Time evolution equations in an ionized fluid using extended irreversible thermodynamics

Patricia Goldstein

Departamento de Física, Facultad de Ciencias,
Universidad Nacional Autónoma de México, 04510 México, D.F.

L.S. García-Colín*

Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa,
Apartado postal 55–534 , 09340 México, D.F.

Abstract. Using Extended Irreversible Thermodynamics we obtain evolution equations for the heat flux and for the stress tensor in an ionized fluid subject to an external magnetic field. These equations may be compared with those obtained from the thirteen moment solution of the Boltzmann equation. This comparison yields explicit expressions for the relevant phenomenological coefficients in terms of molecular quantities. The static correlation functions for the fluctuations of the state variables describing the fluid are also calculated.

PACS: 05.70.Ln; 52.25.Dg

1. Introduction

In the course of this decade, efforts have been made to extend Linear Irreversible Thermodynamics (LIT) to the domain of phenomena which fall outside the scope of such linear theory. Several of these efforts are now known as Extended Irreversible Thermodynamics (EIT). Although these approaches differ among each other on the way in which the non-equilibrium states of the system under study is described, they all coincide in the derivation of time evolution equations for the state variables describing the system. Such equations are of the hyperbolic type in contrast to the parabolic differential equations which are predicted in LIT.

One of these approaches is based on the idea of extending the ordinary space of state variables of LIT by raising the fluxes characterizing the non equilibrium processes in the system to the status of independent variables. This method originated in the so called moment method solution to the Boltzmann equation proposed by H. Grad nearly forty years ago [1]. Although there are in principle conceptual difficulties in accepting the fluxes as state variables [2], the application of the method to a number of different systems has proved to be rather successful [3–14].

In this paper, we address ourselves to study another system, namely, an ionized

---

*Miembro de El Colegio Nacional.

fluid subject to an external magnetic field using the same approach. The reason for undertaking such a study is that this system has been previously studied using the above mentioned solution for the Boltzmann equation by H. Yang [15]. Consequently, the results of EIT which are purely phenomenological may be compared with a more microscopic calculation which will allow us to assess the virtues and shortcomings of the former one.

By raising to the status of independent variables the heat flux and the stress tensor we are able to derive a complete set of time evolution equations for all the state variables.

These equations necessarily contain undetermined phenomenological coefficients. The comparison with the set of equations obtained by Yang will also permit us to study the static correlations functions for the state variables leading us results that, to our knowledge, have not been mentioned in the literature.

The paper is divided in a section 2 containing a summary of the underlying ideas of EIT, together with the derivation of the time evolution equations for the state variables. Next, in section 3 we discuss the physical nature of such equations and we compare them with those obtained by kinetic theory. We follow with section 4, which contains the calculation of the static correlation functions, and finally in section 5 we include some remarks about our results.

2. Extended irreversible thermodynamics time evolution equations

In the framework of Linear Irreversible Thermodynamics (LIT), a system is described by locally conserved variables whose evolution equations, the conserved or balance equations, are complemented by phenomenological constitutive equations for the thermodynamic fluxes which may be obtained either empirically or from more microscopic models. Such is the case of the moment equations obtained from the Boltzmann equations, which is a kinetic model for a dilute gaseous system.

As it was mentioned in the introduction, in EIT one broadens the space of state variables to include the dissipative fluxes. In our case, such space will therefore contain the local internal energy $\epsilon(\mathbf{r}, t)$ the local mass density, $\rho(\mathbf{r}, t)$ as well as the heat flux $q(\mathbf{r}, t)$, and the stress tensor $\mathbb{P}(\mathbf{r}, t)$, as space variables. For purely semantic reasons, we shall refer ourselves to the fluxes as the variables defining the subset $\mathcal{R}$ of non conserved variables. Thus, the complete set of state variables $\mathcal{G}$ may be written as

$$\mathcal{G} = \mathcal{C} \cup \mathcal{R} = \{\epsilon(\mathbf{r}, t), \rho(\mathbf{r}, t); q(\mathbf{r}, t), \mathbb{P}(\mathbf{r}, t)\} \tag{1}$$

where $\mathcal{C}$ is the subset of the locally conserved variables $\epsilon(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$.

