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Higher order equations of motion*

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Abstract. We discuss the possibility that the motion of elementary particles is described by higher order differential equations induced by supersymmetry in higher dimensional space–time. We take the specific example of six dimensions writing the corresponding Lagrangian and equations of motion.

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One of the main sources of ideas and discussions in the last twenty years or so, has been supersymmetry; specially since the proof of the Haag, Lopurzanski, Sohnius-theorem [1]. In particular, the Wess-Zumino model [2] has served as a basis for the construction of "relativistic" lagrangian theories that could describe the physical world.

The usual rule for writing a supersymmetric Lagrangian for a chiral superfield [3], for example, is to take

$$\phi(\theta,\overline{\theta},x)=e^{\frac{i}{2}\theta\partial\overline{\theta}}\phi_0(\theta,x) \tag{1}$$

and write the kinetic lagrangian as

$$\mathcal{L}=(\overline{\phi}\phi)_D, \tag{2}$$

where $(\ )_D$ means the highest component (maximum possible number of Grassmann variables).

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We have

$$\overline{\phi}\phi = \overline{\phi}_0 e^{i\theta\partial\overline{\theta}}\phi_0 + \text{total divergence}$$

$$\cong \overline{\phi}_0 \sum_{s=0} \frac{(i\theta\partial\overline{\theta})^S}{S!}\phi_0. \tag{3}$$

Now, the Grassmann variable θ_α is a Weyl spinor having

$$\omega = 2^{\frac{\nu}{2}-1} \tag{4}$$

independent components in ν dimensions (In what follows we will only consider ν = even number).

The sum over S in Eq. (3) runs then from $S = 0$ to $S = \omega$. In four dimensions S runs from zero to two and this leads us to Lagrangians which imply at most second order wave equations. But this is not so for higher number of dimensions. In fact, the same construction leads again to expression (3) but now in six dimensions. For example, where $\omega = 4$, the equations of motion are of the fourth order and of even greater order for $\nu > 6$. The mass term is introduced via a Lagrangian which is proportional to the square of the chiral field (3)

$$\mathcal{L}_{\mathrm{m}} = C\phi_0^2\big|_F + \mathrm{hc}, \tag{5}$$

where F is the coefficient of the maximum number of θ–variables.

It is easy to see that by defining the components $\psi_{\alpha_1\cdots\alpha_S}$ as

$$\phi_0 = \sum_{S=0} \frac{1}{S!}\theta^{\alpha_1}\cdots\theta^{\alpha_S}\psi_{\alpha_1\cdots\alpha_S}(x) \tag{6}$$

the mass Lagrangian is given by

$$\mathcal{L}_{\mathrm{m}} = C\sum_{S=0}^{\omega} \psi_{\alpha_1\cdots\alpha_S}\psi_{\alpha_{S+1}\cdots\alpha_\omega} + \mathrm{hc}. \tag{7}$$

Eqs. (2) and (5) [or (7)] lead, for each component X to an equation of motion [4]

$$\left(\Box^{\frac{\omega}{2}} - m^\omega\right) X = 0, \tag{8}$$

where C, for dimensional reasons, has been substituted by m^ω.

In four dimensions, (8) is the Klein-Gordon equation, and in six, for example, we have

$$(\Box^2 - m^4)X = 0 \qquad (\Box - m^2)(\Box + m^2)X = 0. \tag{9}$$

The massive propagator is of course

$$\frac{1}{(p^2)^{\omega/2} - m^\omega} \quad \text{and} \quad \frac{1}{(p^2)^{\omega/2}} \quad \text{for massless particles.} \tag{10}$$

The Fourier transforms for the massless case have the form [4]

$$G = \frac{1}{R^2} \quad \text{for } \nu = 4 \text{ or } \nu = 6, \tag{11}$$

$$G = \ln R \qquad \nu = 8, \tag{12}$$

$$G = R^\alpha \ln R \quad \text{with } \alpha = \omega - \nu > 0 \text{ for } \nu > 8. \tag{13}$$

We see that due to the fact that $\alpha > 0$ for $\nu > 8$ the convolution of two of these functions has not ultraviolet divergences. The same happens for the massive cases as for $p \to \infty$ the mass term has no importance.

In this case (massive particle) there are no infrared divergences, neither. This absence of singularities is due to the fact that in a convolution between two Green-functions the number of integration variables grows linearly with the number of dimensions, while the denominators grow exponentially [see Eq. (10)], so the power of p in the denominator outnumbers those in the numerator (for $\nu > 8$). For example, while in eight dimensions we have $\omega = \nu = 8$, in ten dimensions we have $\omega = 16$ ($\nu = 10$) (not to speak of $\nu = 26$!). These facts show that it is worth while to look carefully into theories containing higher order equations such as that given by Eq. (9).

