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Note on supersymmetry in quantum mechanics and the hydrogen atom in one dimension

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Abstract. We point-out and correct some erroneous beliefs associated with the quantum solution of the hydrogen atom in one dimension. We show that no state of infinite energy exist in the problem. We argue against a widely spread belief on the supersymmetric nature of this system. In particular, we show that the one-dimensional Coulomb potential is not its own supersymmetric partner.

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1. Introduction

Recently [1,10,15,19,20,22], there has been a great interest in the use of supersymmetric ideas and techniques [2] in nonrelativistic quantum mechanics. Among the examples discussed in this context, the problem of an atom in an external magnetic field has been studied frequently, specially in the limit of an infinitely strong field [10,19]. In this case the problem is completely equivalent to the so called one-dimensional hydrogen atom [3,14,17], a system described by the Hamiltonian (in atomic units: $\hbar = m_e = e = 1$)

$$H = \frac{p^2}{2} - \frac{1}{|x|}. \tag{1}$$

The hydrogen atom in one dimension has been studied due to the very peculiar and interesting features of its quantum solution [3-12,16,18] arising from the singular nature of H. The singularity has caused a great deal of confusion on the properties of

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its spectrum. For example, it has been argued that this is the only system in which the non-degeneracy theorem for the bound states of one-dimensional systems is violated [10]. Supersymmetry has been invoked as the explanation of the breakdown of the non-degeneracy theorem [10,19]. This is a typical example of the erroneous argumentations appearing in recent works on the subject. Our purpose here is to point out and correct some of such misbeliefs.

The origin of most of the erroneous conclusions on the behaviour of the hydrogen atom in one dimension is the belief that its ground state is a nondegenerate state of infinite binding energy [3,6,8-10,19]. The existence of this state together with the twofold degeneracy of the excited levels has been regarded as producing a supersymmetric pattern in its energy spectrum [10,19]. However, this statement is not correct since the alleged ground state does not exist. Even when the nonexistence of the ground state has been taken into account, it has been claimed that supersymmetry is spontaneously broken [4,5], or that the one-dimensional Coulomb potential [13] is its own supersymmetric partner [9]. Both of these claims are also incorrect as we shall prove in what follows.

2. On the infinite energy ground state

The belief in an infinite energy ground state for the hydrogen atom in one dimension can be traced back to Loudon's work [3] where a mistake in taking a limit gave birth to it. But, if this state were to exist then the Hamiltonian operator of the system would become non-hermitian and hence not well defined in any quantum mechanical sense [4,5]. We have previously proved that such ground state cannot exist [7]. But, given the number of people still believing in its existence, we will next give a very general proof of the nonexistence of any state with infinite energy in the hydrogen atom in one dimension. The most direct way of doing this is to analize, as Loudon originally did, the eigenstates of the potential

$$V_\rho(x) = \frac{-1}{|x| + \rho}, \quad \rho \geq 0, \tag{2}$$

which, in the limit $\rho \to 0$, becomes the one-dimensional Coulomb potential. Thus, we will study the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{|x| + \rho}, \tag{3}$$

and consider it as a function of the parameter ρ. The first thing to notice is that $H(\rho)$ satisfies

$$H(\rho) > H(0), \tag{4}$$

but, since the kinetic energy operator is positive definite, this means that the bound state energy levels must satisfy the inequality

$$E_n(\rho) > -\frac{1}{\rho}. \tag{5}$$

This result proves our point.

The nonexistence of an infinite energy ground state can also be demonstrated by a direct calculation: In the limit $\rho \to 0$, the ground state of the Hamiltonian (3) has been shown to be [3]

$$\Psi_0(x) = \lim_{\alpha \to 0} \alpha^{-1/2} \exp(-\sqrt{2}|x|/\alpha), \tag{6}$$

where we have introduced the parameter $\alpha^2 = -1/E$. Using this expression, it is very easy to show that the alleged ground state vanishes everywhere [7,11]. It is somewhat annoying that despite the proofs that have been given of the nonexistence of such state [9,11,16], it continues to exist in the minds of several authors.