These variables satisfy balance type equations, namely,

$$\frac{d}{dt}\rho = -\rho \operatorname{div} \mathbf{u} \tag{2}$$

$$\rho \frac{d\epsilon}{dt} = -\operatorname{div} \mathbf{q} - p \operatorname{div} \mathbf{u} - \mathbb{P} : \operatorname{grad} \mathbf{u} + \frac{1}{c_0} \mathbf{i} \cdot \mathbf{u} \times \mathbf{H}, \tag{3}$$

where $\mathbf{u}$ is the hydrodynamic velocity, p the hydrodynamic pressure, $\mathbf{i}$ the electric current, $\mathbf{H}$ the external magnetic field and c_0 the velocity of light. It is therefore clear that our task is to derive the time evolution equations for the $\mathcal{R}$ variables.

In order to accomplish this objective, we assume that there exists a sufficiently continuous and differentiable function η defined in $\mathcal{G}$, *i.e.*

$$\eta = \eta(\epsilon, \rho; \mathbf{q}, \mathbb{P}) \tag{4}$$

such that we may construct a differential from,

$$d\eta = \frac{\partial \eta}{\partial \epsilon} d\epsilon + \frac{\partial \eta}{\partial \rho} d\rho + \frac{\partial \eta}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial \eta}{\partial \mathbb{P}} : d\mathbb{P}. \tag{5}$$

Notice should be made of the fact that η is not the entropy associated with the non equilibrium state of the fluid described by the $\mathcal{G}$ variables. Only when the subspace of variables $\mathcal{R}$ turns out to be irrelevant for the description of the system, η reduces to the ordinary local equilibrium entropy as defined in LIT [16].

We may write in our case the analogue of the Gibbs equation,

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial \epsilon} \frac{d\epsilon}{dt} + \frac{\partial \eta}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \eta}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} + \frac{\partial \eta}{\partial \mathbb{P}} : \frac{d\mathbb{P}}{dt}. \tag{6}$$

The derivatives of η are constructed as the most general tensors of the corresponding tensorial order in the space of state variables. The first two derivatives $\partial \eta / \partial \epsilon$ and $\partial \eta / \partial \rho$ must reduce to those defined in local equilibrium when the subspace $\mathcal{R}$ is the void set, and should be constructed as the most general scalars in $\mathcal{G}$. Thus, if we expand these derivatives in terms of the scalar invariants keeping terms which are at most quadratic in the variables, we obtain that

$$\frac{\partial \eta}{\partial \epsilon} = T^{-1} + X_{11} \mathbf{q} \cdot \mathbf{q} + X_{12} \mathbb{P} : \mathbb{P}, \tag{7}$$

$$\frac{\partial \eta}{\partial \rho} = -\frac{T^{-1} p}{\rho^2} + X_{21} \mathbf{q} \cdot \mathbf{q} + X_{22} \mathbb{P} : \mathbb{P}. \tag{8}$$

On the hand, the derivatives $\partial \eta / \partial \mathbf{q}$ and $\partial \eta / \partial \mathbb{P}$ must be the most general vector and tensor constructed in $\mathcal{G}$ respectively, so with the same criterium mentioned above, we may write that

$$\frac{\partial \eta}{\partial \mathbf{q}} = \frac{T^{-1}}{\rho} \alpha_{11} \mathbf{q} + \frac{T^{-1}}{\rho} \alpha_{12} \mathbf{q} \cdot \mathbb{P}, \tag{9}$$

$$\frac{\partial \eta}{\partial \mathbb{P}} = \frac{T^{-1}}{\rho}\alpha_{21}\mathbf{q}\mathbf{q} + \frac{T^{-1}}{\rho}\alpha_{22}\mathbb{P}, \tag{10}$$

where α_{ij} and X_{ij}, $i,j = 1,2$, are scalars defined in $\mathcal{G}$.

