Quantization of the Dirac field in de Sitter space

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Abstract. The Dirac equation in de Sitter space is solved analytically. Explicit expressions for the charge, energy and momentum densities are obtained, following the standard procedures of field quantization.

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Introduction

The quantization of fields in curved space sets several new conceptual and mathematical questions. The standard techniques used in quantum field theory can be generalized to curved space, but the procedure is not always unique due to the lack of privileged coordinate system in general relativity. Nevertheless, the situation can be clarified by working out some specific examples.

The de Sitter space is a particularly simple curved space in which field quantization can be tested. There has been much work on massive scalar fields in de Sitter space [1] but so far little attention has been paid to fields with spin [2].

In the present article, a spin 1/2 massive field in de Sitter space is considered. The Dirac equation is solved in Section II using a particular coordinate system: all possible solutions can be written in terms of Whittaker functions, but there are no eigenvalues of the energy-operator, and therefore a distinction between particles and antiparticles is not as simple as in flat space; this disadvantage is related to the particular coordinate system which has been used. However, the second quatization scheme is pursued in Section III; explicit expressions are obtained for the charge, energy and momentum densities.

2. The Dirac equation in de sitter space

A spin 1/2 massive particle is described by a pair of two-spinors ψ_A and $\phi^{A'}$ (the indices take the values 1 and 2) which satisfy the Dirac equation [3] ($\hbar = G = 1$ hereafter):

$$\nabla^{NA'}\psi_N = -m\phi^{A'}, \tag{1a}$$

$$\nabla_{AN'}\phi^{N'} = m\psi_A, \tag{1b}$$

where $\nabla^{AA'}$ is the spinorial covariant derivative operator and m is the mass of the particle [4]. This form of the Dirac equation is particularly well suited for a treatment in curved space. In a conformably flat space defined by the metric

$$ds^2 = \Omega^2(-dt^2 + dx^2 + dy^2 + dz^2), \tag{2}$$

where $\Omega(t, x, y, z)$ is the conformal factor, the Dirac equation (1) takes on the form

$$(\partial_t + \vec{\sigma} \cdot \nabla)F + m\Omega G = 0, \tag{3a}$$

$$(\partial_t - \vec{\sigma} \cdot \nabla)G + m\Omega F = 0, \tag{3b}$$

where

$$F := \Omega^{3/2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad G := \Omega^{3/2} \begin{pmatrix} \phi^{1'} \\ \phi^{2'} \end{pmatrix}, \tag{4}$$

$\vec{\sigma}$ are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

and ∇ is the divergence operator in Cartesian coordinates.

In the particular case of the de Sitter space, the conformal factor in (2) is

$$\Omega = \frac{\alpha}{t}, \tag{6}$$

where α is a constant (Ricci scalar $R = 12\alpha^{-2}$). The flat space limit can be obtained by performing the transformation $t \to t + \alpha$ and taking the limit $\alpha \to \infty$.

It is convenient to set

$$F = e^{i\mathbf{p}\cdot\mathbf{r}}\mathbf{A}_p u(t), \qquad G = e^{i\mathbf{p}\cdot\mathbf{r}}\mathbf{A}_p v(t), \tag{7}$$

where $\mathbf{A}_p$ is a 2×2 matrix which satisfies Eq. (*A* 1). Thus, Eqs. (3) take the form

$$\frac{d}{dt}u + ip\sigma_3 u + \frac{\mu}{t}v = 0, \tag{8a}$$

$$\frac{d}{dt}v - ip\sigma_3 v - \frac{\mu}{t}u = 0, \tag{8b}$$

where $p := (\mathbf{p} \cdot \mathbf{p})^{1/2}$ and $\mu := m\alpha$. It is easy to decouple each component of u and

v from this last set of equations. Setting

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \tag{9}$$

it follows that

$$t^2 \frac{d^2}{dt^2} u_\pm + t \frac{d}{dt} u_\pm + (p^2 t^2 \pm ipt + \mu^2) u_\pm = 0, \tag{10}$$

and a similar equation applies to $v_\pm$ with only the change $p \to -p$. As discussed in detail in Appendix B, the two linearly independent solutions of Eq. (10) are $w_\pm(pt)$ and $w_\mp^*(pt)$, where the functions $w_\pm$ are certain combinations of Hankel functions. However, the spinors u and v also satisfy Eqs. (8), which must be used together with the recurrence relations ($B4$). Summing up, the most general solutions of Eqs. (8) turn out to be

