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A variational approach to the time evolution equations for nonconserved variables in extended irreversible thermodynamics

Federico Vázquez H. and J. Antonio del Río P.

Area de Física, Universidad Autónoma Chapingo,
56230 Chapingo, Estado de México

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Abstract. We show that a classical variational principle does not exist for a rigid heat conductor in EIT. Then, we formulate a restricted variational principle that would lead to the time evolution equations for the nonconserved variables in the mexican EIT formalism. The principle is illustrated for the case of a rigid heat conductor, and further applied to a simple viscous fluid. The structure of the principle permits us to discuss some aspects of the extended thermodynamic space.

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1. Introduction

A variational formulation of the physical behavior of any system is an alternative description when the equations of evolution balance, boundary conditions, etc., of the system are pursued. It may represent at least two advantages: to summarize a subject suggesting analogies and generalizations, and to lead to a method for obtaining approximate solutions to the problem permitting the use of additional information, such as might be available from intuitional considerations [1,2].

The history of the search for variational principles encompassing fluid mechanics begins in the mid of 19th century. Perhaps, Lord Kelvin's variational principle (valid for an incompressible, inviscid fluid in irrotational flow) was the start of the development of the so called classical variational principles of convective processes [3]. Further efforts were directed to include compressible fluids and rotational flows [4,5,6]. In all these, works the fluid was considered inviscid.

The inclusion of dissipative effects in the transport phenomena leads to a controversy about the existence of classical variational formulations in the framework of linear irreversible thermodynamics (LIT).

Now it seems to be accepted that fluid flows involving dissipation do not admit that kind of variational principles [7]. This resulted in a lot of approaches known as non-classical variational principles, which could have had their roots in Onsager's formulation of Fourier's Law of heat conduction [8]. His principle (formulated for heat conduction in anisotropic media) is that the function

$$A(\mathbf{q},t) = \frac{d}{dt}S(\mathbf{q}) + \frac{d}{dt}S^*(\mathbf{q}) - \phi(\mathbf{q},\mathbf{q})$$

is a maximum with respect to variations of $\mathbf{q}$ when the temperature distribution T is prescribed. The terms on the right hand side are entropy accumulation, out flow and generation, respectively.

Following after Onsager's is the Prigogine's theorem of minimum entropy production [9], which asserts that for prescribed time independent conditions the total entropy production

$$P = \int \sigma \, dV \tag{1}$$

is a minimum at the stationary state (he considered purely dissipative processes), where σ denotes the entropy production per unit time and volume. Rosen [10] was inspired by Prigogine's theorem and he makes the integral $\int(\mathbf{q}\cdot\mathbf{q}/2k)dV$ stationary with respect to variations in $\mathbf{q}$ with certain constraints, for transient heat conduction in isotropic media, $\mathbf{q}$ prescribed on surface, k independent of $\mathbf{q}$, T and $\frac{\partial T}{\partial t}$ held fixed through the region V. Later, Glandsdorff and Prigogine [11] extended Eq. (1) to a general criterion of evolution which includes convection terms. For time independent boundary conditions their criterion reads

$$d\Phi = \int dV \sum \mathbf{J}_i \cdot d\mathbf{X}_i \leq 0, \tag{2}$$

where the thermodynamic forces $\mathbf{X}_i$ and the flows $\mathbf{J}_i$ include mechanical processes. In spite of $d\Phi$ in Eq. (2) being an inexact differential it is possible in many cases to obtain a total differential near stationary states by using the concept of local potential firstly introduced by Glansdorff *et al.* [11]. This concept was further developed by Glansdorff and Prigogine [13] and the underlying idea is to define the functional in terms of two types of dependent variables u and u_0. The latter is called th stationary variable. The functional would be given by

$$I(\mathbf{u},\mathbf{u}_0) = \int \mathcal{L}(\mathbf{u},\mathbf{u}_0)\, dV, \tag{3}$$

where $\mathcal{L}$ is a general lagrangian density whose Euler-Lagrange derivative would reduce to the equations of the system. The variables $\mathbf{u}_0$ are held fixed during the variation process and an additional condition must be satisfied after the variation has been carried out

$$\mathbf{u} = \mathbf{u}_0. \tag{4}$$

Here two comments are required. First, the solutions derived from the variational method with the condition (4) are sufficiently close to the stationary state of the system (or average state) $\mathbf{u}_0$, and second, in order for this stationary state to be a minimum the next inequality must be valid

