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On the theory of ideal relativistic quantum gases

Honorio Vera Mendoza
Escuela de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, Puebla, Pue. 72000, México

Masaki Hayashi
Department of Physics, Saitama Medical College, Moroyama, Saitama 350-04, Japan

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Abstract. The exact formalism of microcanonical distribution is used to describe the ideal relativistic quantum gases. A statistical cluster decomposition is developed for the calculation of the phase space integrals with the correct statistics. Analogies with the real classical gases as well as the applications of the formalism are discussed.

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1. Introduction

The theory of ideal (relativistic) quantum gases has been often described in the literature [1,2]. In particular, the general theory for Bose and Fermi gases has been developed on the basis of the canonical [3] and the grand canonical ensemble [4].

From these approaches, which are the thermodynamical ones, one can develop for the ideal quantum gases a formalism very similar to that for real classical gases, such as a cluster decomposition and a virial expansion. These analogies exist since ideal Bose and Fermi gases have the properties of real gases (*i.e.* with interactions) if considered from the standpoint of classical statistics. Little attention, however, has been paid on the formalism based on the microcanonical distribution, where the conservation of energy and momentum are fully taken into account (see, however, [5,6,7]).

In this paper we develop a general statistical approach for the ideal relativistic quantum gases based on the fully microcanonical distribution. While this formalism can be used for the calculation of the phase space integrals, the other previously considered thermodynamical ones are not applicable to this problem, since they lack of the constraints of energy-momentum conservation. We show that one can develop an exact statistical cluster decomposition for the ideal relativistic quantum gases in the formalism of the microcanonical distribution. Corrections to the usual phase space integrals due to the quantum statistical effects are given. The quantum statistics has effects on problems such as the statistical bootstrap model [8,9], as well. The formalism developed in this paper for the calculation of the phase space integrals with the cluster decomposition can be directly applied in the Frautschi formulation [9] of the statistical bootstrap model [8] with the correct quantum statistics [10]. Systems such as neutron stars should be considered in accord with the correct statistics, due to Pauli's principle.

In Section 2 we study the expressions for the phase space integrals, taking fully into account the quantum statistics. Our treatment is a strict relativistic one and we evaluate them in a closed form for two limiting cases, namely, in the extreme relativistic limit and nonrelativistic limit; in the general relativistic case, we express it in the form of a one dimensional integral over modified Bessel functions.

We derive an exact statistical cluster decomposition for the Bose and Fermi gases. In the most general case, we give the corrections to the usual phase space integral due to the quantum statistical effects.

In Section 3 we discuss the ideal relativistic quantum gas system in the thermodynamical description for completeness. We derive an exact expression for the coefficients of the cluster decomposition in terms of the modified Bessel functions.

The virial expansion then follows directly from these exact expressions. In the nonrelativistic case our exact formulas reduce to the ones already given in the literature [1–4].

2. Microcanonical formalism

Taking fully into account the different type of statistics, we consider an ideal relativistic gas system consisting of ν–kinds of noninteracting particles, each of which has $N_1, N_2, \ldots, N_\nu$ particles with masses $\mu_1, \mu_2, \ldots, \mu_\nu$, respectively. Let a set of vectors $\mathbf{q}_j(\mathbf{q}'_j, \mathbf{q}''_j, \ldots)$ characterize all possible momenta and polarization of every spin states of a particle of j-th kind. Since the particles are noninteracting, the set of vectors $\mathbf{q}_j$ characterizes all the possible states of the given system, and the corresponding set of occupation numbers $n_j(\mathbf{q}_j)$ defines completely the state of the whole system. We start with the precise expression of the phase space volume of the system with fixed total energy-momentum four vector $Q = (E, \mathbf{Q})$; it is given by [6]