However, it is well known that this kind of equations presents considerable difficulties both of a mathematical nature and of physical interpretation (see Hawking [5]). It is advisable then to study first higher order equations in the case of only one significative variable, where it is possible to address the question of unitarity in a controlled way, describing the scattering data in a precise mathematical language [6,7] and looking at the physical implications of the scattering processes.

The general linear differential equation for only one significative variable takes the form

$$\frac{d^n \phi}{dx^n} + q_{n-2}\frac{d^{n-2}\phi}{dx^{n-2}} + \cdots + q_0\phi = z^n \phi, \tag{14}$$

where a possible $q_{n-1}(d^{n-1}\phi/dx^{n-1})$ has been eliminated by means of a transformation $\psi \to f\psi$.

This equation has $n-1$ independent potentials q_i, $(i = 0 \dots n-2)$ when they are well-behaved they tend to zero sufficiently rapid for $x \to \pm\infty$; so that asymptotically (14) tends to

$$\frac{d^n \phi^{(0)}}{dx^n} = z^n \phi^{(0)}, \tag{15}$$

whose solution is

$$\phi_i^{(0)} = e^{\alpha_i z x}, \tag{16}$$

with $\alpha_i^n = 1$, α_i being thus a n^{th} root of unity.

Eq. (15) of course, has n independent solutions that can be expressed in terms of a set of basic linearly independent solutions defined by conditions on $x = \pm\infty$.

We define the first Jost function $f_1(zx)$ as the solution of (15) with the greatest rate of decrease for $x \to -\infty$.

The second Jost function $f_2(zx)$ is that solution of (14) which, for $x \to -\infty$ has the second rate of decrease (and for $x \to \infty$ has the second rate of increase), etc., etc. [8].

In other words

$$f_j(zx) \to e^{\alpha_j z x}, \tag{17}$$

where the roots are ordered in such a way that

$$\operatorname{Re}\alpha_1 z > \operatorname{Re}\alpha_2 z > \cdots > \operatorname{Re}\alpha_n z. \tag{18}$$

It is clear that (18) divides the z-plane in $2n$ regions, within each region the inequality (18) is well defined but as arg z is varied, there are lines for which $\operatorname{Re}\alpha_\ell^z = \operatorname{Re}\alpha_{\ell+1}z$ and this defines a ray on which the order of roots is ill defined. There are $2n$ rays dividing the z-plane in regions with an angle $(2\pi/2n) = (\pi/n)$. In particular for fourth order differential equations, the z-plane is divided in eight "octants" each one with an angle of $\pi/4$ radians [9].

The Jost functions have discontinuities at those rays and furthermore they can have poles with corresponding residues. the set of all discontinuities, including poles and residues, form the so called "scattering data" of the differential equation (14). It is shown by mathematicians that the knowledge of the scattering data is equivalent to the knowledge of the diffential equation. In other words the set of discontinuities (including poles and residues) determines the $n-1$ potentials q_i of the equation.

However from the point of view of a physicist, not all the scattering data are physical. For instance, in a fourth order differential equation, only the real and imaginary axis correspond to physical data (*i.e.*, a plane wave going to plane wave) [9]. The other rays at $\pi/4$ and $\pm 3\pi/4$ correspond to scattering of waves that grow exponentially (or decrease) for $x \to \pm\infty$. These are unphysical data. Therefore, in order to make the equation a physical one, the Jost functions must not present discontinuities on these rays. This imply relations to be satisfied by the potentials of Eq. (14) in order for it to be physically acceptable.

Summarizing: Any higher order equation of motion can not in principle be thought as having physical significance unless some specific relations exist among the coefficients. One hopes that supersymmetric theories may provide the clue to physically meaningful higher order equations. Any way, we want to mention that by using the method $\nu \to \infty$ of Ref. [10] in a higher order equation, we get the static limit in leading approximation and, as a second approximation a second order equation is obtained. The higher order derivatives appear in the following approximations (in a ν^{-1} development). In this sense we can say that a higher order equation has a second order equation as an approximation.

With these motivating ideas we start looking for a supersymmetric theory in six dimensions which is the simplest higher order case. In this respect we like to point out that a more realistic theory that the one we are presenting here can be found in works by P. Fayet [11].

We want to find the coupling between the chiral superfield, Eq. (1), and a gauge superfield [12]

$$V = \sum_{s,t=2}^{4} \overline{\theta}_{\dot{\alpha}_1} \cdots \overline{\theta}_{\dot{\alpha}_4} A^{\dot{\alpha}_1 \cdots \dot{\alpha}_5}_{\alpha_1 \cdots \alpha_t} \theta^{\alpha_1} \ldots \theta^{\alpha_t}, \tag{19}$$

where we have chosen the Wess-Zumino gauge [12].

