langle xy \rangle + \langle xz \rangle - \langle yz \rangle \leq 1, \tag{17}$$

$$\langle xy \rangle - \langle xz \rangle + \langle yz \rangle \leq 1, \tag{18}$$

$$-\langle xy\rangle + \langle xz\rangle + \langle yz\rangle \leq 1, \tag{19}$$

$$-\langle xy\rangle - \langle xz\rangle - \langle yz\rangle \leq 1. \tag{20}$$

Inequalities (17) to (19) are the three possible forms of the trivariate Bell inequality, Eq. (7), and (20) is the additional condition (8). This establishes sufficiency. To prove necessity, we observe that if $p(x, y, z)$ exists, then the three marginals $p'(x, y)$, $p''(x, z)$, and $p'''(y, z)$, exist and are compatible; and if eqs. (7) and (8) are rewritten in terms of the eight components of $p(x, y, z)$, they will be seen to be trivially satisfied. Thus these equations are both necessary and sufficient for the existence of the trivariate jpd.

Suppes and Zanotti [11] write the four inequalities (17) to (20) in the equivalent and more compact form

$$-1 \leq \langle xy\rangle + \langle xz\rangle + \langle yz\rangle \leq 1 + 2\min(\langle xy\rangle, \langle xz\rangle, \langle yz\rangle). \tag{21}$$

We now establish a result needed below, in the form of the following

Lemma

Given two compatible bivariate jpd's for the dichotomic variates x, y and x, z, respectively, a jpd for y and z compatible with them always exists.

The given jpd's satisfy the first two inequalities of (10). The assertion of the lemma is then equivalent to the statement that there always exists a non-empty range of possible values for p_x, such that the last three inequalities of (10) are satisfied. Combining the last two of them, p_z must satisfy

$$p_5 + p_6 \geq p_7 \geq p_5 + p_6 - p_2 - p_4.$$

But this is incompatible with the remaining inequality only if

$$p_5 + p_6 - p_2 - p_4 > p_1 + p_3,$$

that is to say if

$$p_5 + p_6 > p_1 + p_2 + p_3 + p_4 = 1$$

and this is excluded by the normalisation of the jpd of x and z.

Theorem II

This extends the result of Theorem I to the quadrivariate type of Bell inequalities of Eq. (4). Four dichotomic random variables x, y, z, u will satisfy the inequality

$$|\langle xz\rangle - \langle yz\rangle| + |\langle xu\rangle + \langle yu\rangle| \leq 2 \tag{22}$$

(Clauser *et al.* [45]) if and only if the jpd's $p'(x,y,z)$ and $p''(x,y,u)$ exist and are compatible.

To prove sufficiency, we assume p' and p'' to exist and be compatible. Then, by applying (17) to the triple $\{xyz\}$ and (19) to $\{xyu\}$, we find

$$\langle xz\rangle - \langle yz\rangle + \langle xu\rangle + \langle yu\rangle \leq 2, \tag{23}$$

and three similar uses of Eqs. (17) to (20) combine with (23) to complete the derivation of (22). To prove necessity, we observe that (23) may be divided into two inequalities

$$\begin{aligned} \langle xz\rangle - \langle yz\rangle + c &\leq 1, \\ \langle xu\rangle - \langle yz\rangle - c &\leq 1, \end{aligned} \tag{24}$$

where c must evidently satisfy $|c| \leq 1$. But the lemma given above establishes, from each of these two inequalities, the existence of the jpd of x and y, with a covariance $\langle xy\rangle$ which satisfies them and is therefore a possible value of c. Analogous reasoning starting with the other three inequalities resumed in (22) provides a further six inequalities like (24). Four of this total of eight are the conditions (17) to (20) for the existence of the trivariate jpd of x, y and z, the other four for that of the jpd of x, y and u. By construction these jpd's have the same marginal for x and y and so are compatible, as required.