$$\sigma_{N_1\ldots N_\nu}(Q) = \sum_{\{n\}} \delta^4 \left\{ Q - \sum_{j=1}^{\nu} \sum_{\mathbf{q}_j} q_j n_j(\mathbf{q}_j) \right\} \prod_{j=1}^{\nu} \delta \left\{ N_j - \sum_{\mathbf{q}_j} n_j(\mathbf{q}_j) \right\} \Omega_j\{n_j(\mathbf{q}_j)\}, \tag{1}$$

where q_j is the energy-momentum four-vector $q_j = (\epsilon_j, \mathbf{q}_j)$ and the summation over $\mathbf{q}_j$ runs over all possible values of this vector and polarization states. $\Omega_j\{n_j(\mathbf{q}_j)\}$ denotes the degeneracy of a state with given $\mathbf{q}_j$. Both for Bose and Fermi gases, we have $\Omega_j = 1$, on account of the indistinguishability of particles, while for a gas obeying Boltzmann statistics we have

$$\Omega_j\{n_j(\mathbf{q}_j)\} = \frac{N_j!}{\prod_{\mathbf{q}_j} n_j(\mathbf{q}_j)!}. \tag{2}$$

The summation over $\{n\}$ in Eqs. (1) and (2) runs over all possible values of the occupation numbers. In the case of Bose-Einstein and Boltzmann statistics $\{n\}$ runs over $0, 1, 2, \ldots, \infty$, while in the case of Fermi-Dirac statistics $\{n\}$ runs over 0 and 1 only.

The problem consists on the calculation of $\sigma_{N_1 \ldots N_\nu}(Q)$ in Eq. (1). In order to carry this out, we define the 4–dimensional Laplace-transform, or generating function of $\sigma_{N_1 \ldots N_\nu}(Q)$ [5]

$$G_{N_1 \ldots N_\nu}(\beta) = \int e^{-\beta Q} d^4 Q \sigma_{N_1 \ldots N_\nu}(Q)$$

$$= \sum_{\{n\}} \prod_{j=1}^{\nu} \exp\left(-\sum_{\mathbf{q}_j} \beta q_j n_j\right) \delta\left\{N_j - \sum_{\mathbf{q}_j} n_j(\mathbf{q}_j)\right\} \sum_{\mathbf{q}_j} \Omega_j\{n_j(\mathbf{q}_j)\}, \tag{3}$$

where β is a time-like four vector. In thermodynamical language, β is an inverse temperature $\beta = \frac{1}{kT}$, with k denoting the Boltzmann constant. Using the Fourier-representation for the Kronecker δ

$$\delta\left\{N_j - \sum_{\mathbf{q}_j} n_j(\mathbf{q}_j)\right\} = \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{i\alpha N_j - i \sum_{\mathbf{q}_j} n_j(\mathbf{q}_j)\alpha\right\} d\alpha, \tag{4}$$

we get

$$G_{N_1 \ldots N_\nu}(\beta) = \prod_{j=1}^{\nu} G_{N_j}(\beta), \tag{5}$$

where

$$G_{N_j}(\beta) = \frac{1}{2\pi} \int_0^{2\pi} \exp\{iN_j\alpha + \Phi_j(\beta, \alpha)\} d\alpha, \tag{6}$$

and $\Phi_j(\beta, \alpha)$ is defined as

$$e^{\Phi_j(\beta,\alpha)} = \sum_{\{n\}} \prod_{\mathbf{q}_j} e^{-n_j(\mathbf{q}_j)(\beta q_j + i\alpha)} \Omega_j\{n_j(\mathbf{q}_j)\}. \tag{7}$$