$$I(\mathbf{u},\mathbf{u}_0) > I(\mathbf{u}_0,\mathbf{u}_0). \tag{5}$$

During the 70's in a series of three papers, Lebon and Lambermont [14,15,16] applied the local potential concept to purely dissipative processes first, and in the last they worked with multicomponent chemically active mixtures in transient state. In the energy representation the functional to be varied in their principle is written in terms of a lagrangian density $\mathcal{L}$ which depends on the center of mass velocity v and the Legendre transform of the internal energy per unit volume u_v. Thus, the independent variables are the temperature and chemical potential μ

$$\mathcal{L} = \tfrac{1}{2}\rho^0 \mathbf{v}\cdot\mathbf{v} - u_v(T,\mu), \tag{6}$$

here ρ is mass density and the superscript ^{0} indicates that it must be kept fixed during the variation. Through restricted variations taken with respect to μ, $\mathbf{v}$ and T the associated Euler-Lagrange equations are the balance of mass, momentum and energy respectively and the boundary conditions. Three types of different variations are used simultaneously when each of μ, $\mathbf{v}$ or T is varied: no restricted variations, variations leaving constant time derivatives and those leaving constant spatial derivatives. At this point, it must be remarked that neither the Glansdorff and Prigogine nor the Lebon and Lambermont principles are extremal principles if inequality (5) is not satisfied, and in that case they only permit to determine the stationary state of the system [17]. Second, the theoretical importance of the local potential formalism is emphasized when it is introduced as an evolution criterion in order to predict system performance.

In addition to the ones mentioned above, other approaches employing stationary variables have been presented [18]. Venkateswarlu and Deshpande [19] have given a unified local potential formulation of fluid mechanics which includes all mentioned so far. Garrod [20] eliminated the restriction of exactly linear flow equations introducing a variational principle that is useful for nonuniform and nonlinear steady flows.

The interest on variational formulations for irreversible phenomena has reached the theories which are looking up for an adequate description of phenomena beyond LIT [21,22]. As it is well known, no unique version of such theories exists [23,24,25]. In this work the mexican version of the extended irreversible thermodynamics (EIT) [26] will be assumed. The basic ideas behind it are: 1) the extension of the space of state variables to include; as independent variables, the locally conserved densities and the fast or nonconserved variables, 2) a function η is defined in this new space, η is taken as an entropy-like function, 3) starting with these assumptions together with a postulated balance equation for η and the generalized Gibbs equation, it is possible to derive a set of time evolution equations for the nonconserved variables of the extended space, 4) the partial derivatives of η appearing in the generalized Gibbs equations along with the flux of η are functions of the variables of the extended space and therefore of the scalar invariants constructed with the nonconserved variables [26], 5) the production of η may depend additionally on parameters not belonging to the extended space according to the closure assumption [27], and 6) the scalars of the theory are obtained by expanding functions around a local equilibrium state and truncating the expansions with some order criterion. This task must be made with logical consistency [28].

Our work lies in the context described above. The objective is to investigate the existence of variational principles immersed in the framework of EIT. And through these principles to elucidate on the nature of the entropy-like function and the physical properties of the extended thermodynamic space.

In the next Section we begin by showing that no classical principle exists for the time evolution equations in the framework of EIT in the particular case of a rigid heat conductor. Section 3 is devoted to develop a nonclassical variational principle of restricted type for the same system. This principle is formulated in terms of a functional that involves the entropy-like change minus its production. The restricted character of the principle consists in that variations are carried out on fast variables keeping constant both the tangent space and the conserved component of the extended space. In the Section 4, the principle is applied to the well worked system: a simple viscous fluid making evident the importance of the closure assumption [27] in the formalism. Finally some comments are found.

2. Time evolution equation of a rigid conductor and the classical variational principles

For simplicity, we first discuss the ideas for a simple system. Let us consider a rigid infinite heat conductor at rest and with constant density $\rho(\mathbf{r}, t)$. The entropy-like function will depend on a locally conserved density, the internal energy e and a nonconserved quantity, the heat flux $\mathbf{q}$:

$$\eta = \eta(e, \mathbf{q}). \tag{7}$$

The time evolution equation of first order for the heat flux is [26]:

$$-t_q \frac{d\mathbf{q}}{dt} = K\nabla T + \mathbf{q}, \tag{8}$$

where t_q is the relaxation time for $\mathbf{q}$ and T is the local temperature. Eq. (8) is the complement of the energy balance

