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On the geometry for non-abelian gauge fields

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Abstract. A new geometric model for non-abelian gauge fields on the group manifold is proposed. It is constructed on the basis of the gauge interpretation given to the geometry of the Lie group. It is proved that the Noether theorem follows from the geometry of the group manifold.

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In the last few years it has been shown by several authors that the geometrical interpretation of gauge fields in terms of connection forms in the principal fibre bundle is not unique and there are other possible pictures of geometrical setting [1,2]. Furthermore, with the correct geometrical picture, prospects of unification of gauge and Higgs fields are expected and a single vector-scalar scheme of them is obtained [3,5].

In the present paper a geometrical model of gauge fields is proposed on the group manifold. It is made on the basis of the gauge interpretation given to the geometry of the group described in the Cartan setting [6].

Two remarkable features are obtained which consist in the different structure of gauge transformations of this model in comparison with the standard version of the fibre bundle and in the crucial role which the geometrical properties of the group manifold play in the gauge invariance of the theory.

In order to give a gauge interpretation to the geometry of the group manifold we shall first of all establish the geometry on the group manifold M_G. For any differentiable arbitrary manifold M we can put the one-form [7,8]

$$\delta \mathbf{a} = \omega^a \mathbf{e}_a, \qquad \delta \mathbf{e}_a = \omega_a^b \mathbf{e}_b, \tag{1}$$

which determine the infinitesimal translations of a certain movable basis $e_a(x)$, so that $\delta\mathbf{a}$ denotes the translation vector of the co-ordinate origin, and ω^a and ω_b^a are the infinitesimal displacement and affine connection one-forms, respectively. They can depend on the co-ordinate system $\{x^1 \dots x^n\}$ which may parameterize the n-dimensional manifold M. Following Cartan it is said that the geometry of the M is fixed if the structure equations which take place for the differential forms ω^a and

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ω_b^a are given. For generalized spaces they are [9]

$$d \wedge \omega^a + \omega_b^a \wedge \omega^b + \Omega^a = 0, \tag{2}$$

$$d \wedge \omega_b^a + \omega_c^a \wedge \omega_b^c + \Omega_b^a = 0, \tag{3}$$

where $d \wedge \omega$ is the external differential of the ω-form, $\omega \wedge \omega$ is the external product of two ω-forms. Ω^a and Ω_b^a are the two-forms of torsion and curvature of the space, respectively.

If now we put in (1)

$$e_a \to e'_a = M_a^b(x) e_b(x) \tag{4}$$

where $M \in GL(n, R)$, we obtain that the Eqs. (1–3) are invariant ones under the transformation (4) if the following set of transformation laws for ω^a, ω_b^a, Ω^a and Ω_b^a take place

$$\omega'^a = (M^{-1})_b^a \omega^b, \tag{5}$$

$$\omega_b'^a = (M^{-1})_c^a \omega_d^c M_b^d + (M^{-1})_c^a dM_b^c, \tag{6}$$

$$\Omega'^a = (M^{-1})_b^a \Omega^b, \tag{7}$$

$$\Omega_b'^a = (M^{-1})_c^a \Omega_d^c M_b^d, \tag{8}$$

$(M^{-1})M = 1$. Note that we shall only consider real spaces and real algebras. Thus we see that transformation laws inherent to pure physical objects are reproduced on the manifold M. However the question that arises here is when such transformations are integrable ones? In order to answer this question we shall consider only a group manifold M_G.

Let G be a Lie group of dimension n and $\varphi^i(a, b)$ $(i = 1, \ldots, n)$ be a multiplication law in G, a and b are parameters in G. The integration conditions of $\varphi^i(a, b)$ take the form [10]:

$$\frac{\partial \varphi^i(a, b)}{\partial b^k} = \mu_a^i [\varphi(a, b)] \lambda_k^a(b), \tag{9}$$

$$\frac{\partial \varphi^i(a, b)}{\partial a^k} = \xi_a^i [\varphi(a, b)] \zeta_k^a(b), \tag{10}$$

which allow to establish the composition law $\varphi^i(a, b)$ if the function $\mu_a^i(a) \equiv \frac{\partial \varphi^i}{\partial b^a}|_{b=0}$ and $\xi_b^i(b) = \frac{\partial \varphi^i}{\partial a^b}|_{a=0}$ are given

$$\mu_a^i(a) \lambda_k^a(a) = \delta_k^i, \qquad \xi_a^i(a) \zeta_k^a(a) = \delta_k^i. \tag{11}$$