Changing the variables $\lambda = e^{-i\alpha}$ in Eq. (6), we get

$$G_{N_j}(\beta) = \frac{1}{2\pi i} \int_{C+} \lambda^{-N_j - 1} e^{\Phi_j(\beta,\lambda)} d\lambda$$

$$= \frac{1}{N_j!}\left\{\frac{d^{N_j}}{d\lambda^{N_j}}e^{\Phi_j(\beta,\lambda)}\right\}_{\lambda=0}, \tag{8}$$

where the integration is taken counter-clockwise along the contour $|\lambda| = 1$. Summing in Eq. (7) over $\{n\}$ we get

$$\Phi_j(\beta,\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{\infty}\frac{\gamma^{k+1}}{k}\lambda^k\sum_{\mathbf{q}_j}e^{-\beta q_j k}; & \text{Bose-Einstein and Fermi-Dirac} \quad (9a) \\ \displaystyle\lambda\sum_{\mathbf{q}_j}e^{-\beta q_j k+\ln N_j!}; & \text{Boltzmann,} \quad (9b) \end{cases}$$

where $\gamma = 1\ (-1)$ for all particles obeying Bose-Einstein (Fermi-Dirac) statistics. The result for Boltzmann statistics (9*b*) is obtained if one keeps only the $k = 1$ term and adds $\ln N_j!$ in the corresponding expressions of Eq. (9*a*).

We now replace the summation by the covariant integrals according to [11]

$$\sum_{\mathbf{q}_j}e^{-\beta q_j k} \to g_j\int\frac{\Omega_j q_j}{(2\pi)^3}\frac{d^3q_j}{q_j}e^{-\beta q_j k}, \tag{10}$$

where g_j denotes the number of polarization states of the vectors $\mathbf{q}_j$, and Ω_j is the 4-dimensional volume, which in the rest frame is expressed by $\Omega_j = (V_j, \mathbf{0})$, with V_j being the space volume occupied by j-th kind of particles. (We use $\hbar = c = 1$ units). Evaluating the integrals in the β-rest and Ω-rest frame and using the Lorentz covariance we get

$$\Phi_j(\beta,\lambda) = \sum_{k=1}^{\infty}\lambda^k f_j(k,\beta), \tag{11}$$

where

$$f_j(k,\beta) = \frac{g_jV_j}{(2\pi)^3}\frac{\gamma^{k+1}}{\beta k^2}4\pi\mu_j^2K_2(\mu_j\beta k).$$

According to Eq. (8), we have

$$G_{N_j}(\beta) = \frac{1}{N_j!}\left[\frac{d^{N_j}}{d\lambda^{N_j}}\exp\left\{\sum_{k=1}^{\infty}f_j(k,\beta)\lambda^k\right\}\right]_{\lambda=0}$$

$$= \sum_{\{m\}}\prod_{k=1}^{N_j}\frac{1}{m_k!}\{f_j(k,\beta)\}^{m_k}, \tag{12}$$

where $\{m\}$ is the partition number of N_j, satisfying $\sum_{k=1}^{N_j} km_k = N_j$. The summation is taken over all possible $\{m\}$ of N_j. Eq. (12) has the meaning of a cluster decomposition. Now, if we consider a special system consisting of ν-different particles, *i.e.* $N_1 = N_2 = \ldots = N_\nu = 1$ with different masses $\mu_1, \mu_2, \ldots, \mu_\nu$, we have, as is easily shown,

$$G_{1\ldots1}(\beta) = \prod_{j=1}^{\nu} \frac{g_j V_j}{(2\pi)^3} \frac{4\pi\mu_j^2}{\beta} K_2(\mu_j\beta). \tag{13}$$

This formula represents the generating function of an ideal relativistic gas obeying Boltzmann statistics [12]. For the sake of simplicity, in what follows we will calculate the phase space integrals in an explicit form, restricting our attention to the system with N-identical quantum particles. First we consider the extreme relativistic (e.r.) case. Taking the limit $\beta \to 0$ we get from Eq. (12)

$$G_N^{\text{e.r.}}(\beta) = \sum_{\{m\}} \prod_{k=1}^{N} \frac{1}{m_k!} \left\{ \frac{8\pi g \gamma^{k+1}}{\omega \beta^3 k^4} \right\}^{m_k}, \tag{14}$$

where $\omega = (2\pi)^3/V$.