In Theorem II there occurs only a single quadrivariate Bell inequality; but since in (22) any of the four covariances may carry the minus sign, there are another three. (These are the four physically meaningful inequalities; the others would involve the two "forbidden" correlations $\langle aa'\rangle$ and $\langle bb'\rangle$). By applying the theorem to all four, we immediately obtain the following

Corollary

Between four dichotomic random variables x, y, z and u all four possible quadrivariate Bell inequalities not involving either $\langle xy\rangle$ or $\langle zu\rangle$ hold if and only if the four trivariate jpd's $p'(x,y,z)$, $p''(x,y,u)$, $p'''(x,z,u)$, and $p^{iv}(y,z,u)$, exist and are pairwise compatible.

It should be noted that if the conditions of this corollary are satisfied, then the Bell inequalities involving the two "forbidden" correlations will also be satisfied. Furthermore, the corollary does not allow us to conclude that the quadrivariate jpd $q(x,y,z,u)$, exists when we know all the Bell inequalities to hold. The argument leading to Theorem I can be repeated to show that the existence of a quadrivariate jpd, given four trivariate jpd's, requires a further 32 conditions beyond the compatibility conditions; these, combined in suitable ways, yield new Bell-type inequalities, involving now the expectations of the product of three variables, such as

$$-1 \leq \langle xy \rangle + \langle xzu \rangle + \langle yzu \rangle.$$

It is easy to find counterexamples where the four trivariate jpd's are compatible, so that the Bell inequalities are satisfied, but the quadrivariate distribution does not exist.

But if it does exist, then the four trivariate jpd's exist and are compatible. If a trivariate Bell inequality is violated, then Theorem I shows that the jpd of the three variates cannot exist; if a quadrivariate Bell inequality is violated, then the corresponding two trivariate jpd's do not exist; in either case, the quadrivariate jpd cannot exist. Conversely, if the quadrivariate jpd exists, both types of Bell inequality must hold. Of course the non-existence of the relevant jpd is a necessary but not a sufficient condition for the Bell inequality to be violated; sufficient conditions for this are not yet known.

The conclusion is clear: for the Bell inequality to hold, no locality condition is either necessary or sufficient. The only relevant criterion is the existence of the appropriate jpd's. If quantum theory (and the corresponding experiments) violate the Bell inequality, then neither the jpd of a, a' and b nor that of a, a' and b' exists, and so none of higher-order distributions can exist. But this non-existence is not surprising; indeed, quantum theory makes the joint occurrence of a and a' impossible, and similarly for b and b'. Any theory that postulates the existence of, say, $p'(a, a', b)$ conflicts with quantum mechanics; for if such a trivariate distribution existed, it would predict values for expectations like $\langle aa'b \rangle$; but for these, not only does the theory provide no prediction, they are evidently inaccessible in any conceivable experiment. The Clauser-Horne criterion implies the existence of a quadrivariate jpd, Eq. (5), as noted above; but this in its turn implies the existence also of $\langle aa'bb' \rangle$. From these contradictions with what theory yields and experiment confirms the non-existence of these jpd's is clear.

Nor is this failure of the jpd to exist due to any kind of non-locality; a and a' are mutually exclusive alternatives for measurements on the same particle, not on two particles with a possibly spacelike separation, and the non-existence of $p(a, a')$ is unrelated to locality.

That a non-local explanation of the violation of the Bell inequality is untenable is made even more obvious by the fact that what clearly are local situations —in the sense of Bell— may nevertheless violate the inequality [20], while (as mentioned above) non-local situations can satisfy the inequality. A non-local model, because of its potential conflict with special relativity, can even give rise to striking paradoxes [45].