If we now impose the condition that $d\eta$ must be an exact differential, the cross derivatives resulting from Eqs. (7–10) must be equal. This implies automatically that not all the coefficients appearing there are independent. In fact, it is easy to see that X_{11} and X_{22} depend on α_{11} and α_{22} according to the relations

$$X_{11} = \frac{1}{2}\frac{\partial}{\partial \epsilon}\left(\frac{T^{-1}\alpha_{11}}{\rho}\right), \tag{11}$$

$$X_{22} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(T^{-1}\alpha_{22}) - \frac{T^{-1}\alpha_{22}}{\rho^2}\right). \tag{12}$$

Also, we can see that,

$$X_{12} = \frac{1}{2}\frac{\partial}{\partial \epsilon}\left(\frac{T^{-1}}{\rho}\alpha_{22}\right), \tag{11.a}$$

$$X_{21} = \frac{1}{2}\frac{\partial}{\partial \rho}\left(\frac{T^{-1}}{\rho}\alpha_{11}\right), \tag{12.a}$$

which clearly points out that only α_{11} and α_{22} are independent. Notice that the condition $\partial^2\eta/\partial q_i \partial \mathbb{P}_{jk} = \partial^2\eta/\partial \mathbb{P}_{jk}\partial q_i$ leads to the additional relationship that $\alpha_{12} = \alpha_{21} = 0$. Finally, the terms $X_{12}\mathbb{P} : \mathbb{P}$ and $X_{21}\mathbf{q}\cdot\mathbf{q}$ appearing in Eqs. (7) and (8) are also easily manipulated. In fact, $X_{12}\mathbb{P} : \mathbb{P}$, when multiplied by $d\epsilon/dt$, gives rise to terms which are at least of order three in the $\mathcal{R}$ variables and may be neglected. On the other hand, $X_{21}\mathbf{q}\cdot\mathbf{q}$ may be absorbed in an identical term which comes from $X_{11}\mathbf{q}\cdot\mathbf{q}$ by simply redefining the corresponding coefficients.

Therefore, the partial derivatives for the function η may be written as

$$\frac{\partial \eta}{\partial \epsilon} = T^{-1} + X_{11}\mathbf{q}\cdot\mathbf{q}, \tag{13}$$

$$\frac{\partial \eta}{\partial \rho} = -\frac{T^{-1}p}{\rho^2} + X_{22}\mathbb{P} : \mathbb{P} + X_{21}\mathbf{q}\cdot\mathbf{q}, \tag{14}$$

$$\frac{\partial \eta}{\partial \mathbf{q}} = \frac{T^{-1}}{\rho}\alpha_{11}\mathbf{q}, \tag{15}$$

$$\frac{\partial \eta}{\partial \mathbb{P}} = \frac{T^{-1}}{\rho}\alpha_{22}\mathbb{P}. \tag{16}$$

Emphasis should be made on the fact that the vanishing of α_{12} and α_{21} is perfectly consistent with kinetic theory and ordinary magnetohydrodynamics. Indeed,

the resulting time evolution equations for the fluxes are a set of coupled equations in which the homogeneous terms do not contain the mixed expressions $\mathbf{q} \cdot d\mathbb{P}/dt$ or $\mathbb{P} : d\mathbf{q}/dt$.

The second assumption of the theory is introduced with the idea of providing a method to compute the equations that govern the time evolution of the non conserved variables. For this purpose, we assume that η satisfies a balance equation of the form

$$\rho \frac{d\eta}{dt} + \operatorname{div} \mathbf{J}_\eta = \sigma_\eta, \tag{17}$$

where, $\mathbf{J}_\eta$ and σ_η are flux and the source term associated with η, and whose form remains to be determined. We shall impose the restriction that in the case in which the non conserved variables are irrelevant, Eq (17) must reduce to the entropy balance equation in LIT. It is only in this sense that η may be regarded as a "generalized entropy", but we shall avoid this misleading terminology.

The flux $\mathbf{J}_\eta$ is the most general vector that may be constructed in $\mathcal{G}$. Therefore, up to the second order in the $\mathcal{R}$ variables, we have that

$$\mathbf{J}_\eta = \beta_1 \mathbf{q} + \beta_2 \mathbf{q} \cdot \mathbb{P}, \tag{18}$$

where $\beta_i = \beta_i(\epsilon, \rho)$, $i = 1, 2$. We choose

$$\beta_1 = \frac{1}{T}, \tag{19}$$

because as indicated above, $\mathbf{J}_\eta$ must reduce to the entropy flux in LIT when $\mathcal{R}$ is the void set.

