On the statistical behavior of the orbits elements of the logistic equation $4x(1-x)$

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(Recibido el 19 de mayo de 1988; aceptado el 16 de diciembre de 1988)

Abstract. Ulam and von Neumann [1] said that starting with almost every x in the interval [0,1] and iterating the function $Tx = 4x(1-x)$ one obtains a sequence of "random" numbers in that interval with probability distribution arcsin. In this paper it is given *a* demonstration that, in effect, the probability measure arcsin, $\mu(dx) = dx/\pi\sqrt{x(1-x)}$ is *the* probability law that describes the statistical behavior of the numbers in each orbit $O_T(x) = \{x, Tx, T^2x, \ldots\}$ for almost every x in [0,1].

PACS: 02.90.+p; 02.50.+s

We have two contexts and one problem:

First, Ulam and von Neumann [1] said "... Various distributions of such numbers ["random " numbers with a given distribution] can, however, be obtained by deterministic process. For example, starting with almost every x, (in the sense of Lebesgue measure) and *iterating* the function $Tx = 4x(1-x)$ one obtains a sequence of numbers on (0,1) with a computable algebraic distribution..."

Second, in the chaotic regime it has been found the necessity of studying the asymptotic behavior of $T^n x$ when $n \to \infty$; Kai and Tomita [2], Collet and Eckmann [3] among others have found experimentally, making histograms of several sequences of results $x, Tx, \ldots, T^{n-1}x$ for large n that the probabilistic or statistical behavior of the orbits elements $x, Tx, T^2x, \ldots$ is regulated by the probability measure arcsin,

$$\mu(dx) = \frac{dx}{\pi\sqrt{x(1-x)}}.$$

The problem is if this μ is really the probability law that describes that statistical behavior; *i.e*, if it is true that for almost every x

$$\frac{1}{n}\sum_{i=0}^{n-1} I_B(T^i x) \underset{n\to\infty}{\longrightarrow} \mu(B)$$

for every Borel set B of the interval [0,1].

It has been shown [8,9] that this probability measure is invariant under T and that T is ergodic respect to μ, however these characteristics of a probability measure are only necessary but not sufficient to assure this statistical behavior. In fact, there are infinitely many probability measures of this kind. My interest in this paper is to give *a* rigorous proof, using results of Refs. [4,5,6], that in effect, μ is *the* unique positive answer for the question raised above.

We will need the following theorem [4,5,6]:

Theorem:

Let a transformation $\widetilde{T} : [0,1] \to [0,1]$ satisfying the following conditions:

1)$\widetilde{T}$ is continuous and piecewise C^2,

2)$\widetilde{T}'$ does not exist at a point $t \in [0,1]$,

3)$\widetilde{T}$ satisfies

$$\inf_{x \in J_1} \left| \frac{d\widetilde{T}}{dx} \right| > 1 \quad \textit{where} \quad J_1 = \left\{ x \in [0,1];\ \frac{d\widetilde{T}}{dx} \ \textit{exists} \right\}.$$

Then, there is one [4] and only one [5] absolutely continuous probability measure $\widetilde{\mu}$ invariant under $\widetilde{T}$. Moreover [6] for almost all $x \in [0,1]$ we have that

$$\frac{1}{n} \sum_{i=0}^{n-1} I_B(T^i x) \underset{n \to \infty}{\longrightarrow} \widetilde{\mu}(B)$$

for every Borel set B in $[0,1]$. (I_A is the characteristic function of a set $A \subset [0,1]$).

Corollary:

$\widetilde{T}$ is ergodic respect to $\widetilde{\mu}$. If $\widetilde{\mu}(B) > 0$ for every open set $B \subset [0,1]$, then the orbit $O_T(x) = \{x, Tx, T^2x, \ldots\}$ for almost all $x \in [0,1]$ is dense in $[0,1]$.

Consider now the transformation $\widetilde{T} : \widetilde{\Omega} = [0,1] \to \widetilde{\Omega} = [0,1]$ such that

$$\widetilde{T}x = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 2(1-x) & \text{if } x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

here $t = 1/2$. It is well known that the Lebesgue measure ℓ is absolutely continuous with respect to itself and invariant under $\widetilde{T}$. By the foregoing theorem there is only measure $\widetilde{\mu}$ of this type, then we must have $\widetilde{\mu} = \ell$.