Eqs. (9) and (10) in turn, are integrable ones if the Maurer-Cartan equations

$$d \wedge \underset{L}{\Omega}^a(a, da) + \frac{1}{2} C^a_{bc} \underset{L}{\omega}^b \wedge \underset{L}{\omega}^c = 0, \tag{12}$$

$$d \wedge \underset{R}{\omega}^a(b, db) - \frac{1}{2} C^a_{bc} \underset{R}{\omega}^b \wedge \underset{R}{\omega}^c = 0, \tag{13}$$

take place [8]. Here $\underset{L}{\omega}^a(a, da) = \lambda^a_i(a) da^i$ and $\underset{R}{\omega}^a(b, db) = \zeta^a_i(b) db^i$ are the left and right displacement 1-forms in the group.

Functions $\varphi^i(a, b)$ establish the projection of the group manifold into itself

$$a^i L(b) = \varphi^i(a, b), \qquad R(a) b^i = \varphi^i(a, b), \tag{14}$$

determining thus the left action or the second parametric group and the right action or first parametric group, respectively, with (12) and (13) as integration conditions.

As is known, both these parametric groups are isomorphic ones. Due to this fact, we, in order to study the geometry on M_G, shall consider only one of them, *e.g.* the second parametric group of left displacements.

If we now attach a $\underset{L}{\omega}$-form to a basis X_a of the left Lie algebra $\mathcal{L}_L$ of the Lie group G, then an infinitesimal displacement in the group gives rise to an infinitesimal displacement in the Lie algebra [10]

$$\delta X_a = ad(\omega) X_a = \underset{L}{\omega}^b [X_b, X_a] = \underset{L}{\omega}^b C^c_{ba} X_c = \underset{L}{\omega}^c_a X_c. \tag{15}$$

C^a_{bc} denote the structure constants and ω^c_a is the affine connection form corresponding to a certain geometry chosen in the Lie algebra $\mathcal{L}_L$.

Eqs. (1) take the form

$$\delta \mathbf{a} = \underset{L}{\omega}^a X_a, \qquad \delta X_b = \underset{L}{\omega}^c_b X_c. \tag{16}$$

By taking the external differentiation of the latter expression and by using (12), the following equations result

$$d \wedge \underset{L}{\omega}^a + \underset{L}{\omega}^a_b \wedge \underset{L}{\omega}^b + \underset{L}{\Omega}^a = 0, \tag{17}$$

$$d \wedge \underset{L}{\omega}^a_b + \underset{L}{\omega}^a_c \wedge \underset{L}{\omega}^c_d = 0. \tag{18}$$

These structure equations, following (2) and (3), characterize the geometry in $\mathcal{L}_L$ as a geometry with torsion without curvature. Here

$$\underset{L}{\Omega}^a = -\frac{1}{2} C^a_{bc} \underset{L}{\omega}^a \wedge \underset{L}{\omega}^c.$$

By introducing a local frame $\tau_i(a) = \lambda_i^a X_a$ attached to a point $a \in M_G$ we obtain from (17) and (18) that

$$\underset{L}{\widetilde{\omega}_j^i} \wedge da^j + \underset{L}{\widetilde{\Omega}^i} = 0 \tag{19}$$

$$d \wedge \underset{L}{\widetilde{\omega}_j^i} + \underset{L}{\widetilde{\omega}_k^i} \wedge \underset{L}{\widetilde{\omega}_j^k} = 0, \tag{20}$$

where

$$\underset{L}{\widetilde{\omega}_j^i} = \mu_a^i(a)\lambda_{j,k}^a da^k = \underset{L}{\gamma_{jk}^i} da^k \tag{21}$$

$$\underset{L}{\widetilde{\Omega}^i} = \frac{1}{2}\mu_a^i C_{bc}^a \omega^b \wedge \omega^c \equiv \underset{L}{S_{jk}^i} da^j \wedge da^k, \tag{22}$$

where $\lambda_{j,k}^a$ means $\frac{\partial \lambda_j^a}{\partial a^k}$. Eqs. (19) and (20) establish the corresponding local geometry on M_G.