$$\rho \frac{de}{dt} + \nabla \cdot \mathbf{q} = 0. \tag{9}$$

We now inquire whether a classical variational principle exists for the set (8) and (9). In order to answer this question we make use of the method of Fréchet derivatives [29]. Let $N(u)$ be a differential operator

$$N(u) = 0, \tag{10}$$

which may be nonlinear. $N(u)$ is the gradient of a certain functional $F(u)$, if the symmetry condition is satisfied

$$\int \psi N'_u \phi \, dV = \int \phi N'_u \psi \, dV, \tag{11}$$

where the Fréchet derivative N'_u of the operator is defined as

$$\begin{aligned} N'_u \phi &\equiv \lim_{\epsilon \to 0} \frac{N(u + \epsilon\phi) - N(u)}{\epsilon} \\ &= \left[\frac{\partial}{\partial \epsilon} N(u + \epsilon\phi) \right]_{\epsilon=0} . \end{aligned} \tag{12}$$

In the case that u is a set of functions of n parameters

$$u_s = u_s(x_1, \ldots, x_n), \tag{13}$$

the symmetry condition takes the form

$$\frac{\partial f^l}{\partial u_{s,jk}} = \frac{\partial f^s}{\partial u_{l,jk}}, \tag{14.1}$$

$$\frac{\partial f^l}{\partial u_{s,j}} = -\frac{\partial f^s}{\partial u_{l,j}} + 2\nabla_k \left(\frac{\partial f^s}{\partial u_{l,jk}} \right), \tag{14.2}$$

$$\frac{\partial f^l}{\partial u_s} = \frac{\partial f^s}{\partial u_l} - \nabla_j \left(\frac{\partial f^s}{\partial u_{l,j}} \right) + \nabla_j \nabla_k \left(\frac{\partial f^s}{\partial u_{l,jk}} \right), \tag{14.3}$$

where $f^l = 0$ are the differential equations from Eq. (10), latin indices go from one to four, a subscript comma denotes partial differentiation.

Accordingly, if Eqs. (8) and (9) satisfy conditions (14) then a classical variational principle exists for them.

Introducing a 4–vector $\Gamma = (W_1, W_2, W_3, W_4) = (\mathbf{q}, T)$, Eqs. (8) and (9) can be transformed to a suitable form

$$f^\alpha \equiv t_q W_{\alpha,4} + k W_{4,\alpha} + w_\alpha = 0, \tag{15.1}$$

$$f^4 \equiv \rho c_v W_{4,4} + W_{\alpha,\alpha} = 0. \tag{15.2}$$

Here the equality $\frac{d}{dt} W_l = W_{l,4}$ has been used (static conductor), greek indices run from one to three. With this definition for f^l we can now verify if Eqs. (14) are satisfied.

The derivatives for Eq. (14.1) are $\frac{\partial}{\partial W_{s,jk}} f^l = 0$ and $\frac{\partial}{\partial W_{l,jk}} f^s = 0$; and Eq. (14.1) is thus satisfied. Eq. (14.2) reduces to $\frac{\partial}{\partial W_{s,j}} f^l = -\frac{\partial}{\partial W_{l,j}} f^s$, since the second term of the right hand side (rhs) vanishes. The derivatives are $(l = 1, 2, 3)$

$$\frac{\partial f^l}{\partial W_{s,j}} = \begin{cases} t_q \delta_{ls} \delta_{j4} & \text{if } s = 1,2,3 \\ k\delta_{lj} & \text{if } s = 4 \end{cases} \quad \text{and} \quad \frac{f^s}{\partial W_{l,j}} = \begin{cases} t_q \delta_{ls} \delta_{j4} & \text{if } s = 1,2,3 \\ \rho c_v \delta_{ls} \delta_{js} + \delta_{jl} & \text{if } s = 4. \end{cases};$$

therefore Eq. (14.2) is not satisfied when $s = 4$. The remaining derivatives become $(l = 4)$

$$\frac{\partial f^l}{\partial W_{s,j}} = \begin{cases} \delta_{sj}, & \text{if } s = 1,2,3 \\ \rho\delta_{js}, & \text{if } s = 4 \end{cases} \quad \text{and} \quad \frac{\partial f^s}{\partial W_{l,j}} = \begin{cases} k\delta_{sj} & \text{if } s = 1,2,3 \\ \rho\delta_{jl} & \text{if } s = 4 \end{cases};$$

hence, Eq. (14.2) is again not satisfied.