Moreover, by the theorem, we have that for almost all $x \in [0,1]$

$$\frac{1}{n}\sum_{i=0}^{n-1} I_B(\widetilde{T}^i x) \underset{n\to\infty}{\longrightarrow} \widetilde{\mu}(B) = \ell(B)$$

for every Borel set $B \subset [0,1]$.

Consider now the transformation $T : \Omega = [0,1] \to \Omega = [0,1]$ such that $Tx = 4x(1-x)$. By means of the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, such that

$$\varphi(x) = \frac{2}{\pi}\arcsin\sqrt{x}, \qquad \varphi^{-1}(x) = \frac{1-\cos\pi x}{2},$$

we can construct the dynamical system $(\Omega, \mathcal{B}, \mu, T)$ ($\mathcal{B}$ is the class of Borel sets in $[0,1]$) from the dynamical system $(\widetilde{\Omega}, \mathcal{B}, \widetilde{\mu} = \ell, \widetilde{T})$ and transfer the last affirmation of the Theorem about $\widetilde{T}$ to T. In fact:

i) Ruelle [7] constructs by means of φ the probability space $(\Omega, \mathcal{B}, \mu)$ where the probability measure μ is

$$\begin{aligned}\mu(dx) &= \ell\varphi(dx)\\ &= \frac{dx}{\pi\sqrt{x(1-x)}}\ ;\end{aligned}$$

moreover, $\mu(dx)$ is invariant under T.

Furthermore,

ii) $x \in B$ if and only if $\varphi(x) \in \varphi(B)$,

$$T^i x \in B \quad \text{if and only if} \quad \widetilde{T}^i\varphi(x) \in \varphi(B)$$

and the fraction of iterates $\{x, Tx, \ldots, T^{n-1}x\}$ of x that are in B,

$$\frac{1}{n}\sum_{i=0}^{n-1} I_B(T^i x),$$

is the same as the fraction of the iterates $\{\varphi(x), \widetilde{T}\varphi(x), \ldots, \widetilde{T}^{n-1}\varphi(x)\}$ of $\varphi(x)$ that are in $\varphi(B)$,

$$\frac{1}{n}\sum_{i=0}^{n-1} I_{\varphi(B)}(\widetilde{T}^i\varphi(x));$$

by the Theorem this last proportion tends to $\widetilde{\mu}(\varphi(B) = \ell(\varphi(B))$ when $n \to \infty$ for almost all $\varphi(x)$ and $B \in \mathcal{B}$. Then

$$\frac{1}{n}\sum_{i=0}^{n=1} I_B(T^i x) \underset{n\to\infty}{\longrightarrow} \widetilde{\mu}(\varphi(B)) = \ell(\varphi(B)) = \int_B \frac{dx}{\pi\sqrt{x(1-x)}}.$$

In this way we have shown that the statistical behavior of the elements of almost all orbit $O_T(x)$ follow probabilistic law

$$\mu(dx) = \frac{dx}{\pi\sqrt{x(1-x)}}.$$

Besides, we arrive in a different way to results in [8,9]:

Corollary:

The logistic transformation $Tx = 4x(1-x)$ is ergodic respect to $\mu(dx)$ and $O_T(x)$ is dense in $[0,1]$ for almost all $x \in [0,1]$.

Acknowledgements

A la profesora L. Esteva por haberme motivado a aclarar lo que me inquietaba; a los profesores A. Salas, por entusiasmarme a comunicar este enfoque, y D. Moreno, por sus comentarios constructivos.

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Resumen. Ulam y von Neumann [1] dijeron que partiendo con casi cualquier x en el intervalo [0,1] e iterando la función $Tx = 4x(1-x)$ se obtiene una sucesión de números "aleatorios" en ese intervalo con distribución de probabilidad arcsen; en este artículo se da *una* demostración de que, en efecto, la medida de probabilidad arcsen, $\mu(dx) = dx/\pi\sqrt{x(1-x)}$ es *la* ley probabilista que describe el comportamiento estadístico de los números en cada órbita $O_T(x) = \{x, Tx, T^2x, \ldots\}$, para casi toda x en [0,1].