A chiral superfield strength for this field is given by

$$W_{\alpha_1\alpha_2} = \overset{4}{\overline{D}} D_{\alpha_1} D_{\alpha_2} V \tag{20}$$

and the corresponding Lagrangian is

$$\mathcal{L}_W = \epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} W_{\alpha_1\alpha_2} W_{\alpha_3\alpha_4} + \text{hc}.$$

By reducing the spinor components of the gauge superfield defined by (20), using Elie Cartan's reduction recipes, we found the following tensor components.

$A_{(\mu\nu)}$ (Graviton field) $(\mu\nu)$ means symmetric part and the gauge transformation with parameters λ_μ are

$$A'_{(\mu\nu)} = A_{(\mu\nu)} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu - \eta_{\mu\nu} \partial^\alpha \lambda_\alpha \tag{21}$$

This gauge transformation can be used to simplify the Lagrangian by choosing the De Donder gauge in which $\partial_\mu A^\nu_\nu = 2\partial^\nu A_{(\mu\nu)}$. In this gauge

$$\mathcal{L}'_{22} \cong \Box A_{(\mu\nu)} \Box A_{(\mu\nu)}. \tag{22}$$

$A_{[\mu\nu]}$ ($[\mu\nu]$ = antisymmetric part) has the gauge transformation

$$A'_{[\mu\nu]} = A_{[\mu\nu]} + \partial^\rho \lambda_{\rho\mu\nu}, \tag{23}$$

$\lambda_{\rho\mu\nu}$ = self dual and completely antisymmetric three-vector. The corresponding Lagrangian only contains the divergence of this tensor, which is gauge-invariant

$$\mathcal{L}''_{22} \simeq \partial_\rho \partial^\mu A_{[\mu\nu]} \partial^\rho \partial_\sigma A^{[\sigma\nu]}. \tag{24}$$

$-A_\mu^\alpha$ (gravitino field). Under a gauge transformation it transforms as

$$A'^\alpha_\mu = A^\alpha_\mu + (C\tilde{\partial}\gamma_\mu\tilde{\partial})^{\alpha\beta}\lambda_\beta, \tag{25}$$

where λ_β is a spinor parameter, which can be adjusted so as to have a "zero gamma trace" gauge

$$(\gamma^\mu C)_{\alpha\beta} A^\beta_\mu = 0. \quad (C = \text{transposition matrix})$$

In this gauge, the Lagrangian is

$$\mathcal{L}_{2,3} \simeq 4i\partial^\mu A^\alpha_\mu \partial^{\dot\alpha}_\alpha \partial_\nu \overline{A}^\nu_{\dot\alpha} + i\Box \partial^{\dot\alpha}_\alpha \overline{A}^\mu_{\dot\alpha} A^\alpha_\mu. \tag{26}$$

$-B_\mu$ Complex gauge invariant vector field. Its Lagrangian is

$$\mathcal{L}_{2,4} \simeq 2\overline{B}^\mu \partial_\mu \partial^\nu B_\nu - \overline{B}^\mu \Box B_\mu. \tag{27}$$

$-A_\mu$ real vector field with gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \Lambda, \tag{28}$$

and the Maxwell type Lagrangian

$$\mathcal{L}'_{33} = F^{\mu\nu}F_{\mu\nu}; \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{29}$$

$A_{\lambda\mu\rho}$ real gauge invariant self-dual three-vector field. The Lagrangian is

$$\mathcal{L}''_{33} = G^{\mu\nu}G_{\mu\nu}; G_{\mu\nu} = \partial^\rho A_{\rho\mu\nu}.$$

B^α photon field. A complex gauge invariant spinor field with Lagrangian

$$\mathcal{L}_{3,4} \simeq iB^\alpha \partial^{\dot\alpha}_\alpha \overline{B}_{\dot\alpha}.$$

D Finally: an auxiliary gauge invariant scalar field with Lagrangian

$$\mathcal{L}_{44} \simeq D^2.$$

Of course, for a more realistic theory it is necessary to work with non-abelian gauge groups, in particular Yang-Mills type theories for the standard model and also introducing supergravity; but our aim is not so much to construct a realistic theory, but rather to show the plausibility of using higher order equations in a theory that could be renormalizable and unitary.

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Resumen. Discutimos la posibilidad de describir el movimiento de partículas elementales por medio de las ecuaciones diferenciales de mayor orden inducidas por la supersimetría en el espacio-tiempo de un número mayor de dimensiones. Escribimos el Lagrangiano correspondiente y las ecuaciones de movimiento para el ejemplo específico de tomar seis dimensiones.