Using Eqs (6), (13–15) and (17–18) we reach the following expression for the source term σ_η namely

$$\sigma_\eta = \frac{1}{c_0} \mathbf{i} \cdot \mathbf{u} \times \mathbf{H} + \mathbf{q} \cdot \{ -X_{11} \mathbf{q} \operatorname{div} \mathbf{q} - X_{11}^1 \mathbf{q} \operatorname{div} \mathbf{u} - X_{11} \mathbf{q} \mathbf{i} \cdot \mathbf{u} \times \mathbf{H} \frac{1}{c_0}$$

$$+ \frac{T^{-1}}{\rho} \alpha_{11} \frac{d\mathbf{q}}{dt} + \operatorname{grad} \frac{1}{T} + \beta_2 \operatorname{div} \mathbb{P} + \mathbb{P} \cdot \operatorname{grad} \beta_2 \}$$

$$+ \mathbb{P} : \left\{ -T^{-1} \operatorname{grad} \mathbf{u} - X_{22} \mathbb{P} \rho^2 \operatorname{div} \mathbf{u} + T^{-1} \alpha_{22} \frac{d\mathbb{P}}{dt} + \beta_2 \operatorname{grad} \mathbf{q} \right\}. \tag{20}$$

The theory gives us an alternative way to evaluate σ_η. Indeed, σ_η must be the most general scalar we may construct in the $\mathcal{G}$ space, that is

$$\sigma_\eta = \sigma_0 + \mathbf{X}_1 \cdot \mathbf{q} + \mathbf{X}_2 : \mathbb{P}. \tag{21}$$

At this point, it is rather important to point out the way in which we are

constructing our theory results in the omission of several parameters of relevance to our system. Such parameters, by their nature not contained in our primitive set of variables, may have an important role in our description of the system. In our case, the external magnetic field exerts a magnetic force on our fluid. So we must include *ad hoc* this effect in our formalism. This can be done using the closure assumption [17], which considers parameters not belonging to the state variables in the construction of σ_η.

. Having in mind that σ_η must reduce to the local equilibrium entropy production when the non conserved variables are irrelevant, we propose that

$$\sigma_0 = \frac{T^{-1}}{c_0} \mathbf{i} \cdot \mathbf{u} \times \mathbf{H}, \tag{22}$$

$$\mathbf{X}_1 = \mu_{11}\mathbf{i} \times \mathbf{H} + \mu_{12}\mathbf{q} + \mu_{13}\mathbb{P} \cdot \mathbf{i} \times \mathbf{H} + \mu_{14}\mathbf{q} \cdot \mathbb{P}, \tag{23}$$

$$\mathbb{X}_2 = \mu_{21}\mathbf{i}(\mathbf{u} \times \mathbf{H}) + \mu_{22}\mathbf{q}(\mathbf{u} \times \mathbf{H}) + \mu_{23}\mathbb{P} + \mu_{24}\mathbf{q}\mathbf{q}. \tag{24}$$

Here, the terms associated with $\mathbf{u} \times \mathbf{H}$ and $\mathbf{i} \times \mathbf{H}$ are two such parameters of paramount importance to keep our results at grips with both kinetic and electromagnetic theory.

If we introduce the expressions for σ_0, $\mathbf{X}_1$ and $\mathbb{X}_2$ into Eq. (21), omitting for simplicity non linear terms in the non conserved variables and we compare this expression for σ_η with the one we get from the balance Eq. (17), we finally obtain the following time evolution equations for $\mathbf{q}$ and $\mathbb{P}$ valid up to first order in the non conserved variables, namely,

$$\frac{d\mathbf{q}}{dt} = \{\mu_{12}\mathbf{q} + \mu_{11}\mathbf{i} \times \mathbf{H} + \mu_{13}\mathbb{P} \cdot \mathbf{i} \times \mathbf{H} + X'_{11}\mathbf{q}\,\mathrm{div}\,\mathbf{u}$$

$$+ X_{11}\left(\frac{\mathbf{i}}{c_0} \cdot \mathbf{u} \times \mathbf{H}\right)\mathbf{q} - \mathrm{grad}\,\frac{1}{T} - \beta_2\,\mathrm{div}\,\mathbb{P} - \mathbb{P} \cdot \mathrm{grad}\,\beta_2\}, \tag{25}$$

$$\frac{d\mathbb{P}}{dt} = \frac{T}{\alpha_{22}}\{\mu_{23}\mathbb{P} + \mu_{21}\mathbf{i}(\mathbf{u} \times \mathbf{H}) + \mu_{22}\mathbf{q}(\mathbf{u} \times \mathbf{H}) + T^{-1}\,\mathrm{grad}\,\mathbf{u}$$

$$+ X_{22}\mathbb{P}\rho^2\,\mathrm{div}\,\mathbf{u} - \beta_2\,\mathrm{grad}\,\mathbf{q}\}, \tag{26}$$

where $X'_{11} = -p(X_{11} - \rho^2 X_{21})$.