Finally, Eq. (14.3) reduces to

$$\frac{\partial f^l}{\partial W_s} = \frac{\partial f^s}{\partial W_l},$$

since the third term of the rhs vanishes and $\frac{\partial f^s}{\partial W_{l,j}}$ is a constant. It is not difficult to see that Eq. (14.3) is satisfied for all indices.

Therefore we have that there does not exist a classical variational principle for the time evolution equations of a rigid heat conductor for the imposed conditions in the framework of EIT, of course this also occurs in LIT [7]. We are impelled to look for the existence of a nonclassical principle for the same system, which we do in the next Section.

3. A non classical variational principle of the restricted type for a rigid heat conductor

We formulate a restricted variational principle for the rigid heat conductor in the conditions described in Sec. 2. The functional is defined in terms of the change and production of the entropy-like function η as follows,

$$I(e, \mathbf{q}) = \int \left(\rho \frac{d\eta}{dt} - \sigma\right) dV, \tag{16}$$

being σ the entropy-like production per unit time and volume. The entropy-like change is obtained from the generalized Gibbs equation

$$\rho\, d\eta = \alpha_1\, de + \boldsymbol{\alpha}_2 \cdot d\mathbf{q}. \tag{17}$$

α_i are functions of the extended thermodynamic space (e and $\mathbf{q}$). Substituting Eqs. (9) and (17) into Eq. (16), we obtain

$$I(e,\mathbf{q}) = \int \left[-\nabla\cdot(\alpha_1\mathbf{q}) + \nabla\alpha_1\cdot\mathbf{q} + \alpha_2\cdot\frac{d\mathbf{q}}{dt} - \sigma\right] dV, \tag{18}$$

where use has been made of

$$\nabla\cdot(\alpha_1\mathbf{q}) = \alpha_1\nabla\cdot\mathbf{q} + \nabla\alpha_1\cdot\mathbf{q}.$$

If there does not exist perpendicular component of the heat flux on the surface of the conductor ∂B (infinite conductor)

$$\int_{\partial B} \alpha_1\mathbf{q}\cdot d\mathbf{s} = 0,$$

(this term could be retained in order to give the boundary condition for the heat flux, in that case an additional term would appear in Eq. (16) related to a prescribed value for $\mathbf{q}$ on the surface ∂B), then Eq. (19) reduces to

$$I(e,\mathbf{q}) = \int \left[\nabla\alpha_1\cdot\mathbf{q} + \boldsymbol{\alpha}_2\cdot\frac{d\mathbf{q}}{dt} - \sigma\right] dV. \tag{19}$$

What follows now is to search for the consequences on the functional (16) when the nonconserved component of the extended thermodynamic space is slightly modified. By construction it would be expected that $I(e,\mathbf{q})$ remains invariant, since we substract the nonconserved part of the change of entropy-like function. The variations which we carry out are independent of space-time and this implies, as it may be verified, that times derivatives and gradients are fixed during the variation process, *i. e.* these restricted variations take place in the thermodynamic space exclusively. Then the variational principle reads

$$\delta' \int \left[\rho\frac{d\eta}{dt} - \sigma\right] dV = 0, \tag{20}$$

where the superscript $'$ indicates the restricted sense of the principle.

So, equating the restricted variation δ' on the functional of Eq. (19) to zero yields

$$\nabla\alpha_1 + \left[\frac{\partial}{\partial\mathbf{q}}\boldsymbol{\alpha}_2\cdot\frac{d\mathbf{q}}{dt} - \frac{\partial}{\partial g}\sigma\right]\mathbf{q} = 0, \tag{21}$$

here we have used $\delta'(\nabla\alpha_1) = 0$ and $\delta'(\frac{d}{dt}\mathbf{q}) = 0$, as it has been mentioned. This is a nonapproximated time evolution equation for the heat flux in the rigid heat conductor in the framework of EIT, but it is not very useful because we do not know the explicit form of the α_i and σ.