In terms of the Clauser-Horne locality criterion, the non-existence of a jpd for the Bell case implies that the distribution function of the hidden variables, $\mu(\lambda)$ must depend on the measurement angles α and β. Such a dependence seems first to have been postulated by Lochak [46]; that it is physically justified is immediately obvious since the relevant values of the hidden variables μ are those they possess at the time of measurement —when they have been modified by the interaction with whatever inhomogeneous magnetic field splits trajectories according to spin. These modifications cannot be predicted from their starting values, which lie in their backward light cones, as noted above in section 2. An analogous argument holds for the photon polarisation measurements.

More in general, to postulate, as is often done, that $\mu(\lambda)$ cannot depend on the measurement angles is to assume that the hidden variables are not dynamical variables. That such a dependence is also enough to reproduce the quantum predictions has been shown by Cetto [47].

In order to render uncontroversial the violation of the inequality by local models, a recent paper [48] introduces an explicit time dependence; on the basis of the discussion in section 2, this is a plausible notion. But other types of local models that do not satisfy the Bell inequality have also been constructed [49–54,16].

The conclusion that the Bell inequality is irrelevant to the locality problem in quantum mechanics has been drawn also by de Muynck and Abu Zaid [14], by De Baere [4] in his exhaustive review and, from a somewhat different point of view, by Lehr [55]. It has also been attacked [18,56], curiously enough, on the basis of the same logical error in both papers quoted: the authors conclude that, because the Bell inequality can in fact be deduced using a locality condition, its violation implies that the locality condition must be false. For this conclusion to hold it would have to be shown that the Bell inequality cannot be deduced in any other way, *i.e.* that the locality requirement is actually necessary. From the above discussion it should be clear that it is not in fact either necessary or sufficient; on the other hand, the assumption that all subsets of three of the four variables have compatible trivariate jpd's is both necessary and sufficient.

The relation between hidden variables, local or non-local, and the existence of a jpd is also often confused. Thus Fine [12,13] concludes that, in his terminology, a "deterministic hidden-variable model" (composed of a hidden-variable space Λ, a probability function $\mu(\lambda)$ over it, and response functions $A(\alpha, \lambda)$, etc.) is equivalent to the existence of a jpd. Now his particular model guarantees the coexistence of a, a', b and b', simply because it postulates a single $\mu(\lambda)$, with no angular dependence; as we saw above, this is almost trivially equivalent to the existence of the corresponding quadrivariate jpd. But this type of argument ignores the possibility of a hidden-variable model where such a jpd does not exist, which is the case whenever the hidden variables that determine *e.g.* a do not possess a jpd together with those that determine a'; then $\mu(\lambda)$ will contain further parameters, in the Bell case the measurement angles. Such models can be relevant also in classical physics. For instance, in statistical mechanics most ensembles depend on a number of parameters; the distributions corresponding to different values of such parameters are not compatible and so do not give rise to joint distributions.

A related point made by De Baere [28,29] is all too often ignored: random hidden variables must necessarily be irreproducible. This implies that successive measurements whose values depend on such hidden variables do not possess a jpd, and hence cannot satisfy Bell inequalities. But one cannot conclude that these hidden variables can never be made "visible"; however, when we do so, the physical character of the system studied changes, which is why now jpd's for the formerly hidden variables could exist.

4. When do joint probability distributions exist?

A set S of random variables possesses a jpd if and only if together they define a measurable state space. The mathematical content of this statement is trivial. It is physically relevant as soon as the measurable space defined by the variables in S forms the state space for a physical system in a valid theoretical model. If this is the case, then the existence of the jpd implies that the variables are jointly defined and jointly measurable; for if they were not, it would be impossible to determine for instance the various correlations implied by the jpd, and whose value the model predicts; hence the model would require modification. Inversely, if these correlations can be measured, a jpd correctly predicting their values must exist within the model if it is to be theoretically satisfactory. Thus joint measurability of the variables ("joint" here meaning that the determination of one variable does not interfere with the determination of another or alter the value found for it, and not necessarily simultaneous measurement) is the experimental equivalent of the existence of a jpd. This does not necessarily mean that in a given experiment a joint measurement is carried out, but only that such a measurement is feasible, at least in principle. Where it can be shown that —as in the case of two different spin projections for one particle— such a joint measurement is not feasible, the theory should predict the non-existence of a jpd. This is the case for quantum mechanics, since the operator corresponding to the probability of Eq. (5) is not Hermitian.

