Obtention of the invariant for the time-dependent harmonic oscillator via point transformations

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Abstract. We present an alternative method for the obtention of an auxiliary equation and of the invariant for the time-dependent harmonic oscillator using point transformations.

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1. Introduction

It has been shown by some authors [1,3] that a conserved quantity for the time dependent harmonic oscillator is

$$I = \tfrac{1}{2}\left[\frac{x^2}{\sigma^2} + (\sigma\dot{x} - \dot{\sigma}x)^2\right], \tag{1}$$

where x satisfies the harmonic oscillator equation

$$\ddot{x} + \Omega^2(t)x = 0, \tag{2}$$

and σ is a solution of the auxiliary equation

$$\ddot{\sigma} + \Omega^2(t)\sigma = \frac{1}{\sigma^3}. \tag{3}$$

However, there isn't any physical interpretation of equation (1) in the papers cited above, and a transparent method to obtain the auxiliary equation which is needed in order to get the invariant. Precisely, this kind of lacks have motivated the work reported in this paper, where, in addition, we show an alternative way to obtain the invariant using a simple point transformation. Another relevance of this

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work lies on the method proposed to obtain the auxiliary equation, which can be generalized to more complicated systems.

In Section 2 we obtain the invariant and the auxiliary equation. In Section 3 we discuss the results.

2. Obtention of the auxiliary equation and the invariant

We start with the well-known harmonic oscillator equation

$$\ddot{x} + \Omega^2(t)x = 0, \tag{4}$$

where $\Omega(t)$ is the frequency with time dependence.

Substituting the point transformation

$$x = \sigma(t)y, \tag{5}$$

Eq. (4) becomes

$$[\ddot{\sigma} + \Omega^2(t)\sigma]y + \sigma\ddot{y} + 2\dot{\sigma}\dot{y} = 0. \tag{6}$$

The variable y can be eliminated using Eq. (5) again, and multiplication by $(\sigma\dot{x} - \dot{\sigma}x)$ yields

$$(\sigma\dot{x} - \dot{\sigma}x)x[\ddot{\sigma} + \Omega^2(t)\sigma] + \frac{d}{dt}\left(\frac{(\sigma\dot{x} - \dot{\sigma}x)^2}{2}\right) = 0. \tag{7}$$

The right hand term of Eq. (7) is a total derivative and the whole expression will be an invariant if there is a function $F(x,\sigma)$ provided that

$$\frac{dF(x,\sigma)}{dt} = (\sigma\dot{x} - \dot{\sigma}x)x(\ddot{\sigma} + \Omega^2(t)\sigma) \tag{8}$$

is hold.

This request is satisfied by the conditions

$$\frac{dF(x,\sigma)}{dt} = \frac{d\mu}{dt}f(\mu), \tag{9}$$

where μ is a new variable for x/σ. A consequence of (9) is that the functions $F(x,\sigma)$ and $f(\mu)$ must be related by

$$F(x,\sigma) = \int^{x/\sigma} f(\mu)\,d\mu. \tag{10}$$

Also, the following auxiliary equation must be satisfied

$$\ddot{\sigma} + \Omega^2(t)\sigma = \frac{f(x/\sigma)}{\sigma^2 x}. \tag{11}$$

Using Eqs. (8) and (9), one can find immediately the following form for the invariant

$$I = \tfrac{1}{2}\sigma^2 \left(\dot{x} - \frac{\dot{\sigma}}{\sigma}x\right)^2 + \int^{x/\sigma} f(\mu)\, d\mu. \tag{12}$$

3. Concluding remarks

Using only a point transformation [Eq. (5)] we have reproduced the results obtained by Ray and Reid [1,2] via other methods. If we substitute $f(\mu) = k\mu$, with k an arbitrary constant, in Eq. (12), the invariant obtained by Lutzky [4] and Ray and Reid [1,2] follows immediately. Otherwise the auxiliary equation in this work is a differential equation for the scale factor $\sigma(t)$, which makes permissible the point transformation (5). In this context the invariant (12) contains the kinetic energy in mass units, potential energy also in mass units but affected by a new frequency expressed by $\dot{\sigma}/\sigma$ and related with $\Omega(t)$ via the auxiliary equation. The factor $\int^{x/\sigma} f(\mu)\, d\mu$ appears as a compensation to the potential energy and σ^2 as an inflationary term.

References

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Resumen. Se presenta un método alternativo para la obtención de la ecuación auxiliar y del invariante de un oscilador armónico dependiente del tiempo, usando transformaciones puntuales.