If we now reduce the matrices $M \in GL(n,R)$ to matrices $\Lambda \in O(n)$ so that they transform the left functions λ_i^a into right ζ_i^a and vice versa, *i.e.*,

$$\lambda_i^a = \Lambda_b^a \zeta_i^b, \qquad \zeta_i^b = (\Lambda^{-1})_c^b \lambda_i^c, \tag{23}$$

where by (14) it is evident that

$$\Lambda_b^a = \lambda_i^a(a)\xi_b^i(a), \qquad (\Lambda^{-1})_c^b = \zeta_k^b(a)\mu_c^k(a). \tag{24}$$

and now we relate the components of the vectors of the left and right Lie algebras by a linear transformation

$$\underset{L}{\omega}^a(a, da) = D_b^a(a,b)\underset{R}{\omega}^b(b, db), \tag{25}$$

assuming that they both have a common co-ordinate origin, then it is easy to see that such a diffeomorphism is given by the equations of the multiplication law

$$\frac{da^i}{db^k} = \xi_a^i(a)\zeta_k^a(b), \tag{26}$$

only if $D(a,b) = \Lambda(a)$.

Thus we obtain the gauge interpretation of the group G as a projection of the group into itself. Here the elements of the algebra of the one parametric group become elements of the algebra of the other parametric group. These transformations are performed by matrices Λ from $O(n)$ and the geometrical magnitudes on M_G transform themselves according to (5–8).

Thus it is clear that matrices of gauge transformation in our model belong to the group $O(n)$. Now, in order to build a gauge model, we assume the existence of reflection functions $a^i(x)$ projecting the space time R_4 onto the gauge group G. These functions $a^i(x)$ can be given by differentials

$$da^i = K^i_\mu(a, x)dx^\mu$$

whenever the conditions $d\Lambda da^i = 0$ hold.

Functions $a^i(x)$ induce the following evident displacement and connection forms in the Lie algebra

$$\omega^a(x, dx) = \lambda^a_i(a(x))a^i_\mu dx^\mu = \lambda^a_i da^i(x), \tag{27}$$

$$\omega^a_b(x, dx) = C^a_{bc}\lambda^c_n(a(x))a^n_\mu dx^\mu = C^a_{cb}\lambda^c_n da^n(x), \tag{28}$$

which, because of the Eqs. (17) and (18), can only model vacuum gauge configurations.

Due to this fact, we generalize the forms (27) and (28) in the following way

$$\omega^a \to A^a(x, dx) = f^a_\mu(a, x)dx^\mu, \tag{29}$$

$$\omega^a_b \to A^a_b(x, dx) = C^a_{cb}A^c(x, dx), \tag{30}$$

where now, no conditions like $d \wedge f^i_\mu = 0$ hold. By using (29) and (30) and doing several algebraic transformations, the following structure equations in the generalized Lie algebra result

$$(R^a_b)_{\mu\nu}dx^\mu \wedge dx^\nu = C^a_{db}F^d_{\mu\nu}dx^\mu \wedge dx^\nu, \tag{31}$$

$$F^a_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + C^a_{bc}f^b_\mu f^c_\nu. \tag{32}$$

These are invariant ones under gauge transformations like (5–8) if the structure constants C^a_{bc} transform themselves as a tensor, *i.e.*,

$$C^a_{bc} \to \Lambda^a_d C^d_{nm}(\Lambda^{-1})^n_b(\Lambda^{-1})^m_c.$$

However, if matrices Λ are given by (24) then it can be proved [11] that

$$C'^a_{bc} = C^a_{bc}.$$

Thus, structure equations of the generalized Lie algebra model pure gauge potential of Yang-Mills type. As (31) denotes the field strengths may be given either in terms of torsion or in terms of curvature of the group space.

Due to the fact that ω^a forms in turn establish the distant parallelism on the group manifold, in the local frame related to a point $a \in M_a$, that field strength

$F^{(a)}_{\mu\nu}$ becomes

$$F^{(a)}_{\mu\nu} = \lambda^{(a)}_i F^i_{\mu\nu} = \lambda^{(a)}_i \left\{ \partial_\mu f^i_\nu - \partial_\nu f^i_\mu + \underset{L}{\gamma^i_{kl}} \left(a^l_\mu f^k_\nu - a^l_\nu f^k_\mu \right) + 2 \underset{L}{S^i_{kl}} f^k_\mu f^l_\nu \right\}, \quad (33)$$

where now and henceforth indices related to the Lie algebra will be closed in parenthesis if it is necessary.