Now, following the mexican EIT procedure, the coefficients α_i together with the production σ are expanded in terms of the scalar invariant $g = \mathbf{q} \cdot \mathbf{q}$

$$\alpha_1 = \alpha_{10} + \alpha_{11} g + \alpha_{12} g^2 + \cdots \tag{22.1}$$

$$\alpha_2 = (\alpha_{20} + \alpha_{21} g + \alpha_{22} g^2 + \cdots)\mathbf{q} \tag{22.2}$$

$$\sigma = \sigma_1 + \sigma_2 g + \sigma_3 g^2 + \cdots \tag{22.3}$$

If the integrand of Eq. (20) is to be expanded up to some order in the powers of the nonconserved variable, then the series of Eqs. (22) must be developed up to a consistent order [28]. In particular, when the integrand of Eq. (20) is expanded up to order two, the α_i are required up to the lowest order and the production σ up to order two: $\alpha_1 = \alpha_{10}$, $\alpha_2 = \alpha_{20}\mathbf{q}$ and $\sigma = \sigma_2 g$ (by compatibility with LIT $\sigma_1 = 0$). Then, Eq. (21) reduces to the known time evolution equation for the heat flux in the rigid heat conductor [26]

$$\alpha_{20}(\sigma_2)^{-1}\frac{d\mathbf{q}}{dt} = -(\sigma_2)^{-1}\nabla\alpha_{10} + \mathbf{q}, \tag{23}$$

where if we make the next identification

$$\alpha_{10} = T^{-1}, \qquad \alpha_{20}(\sigma_2)^{-1} = -t_q \qquad \text{and} \qquad (\sigma_2)^{-1} = k,$$

Eq. (8) is recovered. Thus, we have seen how variations on the nonconserved variables leaving invariant the tangent space of the extended thermodynamic space give the time evolution equations of the fast variables as "Euler-Lagrange" derivatives of functional (19).

Now we apply the method to a simple viscous fluid, showing the procedure tor the manipulation of a functional which depends on two scalar densities and three nonconserved variables: a scalar, a vector and a second rank tensor. We also make evident the important role which the closure hypothesis [27]is playing in the EIT's formalism.

4. A non classical variational principle of the restricted type for a simple viscous fluid

One of the systems that has been studied more deeply within the framework of EIT is the simple viscous fluid. The closure assumption [27] for this system should be necessarily invoked, not so in the case of the rigid heat conductor, because it is required to consider the spatial unhomogeneties for the description of the fluid. We show how this hypothesis can be used to realize the restricted variation.

Let us consider a simple viscous fluid which is described by the balance equations

$$\frac{d\rho}{dt} = -\nabla \cdot \mathbf{j}, \tag{24.1}$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \nabla \cdot \overleftrightarrow{\tau} \tag{24.2}$$

and

$$\rho \frac{de}{dt} = -\nabla \cdot \mathbf{q} - \overleftrightarrow{\tau} : \nabla \mathbf{v} - p\nabla \cdot \mathbf{v} - \tau \nabla \cdot \mathbf{v}, \tag{24.3}$$

with $\mathbf{j}$ the mass flux, $\mathbf{v}$ the velocity, p the pressure, $\overleftrightarrow{\tau}$ the trace less viscous tensor and τ the trace of viscous tensor.

In this case the generalized Gibbs equation for η is

$$\rho \frac{d\eta}{dt} = \alpha_1 \frac{de}{dt} + \alpha_2 \frac{d\rho}{dt} + \alpha_3 \frac{d\tau}{dt} + \alpha_4 \cdot \frac{d\mathbf{q}}{dt} + \overleftrightarrow{\alpha}_5 : \frac{d\overleftrightarrow{\tau}}{dt}. \tag{25}$$

The generalized state equations α_i must have a determined form accordingly with their tensorial character. Thus the vectors can be written as

$$\alpha_4 = \alpha_{41}\mathbf{q} + \alpha_{42}\mathbf{q} \cdot \overleftrightarrow{\tau}, \tag{26.1}$$

the tensors:

$$\overleftrightarrow{\alpha}_5 = \alpha_{51}\overleftrightarrow{\tau} + \alpha_{52}\mathbf{q}\mathbf{q} + \alpha_{53}\overleftrightarrow{\tau} \cdot \overleftrightarrow{\tau} + \alpha_{54}\overleftrightarrow{\tau} \cdot \mathbf{q}\mathbf{q} \tag{26.2}$$

and the scalars, the entropy-like production and α_{ij} are functions of the six scalar invariants constructed with a scalar τ, a vector $\mathbf{q}$ and a second rank trace less tensor $\overleftrightarrow{\tau}$:

$$g_1 = \tau, \tag{27.1}$$

$$g_2 = \mathbf{q} \cdot \mathbf{q}, \tag{27.2}$$

$$g_3 = \overleftrightarrow{\tau} : \overleftrightarrow{\tau}, \tag{27.3}$$

$$g_4 = \mathbf{q} \cdot \overleftrightarrow{\tau} \cdot \mathbf{q}, \tag{27.4}$$

$$g_5 = \mathrm{tr}(\overleftrightarrow{\tau} \cdot \overleftrightarrow{\tau} \cdot \overleftrightarrow{\tau}), \tag{27.5}$$

$$g_6 = \mathbf{q} \cdot (\overleftrightarrow{\tau} \cdot \overleftrightarrow{\tau}) \cdot \mathbf{q}. \tag{27.6}$$

Remember that σ can depend on other important parameters, *i.e.* the tangent space is not fully spanned by the extended thermodynamic space [27]. In the case we are dealing with we can consider

$$p_1 = \nabla \cdot \mathbf{v}, \qquad \overleftrightarrow{p}_2 = (\nabla \mathbf{v})^s \qquad \text{and} \qquad \overleftrightarrow{p}_3 = (\nabla \mathbf{v})^a,$$

as the relevant parameters, where $(\ )^s$ and $(\ )^a$ are the symmetric and antisymmetric tensor part respectively. These parameters belong to the tangent space and therefore they are not going to be varied.

Thus, substituting Eqs. (24) and (26) in (25) to introduce the result in Eq. (20) and using the independence among the variations on the nonconserved variables, we get the following system of equations

$$\begin{aligned}
0 = {} & \alpha_1 \nabla \cdot \mathbf{v} + \frac{\partial \alpha_3}{\partial g_1}\frac{d\tau}{dt} + \frac{d}{dt}\mathbf{q}\cdot\mathbf{q}\frac{\partial}{\partial g_1}\alpha_{41} + \frac{d}{dt}\mathbf{q}\cdot\mathbf{q}\cdot\overleftrightarrow{\tau}\frac{\partial}{\partial g_1}\alpha_{42} \\
& + \frac{d}{dt}\overleftrightarrow{\tau} : \overleftrightarrow{\tau}\frac{\partial}{\partial g_1}\alpha_{51} + \frac{d}{dt}\overleftrightarrow{\tau} : \mathbf{q}\mathbf{q}\frac{\partial}{\partial g_1}\alpha_{52} + \frac{d}{dt}\overleftrightarrow{\tau} : (\overleftrightarrow{\tau}\cdot\overleftrightarrow{\tau})\frac{\partial}{\partial g_1}\alpha_{53} \\
& + \frac{d}{dt}\overleftrightarrow{\tau} : (\mathbf{q}\overleftrightarrow{\tau}\cdot\mathbf{q})\frac{\partial}{\partial g_1}\alpha_{54} + \overleftrightarrow{\tau} : \nabla\mathbf{v}\frac{\partial}{\partial g_1}\alpha_1 + \tau\nabla\cdot\mathbf{v}\frac{\partial}{\partial g_1}\alpha_1 \\
& + (\nabla\cdot\mathbf{v})\frac{\partial}{\partial g_1}\alpha_2 - \frac{\partial}{\partial g_1}\sigma,
\end{aligned} \tag{28.1}$$