On the other hand, the generating function in Eq. (3) can be written as a K-transform [13]:

$$G_N(\beta) = 4\pi\beta^{-3/2} \int_0^\infty K_1(\beta Q)\,(\beta Q)^{1/2}\, Q^{3/2}\, \sigma_N(Q)\, dQ. \tag{15}$$

Inversion of the integral of Eq. (15) together with Eq. (14) yields a closed expression for $\sigma_N^{\text{e.r.}}(Q)$,

$$\sigma_N^{\text{e.r.}}(Q) = \sum_{\{m\}} \prod_{k=1}^{N} \frac{\gamma^{(k+1)m_k}}{m_k!} \frac{\sigma_{m_k}^{(0)\text{e.r.}}(Q)}{k^{4m_k}}, \tag{16}$$

where

$$\sigma_{m_k}^{(0)\text{e.r.}}(Q) = \frac{\left(\frac{g}{\omega}\right)^{m_k} \left(\frac{\pi}{2}\right)^{m_k - 1} (4m_k - 4)!\, s^{\frac{1}{2}(3m_k - 4)}}{(3m_k - 4)!(2m_k - 1)!(2m_k - 2)!},$$

represents the phase space integral for the system with m_k particles obeying Boltzmann statistics. Keeping the term with $k = 1$, $m_1 = N$, we get the usual phase space integral

$$\sigma_N^{\text{e.r.}}(Q) = \frac{1}{N!} \sigma_N^{(0)\text{e.r.}}(Q). \tag{17}$$

The nonrelativistic (n.r.) limit of the formulae are obtained taking the limit $\beta \to \infty$ in Eq. (12):

$$G_N^{\text{n.r.}}(\beta) = \sum_{\{m\}} \prod_{k=1}^{N} \frac{\left(\frac{g}{\omega}\right)^{m_k} \gamma^{(k+1)m_k} (2\pi k\mu)^{\frac{3}{2}m_k}}{m_k! \beta^{3m_k/2} k^{4m_k}} e^{-\mu\beta m_k k}, \tag{18}$$

$$\sigma_N^{\text{n.r.}}(Q) = \sum_{\{m\}} \prod_{k=1}^{N} \frac{\gamma^{(k+1)m_k}}{m_k! k^{4m_k}} \sigma_{m_k}^{(0)\text{n.r.}}(Q, k\mu), \tag{19}$$

$$\sigma_{m_k}^{(0)\text{n.r.}}(Q, k\mu) = \left(\frac{g}{2\omega}\right)^{m_k} \frac{(2\pi)^{(3(m_k-1)/2} \prod_{j=1}^{m_k} (k_\mu)^{3/2}}{\Gamma\left(\frac{3(m_k-1)}{2}\right) (m_k k\mu)^{3/2}} (Q - k\mu m_k)^{(3m_k-5)/2}.$$

Again Eq. (19) reduces to the usual phase space integral when one restricts oneself to the first term in Eq. (19). A rigorous formula for $\sigma_N(Q)$ can be derived without any approximation

$$\sigma_N(Q) = \sum_{\{m\}} \prod_{k=1}^{N} \frac{\gamma^{(k+1)m_k}}{m_k! k^{4m_k}} \sigma_{m_k}^{(0)}(Q, k\mu), \tag{20}$$

where

$$\sigma_{m_k}^{(0)}(Q, k\mu) = \left(\frac{g}{\omega}\right)^{m_k} (4\pi)^{m_k-2} \frac{4}{Q} \prod_{j=1}^{m_k} (k\mu)^2 \frac{1}{j} \int_{c-i\infty}^{c+i\infty} d\beta \beta^{-m_k+2}$$

$$\times I_1(\beta Q) \prod_{j=1}^{m_k} K_2(\beta k\mu) \qquad \text{with } c > 0.$$

$\sigma_N^{(0)}(Q, \mu)$ represents a rigorous formula for the usual phase space integral, which was obtained earlier in Ref. [14].