To avoid certain misunderstandings, it should be noted that the existence of a jpd for the set S does not imply that these variables are correlated or possess some statistical dependence. On the contrary, the concept of statistical independence is defined only for sets of variables that do possess a jpd; hence random variables that have no physical connection with each other and should therefore be statistically independent will (at least in the non-relativistic approximation) possess a jpd within a theoretical model which includes them, provided an event structure can be defined where one realisation of each variable is associated in a physically meaningful way with an event (time is commonly adequate for this purpose); they must therefore satisfy the Bell inequality.

In contrast, random variables that do not possess a jpd are rather strongly linked to each other, but in a way that cannot be characterised by means of statistical parameters. Their case is analogous to that of mutually exclusive events though more extreme: not just subsets within a common value range exclude each other, but the entire value ranges do so. There is also an obvious connection between the two cases: if a random variable x is conditioned on another random variable y, then the mutually exclusive events that $y = 1$ and that $y = 2$, say, generate two variates which may be written

$$(x|y = 1) \quad \text{and} \quad (x|y = 2) \tag{25}$$

in an obvious notation; these do not possess a jpd. From (25) it follows that given a trivariate jpd $p(x, y, z)$ we may form four correlation coefficients

$$\rho(x, y|z = z_i), \qquad i = 1 \ldots 4 \qquad (z_i \neq z_j, i \neq j)$$

from the corresponding four conditional bivariate distributions, and find that they may violate the Bell inequality. We show in the next section that the Bell inequality may be reformulated in precisely these terms.

The non-existence of a jpd is by no means associated with quantum physics alone, as the example given above should make clear. A case more relevant here is the "mechanism" underlying the classical models (referred to in the preceding section) that do not satisfy the Bell inequality.

To conclude this section, we note that quite in general the existence and compatibility of all n marginals of $n-1$ variates does not imply the existence of the jpd of all n variates. If the n marginals p_{n-1}, are compatible, they contain $2^n - 1$ independent probabilities; hence one of the 2^n that make up p_n can be arbitrarily chosen, provided that then all the others are non-negative. This implies, much as in section 3, a set of 2^{2n-2} inequalities, of which $2^n - 1$ are trivial, $(n-2)2^{n-1}$ are the compatibility conditions for the p_{n-1}, and others are the conditions for all the p_{n-1} to exist; there remain

$$n! \sum_{k=1}^{n} (-1)^{n-k} \frac{1}{k!} 2^{k-1} (2^{k-1} - k)$$

new inequalities, which must be satisfied for the jpd p_n to exist. These inequalities involve expectation values for products of $n-1$ variates; hence only for $n = 3$ do they yield Bell inequalities. In general, we cannot assume for any $n > 2$ that p_n exists merely because the marginals p_{n-1} exist, contrary to what is often supposed. In the Bell situation, the bivariate distributions for the four pairs (a, b), (a, b'), (a', b), and (a', b') exist, because the corresponding correlation coefficients can be determined experimentally; if the trivariate distributions existed, the Bell inequalities, as we have seen, would always be satisfied; but even in the latter case we cannot assume the existence of the quadrivariate distribution.