$$\begin{aligned}
0 = {} & \nabla\alpha_1 + \frac{d}{dt}\tau\sum_i^q \frac{\partial}{\partial g_i}\alpha_3\frac{\partial}{\partial \mathbf{q}}g_i + \alpha_{41}\frac{d}{dt}\mathbf{q} + \frac{d}{dt}\mathbf{q}\cdot\mathbf{q}\sum_i^q\frac{\partial}{\partial g_i}\alpha_{41}\frac{\partial}{\partial\mathbf{q}}g_i \\
& + (\nabla\cdot\mathbf{v})\sum_i^q\frac{\partial}{\partial g_i}\alpha_2\frac{\partial}{\partial\mathbf{q}}g_i + \frac{d}{dt}\mathbf{q}\cdot(q\cdot\overleftrightarrow{\tau})\sum_i^q\frac{\partial}{\partial g_i}\alpha_{42}\frac{\partial}{\partial\mathbf{q}}g_i \\
& + \alpha_{42}\overleftrightarrow{\tau}\cdot\frac{d}{dt}\mathbf{q} + 2\alpha_{52}\mathbf{q}\cdot\frac{d}{dt}\overleftrightarrow{\tau} + 2\alpha_{54}\frac{d}{dt}\overleftrightarrow{\tau} : \mathbf{q}\overleftrightarrow{\tau} \\
& + \frac{d}{dt}\overleftrightarrow{\tau} : \overleftrightarrow{\tau}\sum_i^q\frac{\partial}{\partial g_i}\alpha_{51}\frac{\partial}{\partial\mathbf{q}}g_i + \mathbf{q}\mathbf{q} : \frac{d}{dt}\overleftrightarrow{\tau}\sum_i^q\frac{\partial}{\partial g_i}\alpha_{52}\frac{\partial}{\partial\mathbf{q}}g_i \\
& + \frac{d}{dt}\overleftrightarrow{\tau} : (\overleftrightarrow{\tau}\cdot\overleftrightarrow{\tau})\sum_i^q\frac{\partial}{\partial g_i}\alpha_{53}\frac{\partial}{\partial\mathbf{q}}g_i + \tau\nabla\cdot\mathbf{v}\sum_i^q\frac{\partial}{\partial g_i}\alpha_1\frac{\partial}{\partial\mathbf{q}}g_i \\
& + \frac{d}{dt}\overleftrightarrow{\tau} : (\mathbf{q}\overleftrightarrow{\tau}\cdot\mathbf{q})\sum_i^q\frac{\partial}{\partial g_i}\alpha_{54}\frac{\partial}{\partial\mathbf{q}}g_i + \overleftrightarrow{\tau} : \nabla\mathbf{v}\sum_i^q\frac{\partial}{\partial g_i}\alpha_1\frac{\partial}{\partial\mathbf{q}}g_i \\
& - \sum_i^{q\sigma}\frac{\partial}{\partial g_i}\sigma\frac{\partial}{\partial\mathbf{q}}g_i,
\end{aligned} \tag{28.2}$$

and

$$0 = \frac{d}{dt}\tau \sum_i^t \frac{\partial}{\partial g_i}\alpha_3 \frac{\partial}{\partial \overleftrightarrow{\tau}} g_i + \frac{d}{dt}\mathbf{q}\cdot\mathbf{q}\sum_i^t \frac{\partial}{\partial g_i}\alpha_{41}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ \alpha_{42}\mathbf{q}\frac{d}{dt}\mathbf{q} + \alpha_{51}\frac{d}{dt}\overleftrightarrow{\tau} + \frac{d}{dt}\mathbf{q}\cdot(\mathbf{q}\cdot\overleftrightarrow{\tau})\sum_i^t \frac{\partial}{\partial g_i}\alpha_{42}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ \overleftrightarrow{\tau} : \frac{d}{dt}\overleftrightarrow{\tau}\sum_i^t \frac{\partial}{\partial g_i}\alpha_{51}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i + \frac{d}{dt}\overleftrightarrow{\tau} : \mathbf{q}\mathbf{q}\sum_i^t \frac{\partial}{\partial g_i}\alpha_{52}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ 2\alpha_{53}\overleftrightarrow{\tau}\cdot\frac{d}{dt}\overleftrightarrow{\tau} + \frac{d}{dt}\overleftrightarrow{\tau} : (\overleftrightarrow{\tau}\cdot\overleftrightarrow{\tau})\sum_i^t \frac{\partial}{\partial g_i}\alpha_{53}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ \alpha_{54}\frac{d}{dt}\overleftrightarrow{\tau} : \mathbf{q}\mathbf{q} + \alpha_1\nabla\mathbf{v} + \frac{d}{dt}\overleftrightarrow{\tau} : \mathbf{q}(\overleftrightarrow{\tau}\cdot\mathbf{q})\sum_i^t \frac{\partial}{\partial g_i}\alpha_{54}\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ \overleftrightarrow{\tau} : \nabla\mathbf{v}\sum_i^t \frac{\partial}{\partial g_i}\alpha_1\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i + \tau\nabla\cdot\mathbf{v}\sum_i^t \frac{\partial}{\partial g_i}\alpha_1\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i$$

$$+ (\nabla\cdot\mathbf{v})\sum_i^t \frac{\partial}{\partial g_i}\alpha_2\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i - \sum_i^{t\sigma} \frac{\partial}{\partial g_i}\sigma\frac{\partial}{\partial \overleftrightarrow{\tau}} g_i, \tag{28.3}$$

where$\sum_i^q \equiv \sum_i$ with $i = 3,5,6$; $\sum_i^t \equiv \sum_i$ with $i \neq 1,3$; $\sum_i^{q\sigma} \equiv \sum_i$ with $i = 3,5,6$ and including the terms with parameters; $\sum_l^{t\sigma} \equiv \sum_l$ with $i \neq 1,3$ and including the terms with parameters.