3. Thermodynamical description

For the sake of completeness, we continue our discussions on an ideal relativistic quantum gas system in the thermodynamical description. Let us define the cluster coefficients of the ideal quantum gases as

$$\phi(j, \beta) = \frac{f(j, \beta)}{V}$$

$$= \frac{2g\mu^2}{(2\pi)^2}\frac{\gamma^{j+1}}{\beta j^2}K_2(\beta\mu j). \tag{21}$$

Note that $\phi(j,\beta)$ is a function of the inverse temperature β only, and does not depend on V. Now

$$G_N(\beta) = \sum_{\{m\}}\prod_{j=1}^{N}\frac{1}{m_j!}\{V\phi(j,V)\}^{m_j}; \qquad \left(\sum_{j=1}^{N} jm_j = N\right). \tag{22}$$

For a better understanding of the content of Eq. (22) we introduce the grand partition function, which describes an ensemble of particles with varying N:

$$\begin{aligned}\Xi(\lambda,\beta) &= \sum_{N\geq 0} G_N(\beta)\lambda^N \\ &= \sum_{N\geq 0}\sum_{\{m\}}\prod_{j=1}^{N}\frac{1}{m_j!}\{V\phi(j,\beta)\}^{m_j}; \qquad \left(\sum_{j=1}^{N} jm_j = N\right) \\ &= \exp\left\{\sum_{j\geq 1} V\phi(j,\beta)\lambda^j\right\}.\end{aligned} \tag{23}$$

The average number of particles in the ensemble (multiplicity) is given by

$$\begin{aligned}\overline{N} &= \lambda\frac{\partial \ln \Xi(\lambda,\beta)}{\partial\lambda} = \sum_{j\geq 1} jV\phi(j,\beta)\lambda^j \\ &= \frac{2g\mu^2 V}{(2\pi)^2\beta}\sum_{j\geq 1}\frac{\gamma^{j+1}}{j}K_2(\beta\mu j)\lambda^j.\end{aligned} \tag{24}$$

The average total energy over the ensemble is given by

$$\begin{aligned}\overline{E} &= -\frac{\partial}{\partial\beta}\ln \Xi(\lambda,\beta) \\ &= \frac{2Vg\mu^2}{(2\pi)^2}\sum_{j\geq 1}\frac{\gamma^{j+1}}{j^2}\left[\frac{K_2(\beta\mu j)}{\beta^2} + \frac{\mu j\{K_1(\beta\mu j) + K_3(\beta\mu j)\}}{2\beta}\right]\lambda^j.\end{aligned} \tag{25}$$

In the extreme-relativistic case when $\mu \to 0$, we get

$$\overline{N} = \frac{A}{\beta^3}; \qquad A = \frac{4Vg}{(2\pi)^2} \sum_{j\geq 1} \frac{\gamma^{j+1}}{j^3} \lambda^j, \tag{26a}$$

$$\overline{E} = \frac{B}{\beta^4}; \qquad B = \frac{12Vg}{(2\pi)^2} \sum_{j\geq 1} \frac{\gamma^{j+1}}{j^4} \lambda^j. \tag{26b}$$

If in Eq. (26b) we put $g = 2$, $\gamma = 1$, $\lambda = 1$, *i.e.* we consider the black-body radiation case, we recover the Stefan–Boltzmann law

$$\overline{\epsilon} = \frac{\overline{E}}{V} = \frac{\pi^2}{15} T^4. \tag{27}$$

Setting $\overline{E}$ to be equal to the given energy $\sqrt{s}$ yields

$$\overline{N} = s^{3/8} \frac{A}{B^{3/4}}. \tag{28}$$

Applying the formalism to the multiparticle production $a + b \to c + d + \ldots$ (now s is defined as $s = (p_a + p_b)^2$, where p_a and p_b are the 4–momenta of the incoming particles a and b, respectively), we then have for the average energy $\overline{w}$ of secondaries

$$\overline{w} = \frac{\sqrt{s}}{\overline{N}} = s^{1/8} \frac{B^{3/4}}{A}. \tag{29}$$

Thus the energy dependences of the average multiplicity and the average energy of secondaries in the extreme relativistic case are the same as in the usual statistical theory without quantum effects. The quantum statistical effects manifest themselves only in the energy independent factors A and B.