$$u = \begin{pmatrix} A_{p+} w_+ + B_{p+} \mu w_-^* \\ B_{p-} \mu w_- + A_{p-} \mu w_+^* \end{pmatrix}, \tag{11a}$$

$$v = \begin{pmatrix} -A_{p+} \mu w_- + B_{p+} \mu w_+^* \\ B_{p-} w_+ - A_{p-} \mu w_-^* \end{pmatrix}, \tag{11b}$$

where $A_{p\pm}$ and $B_{p\pm}$ are certain constants. The pair u and v can also be written as a linear combination of four pairs

$$\begin{aligned} u_{(1)+} &= \begin{pmatrix} w_+ \\ 0 \end{pmatrix} \qquad v_{(1)+} = \begin{pmatrix} \mu w_- \\ 0 \end{pmatrix}, \\ u_{(1)-} &= \begin{pmatrix} 0 \\ w_+^* \end{pmatrix} \qquad v_{(1)-} = \begin{pmatrix} 0 \\ -\mu w_-^* \end{pmatrix}, \\ u_{(2)+} &= \begin{pmatrix} \mu w_-^* \\ 0 \end{pmatrix} \qquad v_{(2)+} = \begin{pmatrix} w_+^* \\ 0 \end{pmatrix}, \\ u_{(2)-} &= \begin{pmatrix} 0 \\ \mu w_- \end{pmatrix} \qquad v_{(2)-} = \begin{pmatrix} 0 \\ w_+ \end{pmatrix}. \end{aligned} \tag{12}$$

Eq. ($B7$) implies that they satisfy the orthonormality condition

$$u_{(i)\pm}^\dagger u_{(j)\pm} + v_{(i)\pm}^\dagger v_{(j)\pm} = 2\delta_{(i)(j)}, \tag{13}$$

and also

$$u_{(i)\pm}^\dagger \sigma_3 u_{(j)\pm} + v_{(i)\pm}^\dagger \sigma_3 v_{(j)\pm} = \pm 2\delta_{(i)(j)}. \tag{14}$$

If we define $F_{(i)\pm}$ and $G_{(i)\pm}$ as those spinor pairs constructed according to (7) with each pair $u_{(i)\pm}, v_{(i)\pm}$ separately, then it follows from (8) that

$$\vec{\sigma} \cdot \mathbf{p} F_{(i)\pm} = \pm p F_{(i)\pm} \tag{15}$$

and similarly for $G_{(i)\pm}$. Thus, the spinors given by (12) can be interpreted as defining a basis with definite helicity $\pm 1/2$. Note, however, that none of these spinors, or any linear combination of them, are eigenfunctions of the time derivative operator. It will be seen in the following that the pairs $u_{(i)\pm}, v_{(1)\pm}$, and and $u_{(2)\pm}, v_{(2)\pm}$, correspond to states with opposite charge and that, therefore, they can be defined as "particles" and "antiparticles", but the energies of these states are not well defined.

3. Current and energy-momentum tensor

To proceed further it is necessary to consider the current four-vector [4]

$$J^{\alpha} = \frac{1}{2}\sigma^{\alpha A A'}(\psi_{A'}\psi_A + \phi_A \phi_{A'}) \tag{16}$$

and the energy-momentum tensor [4]

$$T_{\alpha\beta} = \frac{i}{8}\sigma_{\alpha}^{A A'}\left[-\psi_{A'} \overleftrightarrow{\nabla}_{\beta} \psi_A + \phi_A \overleftrightarrow{\nabla}_{\beta} \phi_{A'}\right] + (\alpha \leftrightarrow \beta), \tag{17}$$

where $\sigma^{\alpha A A'}$ are the covariant Pauli matrices and $\overleftrightarrow{\nabla}_{\alpha}$ indicates the covariant derivation to the right minus the covariant derivation to the left. The current J^{α} and the energy-momentum tensor are conserved, $\nabla_{\alpha} J^{\alpha} = 0$ and $\nabla_{\alpha} T^{\alpha\beta} = 0$, by virtue of the Dirac equation (1).