The set of Eqs. (25) and (26) together with Eqs. (1) and (2) constitute the main result of this paper. They form a complete set of differential equations for the state variables which may in principle be solved under given initial and boundary conditions, provided that we know explicit expressions for the so far undetermined coefficients μ_{ij}, X_{ij}, α_{ij} and β_2. It should be emphasized that these coefficients are still space and time dependent functions though the conserved variables ρ and ϵ. Furthermore, their determination relies upon experiment or a microscopic model. For our system under study, the second alternative is feasible since similar equations

have been derived from kinetic theory. The determination of such coefficients will be the subject of the following section.

3. Comparison with kinetic theory

Following the arguments presented in the previous section, we may now compare the time evolution equations we have obtained for $\mathbf{q}$ and $\mathbb{P}$, namely, Eqs. (25) and (26) with those obtained from kinetic theory by Yang [15]. In his work Yang obtains the following time evolution equations for $\mathbf{q}$ and $\mathbb{P}$ for an ionized gas as moment equations of the Boltzmann equation, namely,

$$
\frac{d\mathbf{q}}{dt} = -\frac{2}{3}\frac{p}{\mu}\mathbf{q} - \frac{5}{2}\frac{p}{\rho}\mathbf{i} \times \mathbf{H} - \frac{1}{\rho}\mathbb{P} \cdot \mathbf{i} \times \mathbf{H} - \frac{2}{5}\mathbf{q}\operatorname{div}\mathbf{u} - \frac{5}{2}p\operatorname{grad}\left(\frac{p}{\rho}\right)
$$

$$
- \frac{p}{\rho}\operatorname{div}\mathbb{P} - \frac{7}{2}\mathbb{P} \cdot \operatorname{grad}\left(\frac{p}{\rho}\right) - \frac{\mathbb{P}}{\rho} \cdot \operatorname{grad} p - \frac{7}{2}\mathbf{q} \cdot \operatorname{grad}\mathbf{u}, \tag{27}
$$

$$
\frac{d\mathbb{P}}{dt} = -\frac{p}{\mu}\mathbb{P} + 2\mathbf{i}(\mathbf{u} \times \mathbf{H}) - 2p\operatorname{grad}\mathbf{u} - \frac{4}{5}\operatorname{grad}\mathbf{q} - 2\mathbb{P} \cdot \operatorname{grad}\mathbf{u}. \tag{28}
$$

where μ is the viscosity coefficient. Both sets of Eqs. (25)–(26) and (27)–(28) have the same structure except for the terms $X_{11}(\frac{\mathbf{i}}{c_0} \cdot \mathbf{u} \times \mathbf{H})\mathbf{q}$ and $\mu_{22}\mathbf{q}(\mathbf{u} \times \mathbf{H})$, which are absent in the kinetic results. It is important to point out the fact that similar results have been obtained in other systems treated in EIT [10]. Thus, EIT gives a more general description of the system.

Comparing both sets of Eqs. (25)–(26) and (27)–(28) we may perform a proper identification of our coefficients with the corresponding kinetic quantities. This comparison of our results with those obtained via Grad's moment method provides us a kinetical theoretical justification of the use of EIT in this problem. In order to accomplish this comparison, we shall consider as a first step what happens to our equations when our system reduces to a simple fluid. In that case, Eqs. (25) and (26) reduce to

$$
\frac{d\mathbf{q}}{dt} = \frac{T}{\alpha_{11}}\left\{\mu_{12}\mathbf{q} - \operatorname{grad}\frac{1}{T}\right\}, \tag{29}
$$

$$
\frac{d\mathbb{P}}{dt} = \frac{T}{\alpha_{22}}\left\{\mu_{23}\mathbb{P} + T^{-1}\operatorname{grad}\mathbf{u}\right\}. \tag{30}
$$