3. Supersymmetry and the hydrogen atom in one dimension

It is not difficult to calculate explicitly the supersymmetric partner of the one-dimensional Coulomb potential; to this end we recall [4] that the energy eigenvalues of the one-dimensional hydrogen atom are

$$E_n = -\frac{1}{2n^2}, \quad n = 1, 2, 3, \ldots \tag{7}$$

The corresponding doubly degenerate eigenfunctions, which we give here for the sake of completeness, are [4]:

$$\psi_n^+(x) = \begin{cases} 2n^{-3/2}(-1)^{n-1} x L_{n-1}^1(2x/n) e^{-x/n} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0; \end{cases} \tag{8a}$$

and

$$\psi_n^-(x) = \begin{cases} 0 & \text{if } x > 0, \\ 2n^{-3/2}(-1)^{n} x L_{n-1}^1(-2x/n) e^{x/n} & \text{if } x \leq 0, \end{cases} \tag{8b}$$

where $L_{n-1}^1(x)$ is an associated Laguerre polynomial. Notice the vanishing of the eigenfunctions at $x = 0$, and the explicit separation between the $x > 0$ and the $x < 0$ regions which appears there. This means, in fact, that neither the ψ_m^+ states can be defined for $x < 0$, nor the ψ_m^- states can be defined for $x > 0$ [5,18].

As the ground state energy is $E_1 = -1/2$, it suffices to write an energy-shifted Coulomb potential

$$V_c(x) = -\frac{1}{|x|} + \frac{1}{2} \tag{9}$$

to assure the non-negativity of the problem's spectrum. Now, the equation for the superpotential is [15]

$$W'(x) + W^2(x) + 2V_c(x) = 0, \tag{10}$$

solving it, we obtain $W(x)$ as

$$W(x) = \frac{1}{x} - \ \mathrm{sgn}(x). \tag{11}$$

From this, the partner potential is easily found to be

$$V_p(x) = -\frac{1}{|x|} + \frac{1}{x^2} + \frac{1}{2}. \tag{12}$$

This result shows explicitly that the one-dimensional Coulomb potential is not its own supersymmetric partner. Besides, as potential (9) admit normalizable states of zero energy [4,8], the hydrogen atom in one dimension cannot be considered as a system in which supersymmetry is spontaneously broken [15].

4. Conclusion

The arguments given in this work, taken together with previous results [4], show that all the states of the hydrogen atom in one dimension are degenerate and of finite energy. Therefore, as long as their eigenfunctions are required to vanish at $x = 0$, we can ascertain the Hermiticity of its Hamiltonian [4,7,9]. In fact, the vanishing of eigenfunctions at the origin is intimately related to one of the most curious features of the one-dimensional hydrogen atom, namely the absence of even or odd eigenstates [4] despite of the obvious symmetry of its Hamiltonian. A lack of understanding of this fact has produced a long controversy on the properties of its solution [3-9,11,12,16]. As Núñez Yépez *et al.* have shown recently [5,18], such spontaneous breaking of parity is a consequence of a dynamically induced superselection rule operating on the system. This rule is also the cause of the breakdown of the non-degeneracy theorem, which is in no way related to supersymmetry [21]. We have calculated the supersymmetric partner of the $-1/|x|$ potential and have found that supersymmetry cannot be broken in the hydrogen atom in one dimension —as a glance to the form of $W(x)$ suffices to show.

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Resumen. Señalamos y corregimos algunos de los errores en que se ha incurrido al resolver el problema del átomo de hidrógeno unidimensional. Demostramos que ningún estado de éste tiene energía de enlace infinita. Damos argumentos que contradicen el punto de vista usual sobre la naturaleza supersimétrica del sistema. Demostramos, en particular, que el potencial unidimensional de Coulomb no es igual a su compañero supersimétrico.