Now, the total number of particles in the hypersurface $t = \text{const.}$ is

$$N := \int \left(\frac{\alpha}{t}\right)^4 J^t dx dy dz \tag{18}$$

and is a constant. On the other hand

$$J^t = \frac{1}{2}\frac{t}{\alpha}(\psi_{1'}\psi_1 + \psi_{2'}\psi_2 + \phi^1 \phi^{1'} + \phi^2 \phi^{2'}), \tag{19}$$

and therefore, more explicitly,

$$N = \frac{1}{2}\int dx dy dz \int d\mathbf{p}' \int d\mathbf{p} e^{i(\mathbf{p}-\mathbf{p}')\cdot r}\left[u_{p'}^{\dagger} A_{p'}^{\dagger} A_p u_p + v_{p'}^{\dagger} A_{p'}^{\dagger} A_p v_p\right], \tag{20}$$

where Eqs. [4] and [7] have been used. The integration over the space variables

yields a δ function since

$$\int d\mathbf{r} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} = (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}'), \tag{21}$$

and using (*A*3), (4) and (13) one finally finds that

$$N = 2(2\pi)^3 \int d\mathbf{p} p \left((p-p_3)A^*_{p+}A_{p+} + B^*_{p+}B_{p+}\right) + (p+p_3)\left(A^*_{p-}A_{p-} + B^*_{p-}B_{p-}\right).$$

Following the usual procedure, we normalize the wave function in such a way that there is one particle of each of the four possible types in a cell of volume $(2\pi)^3$ in phase space. This is achieved setting

$$A_{p\pm} = (2\pi)^{-3}\left[2p(p \mp p_3)\right]^{-1/2} a_{p\pm} \tag{23a}$$

$$B_{p\pm} = (2\pi)^{-3}\left[2p(p \mp p_3)\right]^{-1/2} b^*_{p\pm} \tag{23b}$$

where $a_{p\pm}$ and $b_{p\pm}$ are complex numbers of unit moduli; thus

$$N = (2\pi)^{-3} \int d\mathbf{p} \left(a^*_{p+}a_{p+} + a^*_{p-}a_{p-} + b_{p+}b^*_{p+} + b_{p-}b^*_{p-}\right). \tag{24}$$

Consider now the T_{tt} component of the energy-momentum tensor, or more precisely the energy density e measured by an observer with four-velocity $U^\mu = \Omega^{-1}(1,\mathbf{0})$, that is $e = U^\mu U^\nu T_{\mu\nu}$. It follows from Eq. (17) that

$$\begin{aligned} e &= \frac{i}{4}\frac{t}{\alpha}\left[-\psi_{1'}\overleftrightarrow{\partial_t}\psi_1 - \psi_{2'}\overleftrightarrow{\partial_t}\psi_2 + \phi^1\overleftrightarrow{\partial_t}\phi^{1'} + \phi^2\overleftrightarrow{\partial_t}\phi^{2'}\right] \\ &= -\frac{i}{4}\left(\frac{t}{\alpha}\right)^4\left(F^\dagger\overleftrightarrow{\partial_t}F + G^\dagger\overleftrightarrow{\partial_t}G\right) \end{aligned} \tag{25}$$

Using Eqs. (7), (9), (*A*3), (13) and (14), it can be seen that the energy E in the hypersurface $t = \text{const}$ is

$$\begin{aligned} E &= \int dxdydz \left(\frac{\alpha}{t}\right)^3 e \\ &= 8i\pi^3\left(\frac{t}{\alpha}\right)\int d\mathbf{p}\left[-p\left(u^\dagger\frac{du}{dt} + v^\dagger\frac{dv}{dt}\right) + p_3\left(u^\dagger\sigma_3\frac{du}{dt} + v^\dagger\sigma_3\frac{dv}{dt}\right)\right]. \end{aligned} \tag{26}$$