Next, we notice that these last two equations are of the type of the well known Maxwell-Cattaneo relaxion equations [18–20] for these fluxes, *i.e.*

$$
\frac{d\mathbf{q}}{dt} = -\tau^{-1}\left\{\mathbf{q} + K\operatorname{grad}T\right\}, \tag{31}
$$

$$\frac{d\mathbb{P}}{dt} = -\tau_p^{-1} \{\mathbb{P} - \eta_v \operatorname{grad} \mathbf{u}\}, \tag{32}$$

where τ and τ_p, are the relaxation times for $\mathbf{q}$ and $\mathbb{P}$ respectively, K is the thermal conductivity and η_v the viscosity. Examination of Eqs. (29-39) allows us to identify the unknown macroscopic coefficients, namely,

$$\alpha_{11} = -\frac{\tau}{KT}, \tag{33}$$

$$\alpha_{12} = \frac{\tau_p}{\eta_v}, \tag{34}$$

$$\mu_{12} = \frac{1}{K}, \tag{35}$$

$$\mu_{23} = \frac{1}{T\eta_v}. \tag{36}$$

Having relations (33-36) in mind and comparing now Eqs. (25) and (26) with Eqs. (27) and (28), we determine that

$$\beta_2 = -\frac{\tau}{K}\frac{p}{\rho}, \tag{37}$$

$$\mu_{11} = -\frac{5}{2}\beta_2, \tag{38}$$

$$\mu_{13} = \frac{\tau}{K}\frac{1}{\rho}, \tag{39}$$

$$\mu_{21} = \frac{2}{T}\frac{\tau_p}{\eta_v}, \tag{40}$$

$$2\mu = \eta_v. \tag{41}$$

Finally, as we saw in Section 2, coefficients X_{11} and X_{22} depend on the coefficients α_{11} and α_{22} as it can be seen in relation (11) and (12).

Some interesting comments are now pertinent. As we proposed in Eq. (18), the flux $\mathbf{J}_\eta$ was expressed in terms of the coefficients $\beta_1 = \beta_i(\epsilon, \rho)$ $i = 1, 2$, and we have seen that $\beta_1 = \frac{1}{T}$. Furthermore, equation (37) indicates that β_2 is a function of the state variables p and ρ as assumed in our theory. Also, we obtain expressions for the coefficients μ_{ij} which appear in the definition of the source term σ_η. We may see from relations (35), (36), (38-40) that these coefficients are positive. This assertion shows that in this case σ_η turns out to be positive definite which is not necessarily a requirement of the theory.

To complete the description of our system, we may define a steady state de-

scribed by the following conditions:

$$\tau \frac{d\mathbf{q}}{dt} = 0, \tag{42}$$

$$\tau_p \frac{d\mathbb{P}}{dt} = 0. \tag{43}$$

In this case we obtain the following constitutive equations for the steady state fluxes $\mathbf{q}_0$ and $\mathbb{P}_0$ that is

$$\mathbf{q}_0 = K \left\{ \mathrm{grad}\, \frac{1}{T} - \mu_{11}\mathbf{i} \times H \right\}, \tag{44}$$

$$\mathbb{P}_0 = \eta_v \left\{ - \mathrm{grad}\, \mathbf{u} - \mu_{21} T \mathbf{i}(\mathbf{u} \times \mathbf{H}) \right\}. \tag{45}$$

Notice should be made of the fact that Eq. (44) for the heat flux is consistent with the LIT heat flux for this system. This means that besides the usual Fourier heat conduction there is an additional heat flux due to the electric current and the magnetic field. This is known as the Ettinghausen effect [21]. In the constitutive relation for the steady state stress tensor we have a new term, the tensor $\mathbf{i}(\mathbf{u} \times \mathbf{H})$. Apparently, this result has not been thoroughly discussed in the literature.

4. Magnetohydrodynamic fluctuations in EIT

The purpose of this section is to compute the static correlation functions of the fluctuations of the state variables around a conveniently defined steady state using the standard Eisntein-Smoluchowski fluctuation theory [22–26]. This requires a number of important remarks. In the first place, we shall assume that the mathematical function η may be used in the same way as the entropy to define the probability for a given state to prevail around a steady state [26–27].

The fluctuations of the state variables around the steady state will be denoted by

$$\delta x = x - x_0, \tag{46}$$

where x_0 denotes the value of the variable x in the steady state, and x, its instantaneous value, stands here for any of the quantities ϵ, ρ, $\mathbf{q}$, $\mathbb{P}$.

