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# Stability of the Statz-DeMars equations in the description of a laser

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**Abstract.** We present a study of the stability regions for the solutions of the Statz-DeMars equations. It was found that only one of the two critical points of these equations corresponds to a stable solution; an asymptotically stable spiral point. This was done by finding a Liapunov function and by usual stability analysis methods. From this, we found a necessary condition for the stability of a laser. Finally, these results are extended to the case where the $Q$ of the laser cavity suffers a periodic modulation. In particular, it was found that a weak modulation of the quality $Q$ of a laser has as a consequence that an originally asymptotically stable spiral point transforms into a stationary stable limit cycle.

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## 1. Introduction

A quantum mechanically based description of a laser gives rise to a set of three equations known as Bloch or master equations. These equations provide the temporal evolution of the radiation field amplitude, polarization and population inversion within a laser cavity [1,2]. These equations can be obtained for a one-photon [1,2], or multiphoton [3,4] laser, and in both cases studies of the stability of their solutions have been carried out [5,6] .

An alternative and simpler, but inherently less complete, description of a laser can be provided by a set of two phenomenological equations given by Statz and DeMars [7]. In these equations the variables describing laser action are the stimulated photon density of frequency $w$, denoted by $M(t)$, and the population inversion, denoted by $N(t)$. These equations can be written as follows [8]:

$$\frac{dM(t)}{dt} = B'M(t)N(t) - \frac{M(t)}{T}, \tag{1}$$

$$\frac{dN(t)}{dt} = -\beta B'M(t)N(t) + \frac{(N_0 - N(t))}{T_1}, \tag{2}$$

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where $T$ is the photon lifetime inside the cavity, $T_1$ is the relaxation time between the upper and lower levels of the laser transition, $B' = B\hbar w$ where $B$ is the Einstein coefficient. $\beta$ is a constant depending on the number of laser levels and $N_0$ is the equilibrium inversion density *i.e.* the inversion density value when pumping takes place but no oscillation is allowed.

The physical interpretation of the above equations is straightforward. Eq. (1) states that the photon density increases proportionally to the inversion density and decreases inversely to the photon lifetime inside the cavity. Eq. (2) states that population inversion decreases proportionally to the photon density and increases proportionally to the pumping process and inversely to the relaxation time $T_1$.

Denoting by $Ncr$ the value of population inversion when oscillation just starts, we can define the following quantities and write Eqs. (1) and (2) in an adimensional form

$$t' = \frac{t}{T},$$

$$m(t') = B'T_1 M(t', T_1),$$

$$n(t') = \frac{N(t', T_1)}{Ncr},$$

$$\alpha = \frac{No}{Ncr},$$

$$G = \frac{T_1}{T},$$

as follows:

$$\frac{dm}{dt'} = Gm(n-1), \tag{3}$$

$$\frac{dn}{dt'} = \alpha - n(m+1). \tag{4}$$

Typical values for the relaxation time $T_1$, and the photon lifetime are $10^{-3}$–$10^{-6}$ sec for the first and $10^{-8}$ sec for the second. Therefore typical $G$ values are around $10^2$–$10^5$, while typical values for $\alpha$ are $2 \leq \alpha \leq 30$.

In what follows we will find out the stability regions and critical points that satisfy Eqs. (3) and (4) during laser operation. It will be seen that of the two critical points of these equations only one corresponds to a stable solution, an asymptotically stable spiral point. It also will be shown that the introduction of a weak modulation of the quality $Q$ of the laser has as a consequence that an originally asymptotically stable spiral point transforms into a stationary stable limit cycle.

## 2. Critical points

Eqs. (3) and (4) are a system of autonomous, first order, and non-linear differential equations which can be represented as follows:

$$\frac{dm}{dt'} = M(m,n) = am + bn + M'(m,n), \tag{5}$$

$$\frac{dn}{dt'} = N(m,n) = cm + dn + N'(m,n), \tag{6}$$

where $a$, $b$, $c$ and $d$ are real constants and $M'(m,n)$, $N'(m,n)$ are non-linear functions.