From Eqs. (11) and (23) it further follows that

$$E = (2\pi)^{-3}\left(\frac{t}{\alpha}\right)\int d\mathbf{p}p\left[\left(a^*_{p+}a_{p+} + a^*_{p-}a_{p-} - b_{p+}b^*_{p+} - b_{p-}b^*_{p-}\right)F\right.$$
$$\left. + \left(b_{p-}a_{p-} - b_{p+}a_{p+}\right)\mu G + \left(a^*_{p-}b^*_{p-} - a^*_{p+}b^*_{p+}\right)\mu G^*\right], \tag{27}$$

where

$$F := \frac{i}{2}\left(w^*_+ w'_+ + \mu^2 w^*_- w'_-\right)$$
$$= 1 + \mu^2\left[\frac{i}{2x}\left(w^*_+ w_- w^*_- w_+\right) - w^*_- w_-\right], \tag{28a}$$

$$G := \frac{i}{2}\left(w_- w'_+ - w_+ w'_-\right). \tag{28b}$$

A similar calculation permits us find the momentum vector. Due to the symmetry of the problem, it is enough to consider the z direction only; the total momentum in that direction is

$$P_3 = \int dxdydz\left(\frac{\alpha}{l}\right)^3 U^\mu e^\nu_3 T_{\mu\nu}, \tag{29}$$

where e^ν_3 is a unit vector in the z direction. Following algebraic calculations similar to Eq. (26), one finds that

$$P_3 = (2\pi)^3\frac{t}{\alpha}\int d\mathbf{p}pp_3\left[p(u^+u - v^+v) - p_3(u^+\sigma_3 u - v^+\sigma_3 v)\right], \tag{30}$$

and more explicitly

$$P_3 = (2\pi)^{-3}\frac{t}{\alpha}\int d\mathbf{p}p_3\left[\left(a^*_{p+}a_{p+} + a^*_{p-}a_{p-} - b_{p+}b^*_{p+} - b_{p-}b^*_{p-}\right)\mathcal{H}\right.$$
$$\left. + \mu\left(b_{p+}a_{p+} + a_{p-}b^*_{p-}\right)w_+ w_- + \mu\left(a^*_{p+}b^*_{p+} + b_{p-}a_{p-}\right)w^*_+ w^*_-\right], \tag{31}$$

where

$$\mathcal{H} := \frac{1}{2}\left(w_+ w^*_+ - \mu^2 w_- w^*_-\right). \tag{32}$$

An exactly similar expression results for p_1 and p_2, with only p_1 and p_2 in Eq. (31) instead of p_3.

Summing up, we have obtained expressions for the particle number, Eq. (31), of the field. In order to extract meaninful results, the second quatization of the field

must now be considered. Following the usual procedure [5], one takes $a_{p\pm}$ and $b_{p\pm}$ as operators in Fock space which satisfy the anticommutation relations

$$\{a_{p\pm}, a^*_{p\pm}\} = 1, \qquad \{b_{p\pm}, b^*_{p\pm}\} = 1, \tag{33}$$

and define the "particle" and "antiparticle" number operators as

$$\begin{aligned} N_{p\pm} &= a^*_{p\pm} a_{p\pm}, \\ \bar{N}_{p\pm} &= b^*_{p\pm} b_{p\pm}, \end{aligned} \tag{34}$$

respectively. Furthermore, it must be assumed that states with only "particles"are orthogonal to states with only "antiparticles", that is, the vacuum expectation values of products such as $a_+ b_+$ must be zero.

With these assumptions, Eqs. (24), (27) and (31) take the forms

$$Q := N = (2\pi)^{-3} \int d\mathbf{p} \left(N_{p+} + N_{p-} - \bar{N}_{p+} - \bar{N}_{p-} - 2\right), \tag{34}$$

$$E = (2\pi)^{-3} \int d\mathbf{p} p F(pt) \frac{t}{\alpha} \left(N_{p+} + N_{p-} + \bar{N}_{p+} + \bar{N}_{p-} - 2\right), \tag{35}$$

$$P_n = (2\pi)^{-3} \int d\mathbf{p} p_n H(pt) \frac{t}{\alpha} \left(N_{p+} + N_{p-} + \bar{N}_{p+} + \bar{N}_{p-} - 2\right). \tag{36}$$

Q is now the charge density, simply the number density of "particles "minus "antiparticles". Eqs. (35) and (36) imply a complicated relation between the energy and the momentum of a particle which has not been possible to elucidate. Finally, note the appearance of the term -2 which corresponds to the vacuum energy in Eq. (35) and the "Dirac sea"in Eq. (34) (there is one particle of given helicity and negative energy in each cell of volume $(2\pi\hbar)^3$ in phase-space). As in the flat-space case, the vacuum energy can be eliminated if normal ordering of operators in all vacuum expectation is taken.