Under these assumptions, one may formally manipulate the Einstein-Smoluchowski equation by expanding η around the steady state to show that in this case the probability ω that a fluctuation around a steady state occurs is given by [24].

$$\omega = \omega_0 \exp \left\{ -\frac{1}{2k} \delta^2 \eta \right\} \tag{47}$$

where ω_0 is a normalization constant. For our system, the probability ω is given by

the following expressions, namely,

$$
\omega \sim \exp\left\{-\frac{1}{2k}\left[\left(-\frac{1}{T^2}\frac{\partial T}{\partial \epsilon} + \frac{\partial X_{11}}{\partial \epsilon}\mathbf{q}_0\cdot\mathbf{q}_0\right)(\delta\epsilon)^2\right.\right.
$$

$$
+ \left(\frac{2T^{-1}p}{\rho^3} - \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(T^{-1}p) + \frac{\partial X_{22}}{\partial\rho}\mathbf{P}_0:\mathbf{P}_0\right)(\delta\rho)^2
$$

$$
+ \frac{T^{-1}}{\rho}\alpha_{11}\delta\mathbf{q}\cdot\delta\mathbf{q} + \frac{T^{-1}}{\rho}\alpha_{22}\delta\mathbf{P}:\delta\mathbf{P} + \left(\frac{\partial}{\partial\epsilon}\left(\frac{T^{-1}\alpha_{11}}{\rho}\right)\mathbf{q}_0\right)\cdot\delta\epsilon\delta\mathbf{q}
$$

$$
\left.\left.+ \left(\frac{\partial}{\partial\rho}\left(\frac{T^{-1}\alpha_{22}}{\rho}\right)\mathbf{P}_0\right):\delta\rho\delta\mathbf{P}\right]\right\}. \tag{48}
$$

From this last expression, introducing the steady state fluxes $\mathbf{q}_0$ and $\mathbf{P}_0$ defined in Eqs. (44) and (45) we may finally evaluate the static correlation functions for our variables, *i.e.*

$$
\langle\delta\epsilon\delta\epsilon\rangle = k\left(\frac{1}{T^2}\frac{\partial T}{\partial\epsilon} - \frac{\partial X_{11}}{\partial\epsilon}\mathbf{q}_0\cdot\mathbf{q}_0\right)^{-1}, \tag{49}
$$

$$
\langle\delta\rho\delta\rho\rangle = k\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(T^{-1}p) - \frac{2T^{-1}p}{\rho^3} - \frac{\partial X_{22}}{\partial\rho}\mathbf{P}_0:\mathbf{P}_0\right)^{-1}, \tag{50}
$$

$$
\langle\delta q_i\delta q_j\rangle = \frac{\rho k T^2 K}{\tau}\delta_{ij}, \tag{51}
$$

$$
\langle\delta P_{ij}\delta_{em}\rangle = -\frac{\rho k T\eta_v}{\tau_p}\left(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm}\right). \tag{52}
$$

It is interesting to see that the function for both the fluctuations in the heat flux and in the stress tensor are the same as the ones obtained in LIT by Landau-Lifshitz [23] for a simple fluid in the absence of a magnetic field. This means that the existence of an external magnetic field does not effect the value of these correlation functions in the case of an ionized fluid, a fact which seems not to be mentioned in the literature. On the other hand, the correlation functions for the fluctuations of the conserved variables have each an extra term if compared with the classical results. These new terms are in both cases second order terms of the steady state fluxes which include the external magnetic field. This implies that the influence of the latter one is exerted only through the fluctuations of the energy and the density. This also seems to be a new result.

5. Conclusions

The main issue of this paper is to show that EIT is a suitable theory to obtain the time evolution equations for the non conserved variables that are chosen to describe the state of a thermodynamic non equilibrium state of a given system. In particular, the case of an ionized fluid subject to an external magnetic field is treated in full detail. The resulting evolution equations may be compared in a satisfactory way with equations for the same quantities obtained from the thirteen moment method solution to the Boltzmann equation. This comparison enables us to associate the phenomenological coefficients we introduce in our theory with well established kinetic quantities. By using an ad hoc extension of the standard Einstein-Smoluchowski theory of thermodynamic fluctuations, we may devise a method which allows us to obtain the static correlation functions for the state variables. The results of this calculations show some features of these functions which to our knowledge have not been mentioned in the literature.