It can be verified that Eqs. (3) and (4) satisfy the following two conditions:
*i)* $(ad - bc) \neq 0$,
*ii)* $M'(m,n)$ and $N'(m,n)$ have continuous first partial derivatives for all $(m,n)$.

It can also be shown that the critical points, *i.e.* points satisfying $M(m,n) = N(m,n) = 0$, of Eqs. (3) and (4) are:

$$Q_1 = (0,\alpha),$$

$$Q_2 = (\alpha - 1, 1).$$

Considering the critical point $Q_1$ and making the change of variable

$$m(t') = u(t'), \tag{7}$$

$$n(t') = v(t') + \alpha. \tag{8}$$

Eqs. (3) and (4) can be written as follows:

$$\frac{du}{dt'} = Gu(v + \alpha - 1), \tag{9}$$

$$\frac{dv}{dt'} = -u(v + \alpha) - v. \tag{10}$$

Taking the linear terms of Eqs. (9) and (10) and since $(ad - cd) \neq 0$ we obtain the characteristic equation

$$p^2 - (G(\alpha - 1) - 1)p - G(\alpha - 1) = 0 \tag{11}$$

with solutions $p_1 = G(\alpha - 1)$ and $p_2 = -1$. Since $G$ and $\alpha$ are always positive quantities (and $\alpha > 1$) the roots of the characteristic equation (11) are real and of opposite signs. Therefore the critical point $Q_1 = (0, \alpha)$ is a saddle point and since the previous conditions *i)* and *ii)* are satisfied, this is a saddle point for both the linear and the non-linear system [Eqs. (9) and (10)]. This result states that the

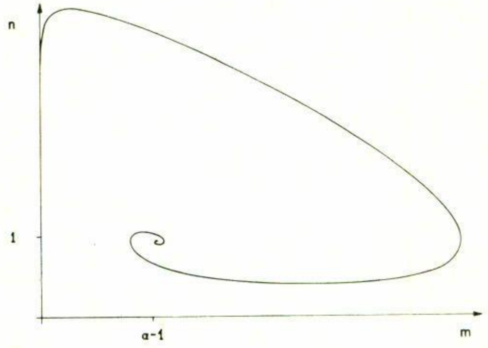


FIGURE 1. Phase space solution of the Statz-DeMars equations [Eqs. (3), (4)]. As it is shown, the solution is asymptotically stable towards the critical point $(\alpha - 1, 1)$.

dynamic system described by Eqs. (9) and (10) will never remain at the critical point $Q_1$, unless the initial conditions are $(0, \alpha)$, but in any case this point lacks any useful interest for laser application.

Considering the critical point $Q_2$ and making the change of variable

$$m(t') = u(t') + (\alpha - 1), \tag{14}$$

$$n(t') = v(t') + 1, \tag{15}$$

Eqs. (3) and (4) can be written as follows:

$$\frac{du}{dt'} = Gv(u + \alpha - 1), \tag{16}$$

$$\frac{dv}{dt'} = -v(u + \alpha) - u. \tag{17}$$

Taking the linear terms of Eqs. (16) and (17) we obtain, for the characteristic equation of the system

$$p^2 + \alpha p + G(\alpha - 1) = 0 \tag{20}$$

with solutions

$$p_{1,2} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4G(\alpha - 1)}}{2}. \tag{21}$$

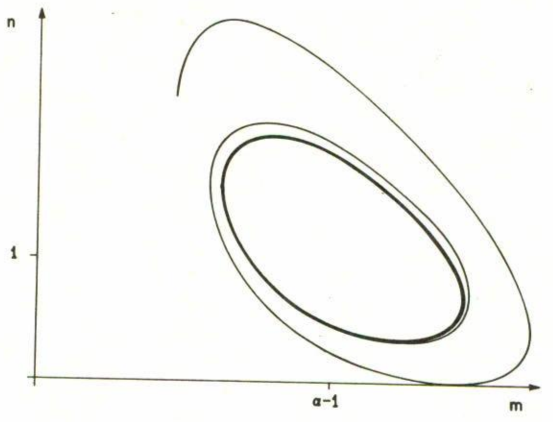


FIGURE 2. Phase space solution of the Statz-DeMars equations when a periodic variation of the quality of the cavity $Q$ is introduced [Eqs. (29), (30)]. As it is shown the solution is asymptotically stable towards a limit cycle around the critical point $(\alpha - 1, 1)$