5. The physical description of the Bell situation

In the Bell-type experiments, as the discussion above has shown, the Suppes-Zanotti theorem and its extensions imply that the commonly used variables do not possess a jpd. The discussion of the preceding section implies that this case is not at all exceptional; it may nevertheless be made much more comprehensible by the following two arguments:

(a) The variables normally used to describe the Bell problem may be considered to be somewhat misleading, in that a and a' bear two related but distinct pieces

of information: the angle of measurement (α or α') is indicated by the choice of variable, while the outcome of the measurement is indicated by the value of the variable. Common laboratory practice requires that these elements be separated, since one is determined by the experimenter at the time of initiating the measurement procedure, while the other depends on the measured system and its environment, and not on the experimenter. We therefore consider, instead of the pair a and a', a pair m_A and θ_A. Here $\theta_A = \alpha$ or α', according to what measurement will be done, while $m_A = \pm 1$, according to the outcome. We write, similarly, m_B and θ_B instead of b and b'. The four new variables contain exactly the same information as the original four, and the two sets are interconvertible. But now the four correlation coefficients required for the Bell inequality (4) appear as

$$\rho_{\alpha\beta} = \rho(m_A, m_B | \theta_A = \alpha, \theta_B = \beta) \tag{26}$$

and so on; they are obtained from conditional probability distributions such as $P(m_A, m_B | \theta_A = \alpha, \theta_B = \beta)$, derived from the jpd of m_A, m_B, θ_A, and θ_B, rather than from the marginal distributions of the (non-existent) jpd of a, a', b and b'. There is then no reason to suppose them to satisfy any Bell inequality, for now the situation is analogous to that of the previous section.

Of course the jpd $p(m_A, m_B, \theta_A, \theta_B)$ exists, and the correlation coefficients of any four of its six marginal distributions, $\rho(m_A, m_B)$ etc., satisfy the Bell inequality. This is obvious, since

$$\rho(m_i, \theta_j) = 0, \qquad i, j = A, B$$

while $\rho(\theta_A, \theta_B)$ is determined by the methodology of the experiment. Only $\rho(m_A, m_B)$ carries any information derived from the two-particle system, and by itself cannot form a Bell inequality.

This argument has physical content: the spin projection of a particle does not have a value in the absence of a corresponding angle; it is also undefined if more than one angle is specified, because the inhomogeneous magnetic field (or equivalent set-up used to measure it) can have only one orientation at a given instant. (A point commonly overlooked is that a Stern-Gerlach magnet or similar arrangement serves only to measure the spin projection of neutral particles; however, the present argument applies also to the measurement of spins of charged particles, and a similar one applies to the measurement of photon polarisations.) This is a consequence of the fact that the measurement of spin projections is really a preparation procedure that has been adapted to the purposes of a measurement. A dependence of the outcome on an instrumental parameter such as the angular orientation is therefore necessary and indeed unsurprising.

This rather obvious point is sometimes presented as evidence for what is called a "contextual" point of view, and then interpreted as characteristic of the Copenhagen interpretation of quantum mechanics [27]. Here, however, a confusion has crept in. Contextualism —which does have a close relation to the Copenhagen school, though it should not be identified with it— attributes a measuring-device dependence to the outcome of all measurements. It is the opposite of objectivism, *i.e.* the view that a measurement outcome must always be interpreted as an attribute of the measured system (the "object" in objectivism). Both views are unacceptable oversimplifications of the real situation, for it is evident that in fact outcomes are in many cases correctly identified as being essentially due to the system under study, in other cases as due to both the system and the measuring apparatus, in still others as due only to the measurement device, and finally in some cases are to be attributed to still other parts of the world (including erroneous operation of the equipment). It is part of the experimenter's job to carry out a proper separation of these cases and to correct for undesired "outside" influences; but he can do this only insofar as he has been provided with an adequate theoretical picture, and when it is precisely this adequacy which is in question, then it is important to recognise and apply correctly these distinctions.