4. Concluding remarks

Though the coordinate system used in this paper permits to solve analytically the Dirac equation, it does not permit a clear distinction between positive and negative frequency solutions. Thus, the concept of particle and antiparticle is based only on the charge sign: Eq. (34) implies that the operators $a_{p\pm}$ and b_{p+} are associated to particles of opposite charge. Moreover, it must be postulated that "particle" and "antiparticle" states are orthogonal to each other, as mentioned just after Eq. (33). It is likely that a coordinate system which does not depend explicitly on time leads

to solutions of the Dirac equation such that positive and negative energy states are well defined. This point will be investigated in a future article.

Appendix A

The matrix $\vec{\sigma} \cdot \mathbf{p}$ can be diagonalized in the following form

$$\mathbf{A}_p^{-1} \vec{\sigma} \cdot \mathbf{p} \mathbf{A}_p = p\sigma_3, \tag{A1}$$

where $p := (\mathbf{p} \cdot \mathbf{p})^{1/2}$. Explicitly

$$\mathbf{A}_p = \begin{pmatrix} p_1 + ip_2 & p_1 + ip_2 \\ p - p_3 & -p - p_3 \end{pmatrix} \tag{A2}$$

and it follows that

$$\mathbf{A}_p^\dagger \mathbf{A}_p = 2p(p - p_3\sigma_3). \tag{A3}$$

Appendix B

Equation (10) is of the form

$$x^2 f_\pm'' + x f_\pm' + (x^2 \pm ix + \mu^2) f_\pm = 0. \tag{B1}$$

The two linearly independent solutions are

$$w_\pm(x) \equiv x^{-\frac{1}{2}} W_{\pm 1/2, i\mu}(2ix) \tag{B2}$$

and

$$w_\mp^*(x) \equiv x^{-\frac{1}{2}} W_{\mp 1/2, i\mu}(-2ix) = x^{-\frac{1}{2}} \left[W_{\mp -1/2, i\mu}(2ix) \right]^*, \tag{B3}$$

where $W_{\pm 1/2, i\mu}(2ix)$ is the Whittaker function of imaginary argument [6]. [In general, $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are two linearly independent solutions of the Whittaker equation, and $W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z)$].

From the standard recurrence relations of the Whittaker function [6], it can be shown that

$$\begin{aligned} \frac{d}{dx} w_+ &= -iw_+ + \frac{\mu^2}{x^2} w_-, \\ \frac{d}{dx} w_- &= iw_- - \frac{1}{x} w_+; \end{aligned} \tag{B4}$$

and also that

$$w_+ \pm i\mu w_- = (\pi x)^{1/2} e^{\frac{\pi}{2}\mu \mp i\frac{\pi}{4}} H^{(2)}_{i\mu\pm 1/2}(x), \tag{B5}$$

where $H^{(2)}_{i\mu\pm 1/2}(x)$ is the Hankel function of the second kind.

The Wronskian of the two solutions $w_\pm$ and $w^*_\mp$ is

$$w'_+ w^*_- - w_+ w^{*\prime}_- = \frac{2}{x}\,, \tag{B6}$$

which together with Eqs. (A4) implies the important property

$$w_+ w^*_+ + \mu^2 w_- w^*_- = 2. \tag{B7}$$

Note, finally, the asymptotic forms of $w_\pm$ for large x:

$$\begin{aligned} w_+ &= (2)^{\frac{1}{2}} e^{\frac{\pi i}{4}} e^{-ix}\left[1 + i\frac{\mu^2}{2x} + O(x^{-2})\right] \\ w_- &= (2)^{-\frac{1}{2}} e^{-\frac{\pi i}{4}} \frac{e^{-ix}}{x}\left[1 + i\frac{\mu^2+1}{2x} + O(x^{-2})\right] \end{aligned} \tag{B8}$$

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Resumen. Se resuelve analíticamente la ecuación de Dirac en espacio de Sitter. Se sigue el método estándar de cuantización de campos para obtener explícitamente las densidades de carga, energía y momento.