Assuming validity of the condition

$$4G(\alpha - 1) \gg \alpha^2 \tag{22}$$

we obtain:

$$p_{1,2} = -\frac{\alpha}{2} \pm i\sqrt{G(\alpha - 1)}. \tag{23}$$

These solutions are complex with negative real part. Once again, since the above conditions *i)* and *ii)* are satisfied, the critical point $Q_2 = (\alpha - 1, 1)$ is an asymptotically stable spiral point for both, the linear and the non-linear system [Eqs. (16) and (17)]. This means that independently of where the dynamic system starts, it will always finish, in phase space, at the critical point $Q_2$. A computer generated plot of the phase space for this case is shown in Fig. 1 for $\alpha = 30$ and $G = 10^5$. Any other typical value of $\alpha$ and $G$ will give qualitatively similar plots.

It can be seen that condition (22) will, in general, be satisfied for any laser. Therefore its oscillation will also be stable and towards the critical point $Q_2$.

## 3. Liapunov function

The stability region of the Statz-DeMars equations [Eqs. (3) and (4)] around the critical point $Q_2 = (\alpha - 1, 1)$ can be found by using the Liapunov direct second method [9].

Defining the function

$$L(m,n) = \frac{n^2 - 1}{m^2}, \tag{24}$$

with domain $D = \{m, n | m, n \geq 1\}$, which contains the critical point $Q_2$.

The function $L(m,n)$ satisfies:

*i)* $L(\alpha - 1, 1) = 0$,

*ii)* $L(m,n) > 0$ for any other point $(m,n) \in D$.

The derivative of $L(m,n)$ is:

$$\dot{L}(m,n) = \frac{\partial L(m,n)}{\partial m} M(m,n) + \frac{\partial L(m,n)}{\partial n} N(m,n). \tag{25}$$

So $\dot{L}(m,n)$ satisfies:

*i)* $\dot{L}(\alpha - 1, 1) = 0$,

*ii)* $\dot{L}(m,n) < 0$ for any other point $(m,n) \in D$.

Therefore $\dot{L}(m,n)$ is a negative definite function in the domain $D$. It follows that $L(m,n)$ is a Liapunov function of the Statz-DeMars equations. These equations are valid for any $(m,n) \in D$ and are asymptotically stable towards the point $(\alpha - 1, 1)$.

## 4. $Q$-switching

As it is known, the $Q$-switching technique consists in that if there is initially a very high loss in the laser cavity, *i.e.* low $Q$ value, and pumping is taking place, the population inversion can reach a very high value without laser oscillations occurring. If cavity losses are suddenly reduced, *i.e.* $Q$ is switched to a higher value, laser oscillations will start and a rapid depopulation of the population inversion will occur with the consequent emission of a short, intense laser pulse.

Following [8] we will assume a periodic variation of frequency $w_m$ of the quality $Q$ of the cavity as

$$Q(t) = Q_0(1 - \gamma \cos(w_m t)). \tag{26}$$

Since the photon lifetime $T$ inside the cavity is $T = Q/w$ we get for Eq. (1)

$$\frac{dM(t)}{dt} = B' M(t) N(t) - \frac{M(t) w}{Q_0(1 - \gamma \cos(w_m t))}. \tag{27}$$

Defining $T_0 = \frac{Q_0}{w}$ and assuming a weak modulation $\gamma < 1$ we may approximate the above equation as

$$\frac{dM(t)}{dt} = B' M(t) N(t) - \frac{M(t)}{T_0}(1 + \gamma \cos(w_m t)). \tag{28}$$