We derive once more Eq. (24) by a different method which allows to obtain a better insight into the physical interpretation of decomposition appearing in Eqs. (12) and (22). From Eq. (23) we obtain

$$\phi(k,\beta) \frac{\partial \ln \Xi(\lambda,\beta)}{\partial \phi(k,\beta)} = \frac{\sum_{N\geq 0} \sum_{\{m\}} m_k \prod_{j=1}^{N} \frac{\{V\phi(j,\beta)\lambda^j\}^{m_j}}{m_j!}}{\sum_{n\geq 0} \sum_{\{m\}} \prod_{j=1}^{N} \frac{\{V\phi(j,\beta)\lambda^j\}^{m_j}}{m_j!}}$$

$$= \overline{m}_k; \qquad \left(\sum_{j=1}^{N} j m_j = N \right) \tag{30}$$

Formally, $\overline{m}_k$ can be interpreted as the average of m_k, appearing in various ensembles of $\{m\}$, satisfying the condition $\sum_{j=1}^{N} j m_j = N$. Average is taken, in this case, over the grand canonical function. From Eqs. (30) and (23) we get

$$\overline{m}_k = V\phi(k,\beta)\lambda^k. \tag{31}$$

Then the average number of particles is given by

$$\overline{N} = \sum_{j\geq 1} j\overline{m}_j = \sum_{j\geq 1} jV\phi(j,\beta)\lambda^j, \tag{32}$$

coinciding with Eq. (24). this suggests that we can give to m_k the physical meaning of a number of clusters with k particles. In other words, the system of N–identical free particles can be decomposed into $m_1, m_2, \ldots, m_N$ clusters in each of which are contained $1, 2, \ldots, N$ particles, satisfying the general relation $\sum_{j=1}^{N} jm_j = N$. This situation is very similar to that of the real classical gases. This happens because ideal Bose and Fermi gases possess the properties of real gases on account of indistinguishabililty of particles, and behave as if they formed clusters in the system. Thus, $\phi(j,\beta)$ corresponds to the cluster coefficients of the real classical gases, while λ can be formally interpreted as the "absolute activity" [15] of a particle, *i.e.* $\lambda = e^{\mu\beta}$, where μ is the chemical potential of the system (not to be confused with the mass of the particle). In this way, one can develop for the ideal quantum gases a formalism in complete analogy with the one of the classical real gases.

One can further go on by introducing the pressure p as

$$\frac{p}{kT} = \lim_{V\to\infty} \sum_{j\geq 1} \phi(j,\beta)\lambda^j = \sum_{j\geq 1} \phi(j,\beta)\lambda^j. \tag{33}$$

From Eq. (24) we have

$$\frac{\overline{N}}{V} = \rho = \sum_{j\geq 1} j\phi(j,\beta)\lambda^j. \tag{34}$$

From Eqs. (33) and (34) one can obtain, without approximation, the virial expansion for the ideal relativistic quantum gases

$$\frac{pV}{\overline{N}kT} = 1 - \sum_{k\geq 1} \frac{k}{k+1}\beta_k\rho^k; \qquad \rho = \frac{\overline{N}}{V}, \tag{35}$$

where

$$\beta_k = \sum_{\{m\}} (-)^{\sum_j m_j - 1} \frac{(k-1+\sum_j m_j)!}{k!} \prod_j \frac{\{j\phi(j,\beta)\}^{m_j}}{m_j!}; \quad \left(\sum_{j=2}^{k+1} (j-1)m_j = k\right),$$

with $\phi(j,\beta)$ defined as in Eq. (21). In the nonrelativistic approximation with $\beta \to \infty$ our exact formula, Eq. (35), reduces to the one already given in the literature [1–4].