References

1. H. Grad, in *Handbuch der Physik*, ed. by S. Flugge, Vol. XII Springer, Berlin, (1958).
2. J. Kestin, "Irreversible Thermodynamics" in *Jahrbuch 1984/85 Wissenschaftkollege zu Berlin*, P. Wapnewski, ed., Siedler Verlag, Berlin (1986).
3. D. Jou, C. Pérez-García, L.S. García-Colín, M. López de Haro and R.F. Rodríguez, *Phys. Rev.* **31A** (1985) 2502.
4. C. Pérez-García, D. Jou, M. López de Haro, R.F. Rodríguez and L.S. García-Colín, *Physica* **135A** (1986) 251.
5. R.F. Rodríguez, L.S. García-Colín and M. López de Haro, *J. Chem. Phys.* **83** (1985) 4099.
6. R.F. Rodríguez, M. López de Haro and L.F. del Castillo, J. *Non-Equilib. Thermodyn.* **10** (1985) 1.
7. G. Lebon, C. Pérez-García and J. Casas-Vázquez, *Physica* **137A** (1986) 531.
8. L.S. García-Colín, S. M. T. de la Selva and E. Piña, *J. Phys. Chem.* **90** (1986) 953.
9. S.M.T. de la Selva and L.S. García-Colín, *J. Chem. Phys.* **85** (1986) 2140.
10. R.F. Rodríguez, L.S. García-Colín and L.F. del Castillo, *J. Chem. Phys.* **86** (1987) 4208.
11. M. López de Haro, R.F. Rodríguez and L.S. García-Colín, *J. Chem. Phys.* **86** (1987) 4216.
12. L.F. del Castillo and L.S. García-Colín, *Phys. Rev.* **B34** (1986) 4944.
13. L.S. García-Colín and L.F. del Castillo, *Phys. Rev.* **B37** (1988) 448.
14. M. López de Haro, L.F. del Castillo and R.F. Rodríguez, *Reol. Acta* **25** (1986) 207.
15. H.T. Yang, *Phys. of Fluids* **5** (1962) 1580.
16. S.R. de Groot and P. Mazur, *Non Equilibrium Thermodynamics*. Dover Publications, New York, (1984).
17. M. López de Haro and R.F. Rodríguez, *J. of Non-Equil. Thermodyn.* **13** (1988) (in press).
18. J.C. Maxwell, *The Collected Papers of J.C. Maxwell* ed. by W.D. Niven, Dover Publications, New York (1965).
19. C. Cattaneo, *Atti del Seminario Matemático e Físico della Universitá di Moderna*, **3** (1948); *C.R. Acad. Sci. Paris* **247** (1958) 431.
20. P. Vernotte, *C.R. Acad. Sci. Paris* **246** (1958) 3154.

21. S. Chapman and T.G. Cowling, *The Mathematical Theory of Non Uniform Gases.* Cambridge Unviersity Press, Cambridge (1952).
22. L. Onsager and S. Machlup, *Phys. Rev.* **91** (1953) 1505.
23. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics.* Pergamon Press Ltd. London (1979).
24. H.B. Callen, *Thermodynamics and an Introduction to Thermostatics.* John Wiley and Sons, New York 2nd. edtion (1985).
25. L.D. Landau and E.M. Lifshitz, *Statistical Physics.* Vol. I Pergamon Press, London (1980).
26. L.S. García-Colín, *Selected topics in Non Equilibrium Phenomena.* Universidade Stadual de Campinas, Sao Paulo, Brasil, (1984).
27. D. Jou, E. Llebot and J. Casas Vázquez, *Phys. Rev.* **25 A** (1982) 508.

Resumen. Utilizando la termodinámica irreversible extendida, se obtienen ecuaciones de evolución temporal para el flujo de calor y el tensor de esfuerzos en un fluido ionizado sujeto a un campo magnético externo. Estas ecuaciones pueden ser comparadas con aquellas que se obtienen a partir de la solución de los trece momentos de la ecuación de Boltzmann. Esta comparación proporciona expresiones explícitas para los coeficientes fenomenológicos relevantes en términos de cantidades moleculares. Se calculan además las funciones de correlación estáticas de las fluctuaciones de las variables de estado que describen al fluido.