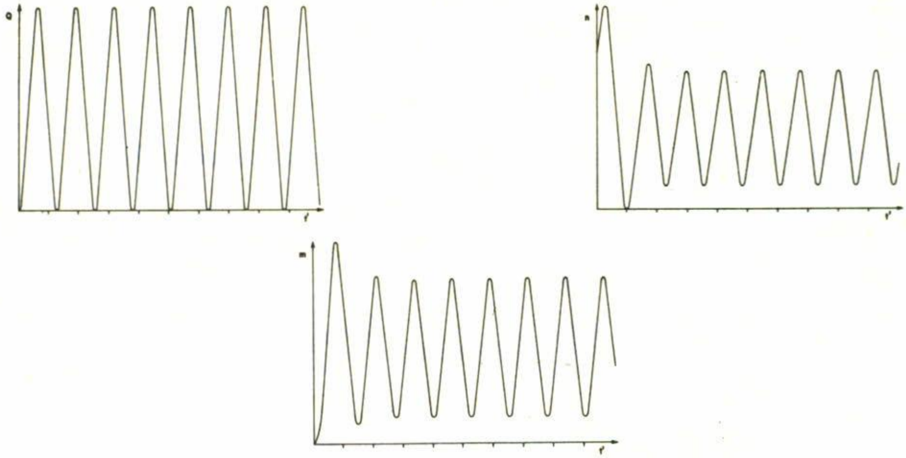

FIGURE 3. The solution obtained when a periodic variation of the quality $Q$ of the cavity is introduced (as shown in (a)) for the population inversion density $n$ (shown in (b)) and the normalized photon density $m$ (shown in (c)) as a function of time $t'$. We can see that when $Q$ has a high value, the population inversion decreases while the photon density increases in complete agreement with what is expected.

Introducing the above adimensional quantities and defining $w'_m = w_m T_1; G = \frac{T_1}{T_0}$ we can write the Statz-DeMars equations [Eqs. (28) and (2)] as

$$\frac{dm}{dt'} = Gm(n - 1 - \gamma \cos(w'_m t')), \tag{29}$$

$$\frac{dn}{dt'} = \alpha - n(m + 1). \tag{30}$$

These equations are an approximation of the Statz-DeMars equations for the case of $Q$ weak modulation. They are a system of non-autonomous first order non-linear equations. An analytic solution for these equations is not known yet. However, for a weak modulation the numeric solution in phase space using the same $\alpha$ and $G$ values as in Fig. 1, is shown in Fig. 2. In this case it is shown that the solution tends to a stationary limit cycle. It can be seen that the systems of Eqs. (29), (30) and (3), (4) are essentially the same, with exception of the term $-\gamma \cos(w'_m t')$ for Eq. (29) which acts as a small perturbation term. We can see that this perturbation causes an asymptotically stable spiral point in the system (3), (4) to transform into a stationary limit cycle in the system (29), (30).

Fig. 3 shows a plot of the quality $Q$ of the cavity, the normalized inversion density $n$ and the normalized photon density $m$ *vs.* time $t'$ . We can see that when $Q$ has a high value, the population inversion decreases while the photon density increases in complete agreement with what is expected.

## 5. Conclusions

It was found that only one of the two critical points of the Statz-DeMars equations corresponds to a stable solution, an asymptotically stable spiral point. The stability of this point was confirmed by a Liapunov function obtained for these equations.

Also, a condition necessary to have a stable solution was found [Eq. (22)]. Nevertheless, this condition will in general always be fulfilled by any laser.

Finally, it was found that a weak modulation of the quality $Q$ of a laser cavity has as a consequence that an asymptotically stable spiral point of the original equations transforms into a stationary limit cycle.

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**Resumen.** Se presenta un estudio de la estabilidad de las soluciones de las ecuaciones de Statz-DeMars. Se encontró que de los dos puntos críticos de estas ecuaciones sólo uno corresponde a una solución estable que es un punto espiral asintóticamente estable. Esto fue hecho encontrando una función de Liapunov y aplicando métodos estándar para análisis de estabilidad. De aquí se encontró una condición necesaria para la estabilidad de un láser. Finalmente estos resultados son extendidos al caso en que el $Q$ de la cavidad láser sufre una modulación en amplitud periódica. Se encontró que una modulación débil del $Q$ de la cavidad transforma un punto espiral asintóticamente estable en un circulo límite estable.