We end the paper by showing that Eqs. (24) and (33) are consistent with the conventional thermodynamical formula

$$\frac{\overline{N}}{V} = \left(\frac{\partial p}{\partial \mu}\right)_{T,V}. \tag{36}$$

This can easily be shown from Eqs. (23), (24) and (33) as follows:

$$\begin{aligned} \frac{\overline{N}}{V} &= \frac{1}{V}\lambda\frac{\partial \ln \Xi(\lambda,\beta)}{\partial\lambda} \\ &= \lambda\frac{\partial}{\partial\lambda}\left[\sum_{j\geq 1}\phi(j,\beta)\lambda^j\right] = \frac{\partial}{\partial\mu}\left[\beta\sum_{j\geq 1}\phi(j,\beta)\lambda^j\right] \\ &= \left(\frac{\partial p}{\partial\mu}\right)_{\beta,V}, \end{aligned} \tag{37}$$

where we used the relation $\lambda = e^{\beta\mu}$.

Furthermore, from Eq. (24) we have

$$\phi(j,\beta) = \frac{g\mu^2}{2\pi^2\beta}\frac{\gamma^{j+1}}{j^2}K_2(\beta\mu j), \tag{38}$$

hence, we have explicitly that for the pressure:

$$p = \sum_{j\geq 1}\frac{g\mu^2}{2\pi^2}\frac{\gamma^{j+1}}{j^2}K_2(\beta\mu j)\lambda^j. \tag{39}$$

This is in agreement with the formula derived by Miller and Karsch [16].

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References

1. See, *e.g.*, J.L. Synge, *The Relativistic Gas*. North Holland, (1957); S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover Pub., N.Y. (1957).
2. P.T. Landsberg and P. Havas, in: *Statistical Mechanics of Equilibrium and Non*

Equilibrium, ed. by J. Meixner, North Holland (1965); P.T. Landsberg and J. Dunning Davies, *Phys. Rev.* **138A** (1965) 1049.
3. S. Katsura, *Prog. Theor. Phys.* **16** (1956) 589.
4. T.D. Lee and C.N. Yang, *Phys. Rev.* **117** (1960) 22 and references therein.
5. A.Ya. Khinchin, *Analytical Foundations of Physical Statistics.* Gordon and Breach (1961).
6. V.B. Magalinskii and Ya. P. Terletskii, *Žurn. Ėksp. Teor. Fiz.* **32** (1957) 584.
7. M. Hayashi, *Bulletine de L'Academie Polonaise des Sci. Sér. des Math Astr. et Phys.* **14** (1966) 261.
8. R. Hagedorn, *Nuovo Cimento Suppl.* **3** (1965) 147.
9. S. Frautschi, *Phys. Rev.* **D3** (1971) 2821.
10. W. Nahm, *Nucl. Phys.* **B45** (1972) 525.
11. B. Touschek, *Nuovo Cimento* **58** (1968) 295 and private communication of L. Sertorio with R. Hagedorn.
12. If we could have taken the covariant phase space $d^3p/2p_0$, we would have recovered the Boltzmann statistics case of F. Lurçat and P. Mazur, *Nuovo Cimento* **31** (1964) 140.
13. A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms.* New York (1953).
14. V.A. Kolmunov, *Žurn. Ėksp. Teor. Fiz* **43** (1962) 1448.
15. A. Münster, *Statistical Thermodynamics.* Springer-Verlag (1969).
16. D.E. Miller and E. Karsch, *Phys. Rev.* **D24** (1981) 2564.

Resumen. Se usa el formalismo exacto de la distribución microcanónica para describir los gases ideales cuántico-relativistas. Se desarrolla una descomposición estadística de cúmulos para el cálculo de las integrales de espacio fase con la estadística correcta. Se discuten analogías con los gases clásicos reales y aplicaciones del formalismo