Just in this way, *i.e.* by using (33), it is interesting to study the geometrical features of this model. Once the gauge invariance has been provided, we put

$$S = -\frac{1}{4} \int d^4x g_{ik} F^i_{\mu\nu} F^k_{\mu\nu}, \quad (34)$$

where $g_{ik}(a) = C^a_{bc} C^b_{ad} \lambda^c_i \lambda^d_k$ is the Killing-Cartan metric tensor on M_G. Using standard variational methods the Yang-Mills-like equations result in the following way

$$\partial_\mu F^{\mu\nu}_i - \gamma^l_{ik} A^k_\mu F^{\mu\nu}_l - 2 S^l_{ki} f^k_\mu F^{\mu\nu}_l = 0. \quad (35)$$

In order to prove how the gauge invariance is controlled we follow how the Noether identities $\delta S / \delta a^i \equiv 0$ take place. Taking the variation of S over a^i and considering that f^i_μ do not depend on the fields a^i, we have

$$4\partial_\mu \left(g_{ik} \gamma^i_{nl} F^{k\mu\nu} f^n_\nu \right) + g_{ik,l} F^i_{\mu\nu} F^{k\mu\nu}$$
$$+ 2 g_{ik} F^{i\mu\nu} \left[\gamma^k_{nm,l} \left(a^m_\mu f^n_\nu - a^m_\nu f^n_\mu \right) + 2 S^k_{mn,l} f^m_\mu f^n_\nu \right] = 0. \quad (36)$$

At the level of equation of motion (35) it is easy but tedious to confirm that the Noether theorem (36) becomes controllable by the set of the following well-known properties of the group manifold M_G [9]:

a) Nullity of the curvature tensor on M_G

$$R^k_{lmn} = \gamma^k_{lm,n} - \gamma^k_{ln,m} + \gamma^j_{lm} \gamma^k_{jn} - \gamma^j_{ln} \gamma^k_{jm} = 0,$$

b) Jacobi identities for the torsion tensor S^i_{kl} take place

$$S^k_{mj} S^j_{nl} + S^k_{mj} S^j_{lm} + S^k_{lj} S^j_{mn} = 0,$$

c) Nutillity of the convariant derivate of S^i_{kl}

$$S^k_{ln;m} \equiv S^k_{ln,m} + \gamma^k_{jm} S^j_{ln} - \gamma^j_{lm} S^k_{jn} - \gamma^j_{nm} S^k_{lj} = 0,$$

d) Nullity of the convariant derivate of g_{ik}

$$g_{ik;l} \equiv g_{ik,l} - \gamma_{ik}^{m} g_{mk} - \gamma_{kl}^{m} g_{im} = 0.$$

Thus it is proved that gauge invariance is provided by the geometrical properties of the group manifold. The properties mentioned below take place in the compact Lie group with an absolutely antisymmetric torsion tensor. The proof for the case when $f_\mu^a = f_\mu^a(a,x)$ is straightforward.

By starting from the gauge interpretation given to the Lie group as projections of the two parametric groups into themselves and by means of a generalization of the Lie algebra, it was possible to formulate a new geometrical model for non-abelian gauge fields. The projections of the two parametric groups are performed by equations of composition law and their integration conditions. Precisely, because the transformation matrices performing such projections belong to $O(n)$, the main remarkable feature of the model consists in the structure of the gauge transformations. It is easy to see that at infinitesimal level, when $a^i(x) \to 0$, matrices $\Lambda(a) \in O(n)$ become reducible matrices $\Lambda_b^a = \exp\{a^a(x) C_{ac}^b\}$ from the adjoint representation of G. This fact in addition to the fulfillment of the Noether identities make possible to perform the renormalization procedure of the model [12].

As regards the fact that the gauge invariance of this model is a consequence of the geometry of the group manifold, one can think that this is an interesting property which the standard Yang-Mills version in the fibre bundle does not offer.

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Resumen. Se propone un nuevo modelo geométrico de los campos de calibración no abelianos en la variedad del grupo. Este se construye en base a la interpretación de norma dada a la geometría de un grupo de Lie. Se demuestra que el teorema de Noether es una consecuencia de la geometría de la variedad del grupo.