(b) An alternative way of seeing this problem takes the hidden-variable formalism seriously and considers them to be dynamical variables with a corresponding time dependence. If we take the break-up of the original spin-0 system to occur at time 0 and the two measurements to be made at times t_α and t_β (which need not be equal), then the hidden-variable set must be broken up into $\lambda_\alpha(t_\alpha)$ and $\lambda_\beta(t_\beta)$, and the correlation between the measurements becomes

$$\rho_{\alpha\beta} = \int_{\Lambda_\alpha}\int_{\Lambda_\beta} A(\alpha, \lambda_\alpha(t_\alpha))B(\beta, \lambda_\beta(t_\beta))d\mu_\alpha(\lambda_\alpha)d\mu_\beta(\lambda_\beta) \tag{27}$$

and similarly

$$\rho_{\alpha\beta'} = \int_{\Lambda_\alpha}\int_{\Lambda'_\beta} A(\alpha, \lambda_\alpha(t_\alpha))B(\beta', \lambda_{\beta'}(t_{\beta'}))d\mu_\alpha(\lambda_\alpha)d\mu_{\beta'}(\lambda_{\beta'}). \tag{27}$$

It is evident that from equations of the type of (27) and (28) no Bell inequality can be deduced. This is so even if $\Lambda_\beta = \Lambda_{\beta'}$ and $\mu_\beta = \mu_{\beta'}$; a Bell inequality can be derived only if we also necessarily have $t_\beta = t_{\beta'}$, *i.e.* if the measurements are carried out jointly, so that a jpd exists.

The last point of *(a)* above now implies that λ_α must be written as $\lambda(t_\alpha, \alpha)$, and so on; in this case not even $t_\alpha = t_{\alpha'}$ can guarantee the existence of a jpd and so permit the derivation of a Bell inequality. This argument is closely related to the irreproducibility of hidden variables commented on above.

In conclusion it may be said that the locality criteria commonly employed are neither necessary nor sufficient to establish the Bell inequality, which depends only on whether or not a joint probability distribution exists. In classical and in quantum physics situations are found where no such distribution exists; this occurs when two (or more) quantities have been so defined theoretically that they cannot coexist; and as dicussed above for the Bell case, it may usually be avoided by using quantities more suitably defined. The relevance of the Bell inequality to problems of understanding quantum physics is thus doubtful, and the large conclusions that, as mentioned in the introduction, are often drawn from supposed violations of locality cannot be maintained.

References

1. J.S.Bell, *Physics* **1** (1964) 195.
2. M. Bruno, M. d'Agostino and C. Maroni, *Nuovo Cim.* **40B** (1977) 143.
3. J.F.Clauser and A. Shimony, *Reps. Prog. Phys.* **41** (1978) 1881.
4. W. De Baere, *Adv. Electronics and Electron Physics* **68** (1986) 245.
5. F. Rohrlich, *Science* **221** (1983) 1251.
6. E. Fry, *Phys. Today* **37(1)** (1984) S26.
7. A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* **47** (1935) 777.
8. N. Bohr, *Phys. Rev.* **48** (1935) 696.
9. B. d'Espagnat, *Phys. Reports* **110** (1984) 201.
10. D.T.Gillespie, *Am. J. Phys.* **54** (1986) 889.
11. P. Suppes and M. Zanotti, *Synthese* **48** (1981) 191.
12. A. Fine, *Phys. Rev. Lett.* **48** (1982) 291.
13. A. Fine, *J. Math. Phys.* **23** (1982) 1306.
14. W.M.de Muynck and O. Abu-Zeid, *Phys. Lett.* **100A** (1984) 485.
15. W.M. de Muynck, *Phys. Lett.* **114A** (1986) 65.
16. T.A.Brody (1986), preprint.
17. P. Rastall, *Phys. Lett.* **86A** (1982) 85.
18. D. Dieks, *Phys. Lett.* **117A** (1986) 433.
19. A. Rae, *New Scientist* **112:1536** (1986) 36.
20. D. Home and S. Sengupta, *Phys. Lett.* **102A** (1984) 159.
21. H.P.Stapp, *Nuovo Cim.* **40B** (1977) 171.
22. H.P.Stapp, *Am. J. Phys.* **53** (1985) 306.
23. H.P.Stapp, LBL preprint 20094 (1985).
24. P. Eberhard, *Nuovo Cim.* **38B** (1977) 75.
25. P. Eberhard, *Nuovo Cim.* **46B** (1978) 392.
26. A. Peres, *Am. J. Phys.* **46** (1982) 1470.
27. E. Santos, Universidad de Santander, preprint (1985).
28. W. De Baere, *Lett. Nuovo Cim.* **39** (1984) 234.
29. W. De Baere, *Lett. Nuovo Cim.* **40** (1984) 488.
30. J.F.Clauser and M.A.Horne, *Phys. Rev.* **D10** (1974) 526.
31. T.A.Brody and A.K.Theophilou, preprint (1987).
32. J.F.Clauser, M.A.Horne, A. Shimony and R.A.Holt, *Phys. Rev. Lett.* **23** (1969) 880.
33. P.M.Pearle, *Phys. Rev.* **D2** (1970) 1418.
34. N. Herbert and J. Karush, *Found. Phys.* **8** (1978) 313.
35. A. Garuccio, *Lett. Nuovo Cim.* **23** (1978) 559.
36. A. Garuccio and F. Selleri, *Found. Phys.* **10** (1980) 209.
37. M. Froissart, *Nuovo Cim.* **64B** (1981) 241.
38. S. Bergia and F. Cannata, *Found. Phys.* **12** (1982) 843.
39. N.D.Mermin and G.M.Schwarz, *Found. Phys.* **12** (1982) 101.
40. A. Garg and N.D.Mermin, *Phys. Rev. Lett.* **49** (1982) 901.
41. A. Garg and N.D.Mermin, *Found. Phys.* **14** (1985) 1.
42. T.A.Brody and L. de la Peña, *Nuovo Cim.* **54B** (1979) 455.
43. T.A.Brody, in E.I. Bitsakis and N. Tambakis (eds.), *Determinism in Physics*, Gutenberg, Athens (1985) 243.
44. E.P.Wigner, *Am. J. Phys.* **38** (1970) 1005.
45. R.I.Sutherland, *Nuovo Cim.* **88B** (1985) 114.

46. G. Lochak, *Found. Phys.* **6** (1976) 173.
47. A.M.Cetto, IFUNAM preprint (1987).
48. S. Pascazio, *Phys. Lett.* **118A** (1986) 47.
49. G.C.Scalera, *Lett. Nuovo Cim.* **38** (1983) 16.
50. G.C.Scalera, *Lett. Nuovo Cim.* **40** (1984) 353.
51. S. Notarrigo, *Nuovo Cim.* **83B** (1984) 173.
52. A.O.Barut and P. Meystre, *Phys. Rev. Lett.* **53** (1984) 1021.
53. A.O.Barut and P. Meystre, *Acta Phys. Aust.* **56** (1984) 13.
54. H.P.Seipp, *Found. Phys.* **16** (1986) 1143.
55. W.J.Lehr, *Nuovo Cim.* **90B** (1985) 185.
56. L.C.B.Ryff, *Phys. Lett.* **118A** (1986) 3.

Resumen. Se da una demostración alternativa y más general de que la existencia de las cuatro distribuciones de probabilidad trivariadas posibles entre las cuatro variables dicotómicas involucradas es condición necesaria y suficiente para que sean válidas las desigualdades de Bell. En las situaciones experimentales habitualmente contempladas (correlaciones de proyecciones de espines en pares de partículas con espin total cero o pares de fotones en cascada) esta condición no se cumple. Se muestra que los tres criterios de localidad más citados no guardan relación alguna con esta condición y ni siquiera son muy plausibles, de modo que las conclusiones extremosas que se suelen derivar de su aparente violación no se pueden mantener. Finalmente se propone una formalización más natural para los experimentos tipo Bell, tal que no pueda surgir lo que parece ser una contradicción con la mecánica cuántica.