This system is a formal set of time evolution equations for the nonconserved variables. But again, it is not profitable in order to solve a specific problem.

Now, proceeding in a similar way as in the last section, we develop $[d\eta/dt - \sigma]$ up to second order, so that

$$\alpha_1 = \alpha_{10}, \tag{29.1}$$

$$\alpha_2 = \alpha_{20}, \tag{29.2}$$

$$\alpha_3 = \alpha_{31}\tau, \tag{29.3}$$

$$\alpha_4 = \alpha_{40}\mathbf{q}, \tag{29.4}$$

$$\alpha_5 = \alpha_{50}\overleftrightarrow{\tau}. \tag{29.5}$$

Here, we have used compatibility with LIT [28], and moreover for the sake of mathematical simplicity, both local temperature and pressure have been considered in α_1 and α_2. The coefficients α_{ij} are only functions of the local equilibrium variables. In the case of the production term the expansion considered is

$$\sigma = \sigma_1 g_1 + \sigma_2 g_2 + \sigma_3 g_3 + \sigma_4 p_1 g_1 + \sigma_5 p_1 g_2 + \sigma_6 p_1 g_3 + \sigma_7 \mathbf{q}\mathbf{q} : \overleftrightarrow{p}_2 + \sigma_8 \mathbf{q}\mathbf{q} : \overleftrightarrow{p}_3 + \sigma_9 \overleftrightarrow{\tau} : (\overleftrightarrow{\tau} \cdot \overleftrightarrow{p}_2) + \sigma_{10} \overleftrightarrow{\tau} : (\overleftrightarrow{\tau} \cdot \overleftrightarrow{p}_3), \tag{30}$$

again the σ_i are only functions of the conserved variables. Thus, if Eqs. (29) and (30) are substituted in Eq. (28) we recover the known time evolution equations for the fluxes of the fluid, namely,

$$\alpha_{30} \frac{d}{dt} \tau = -T^{-1} \nabla \cdot \mathbf{v} + 2\sigma_1 \tau + 2\sigma_4 \tau \nabla \cdot \mathbf{v}, \tag{31.1}$$

$$\alpha_{40} \frac{d}{dt} \mathbf{q} = -\nabla(T^{-1}) + 2\sigma_2 \mathbf{q} + 2\sigma_5 \nabla \cdot \mathbf{v}\mathbf{q} + 2\sigma_7 \mathbf{q} \cdot \overleftrightarrow{p}_2 + 2\sigma_8 \mathbf{q} \cdot \overleftrightarrow{p}_3, \tag{31.2}$$

$$\alpha_{50} \frac{d}{dt} \overleftrightarrow{\tau} = -T^{-1} \nabla \mathbf{v} + 2\sigma_3 \overleftrightarrow{\tau} + 2\sigma_6 \nabla \cdot \mathbf{v} \overleftrightarrow{\tau} + 2\sigma_9 \overleftrightarrow{\tau} \cdot \overleftrightarrow{p}_2 + 2\sigma_{10} \overleftrightarrow{\tau} \cdot \overleftrightarrow{p}_3. \tag{31.3}$$

In these equations we observe the coupling between the heat and moment fluxes and the closure terms. It is necessary to point out that a comparison of Eqs. (31) with other results [28,30] is not complete, because the spatial inhomogeneities of the fluxes are absent in our results. Yet, the principal terms are present in Eqs. (31). If the integrand $[d\eta/dt - \sigma]$ is developed up to zeroth order, the results of local equilibrium would be found again. Up to first order, we cannot construct the entropy-like production. Thus, in these cases we are reproducing previous results [28]. The structure of the Eqs. (28) permits to predict a coupling between the three equations when we go to the third order, but this development will not contribute significantly to clarify the physical meaning of η.

5. Remarks

We have seen that a classical variational principle does not exist for the equations that describe the evolution of a system in EIT. This was not surprising since the same fact occurs in the framework of LIT [7]. Thus, we were impelled to search a nonclassical principle for those equations.

We postulated a variational principle of the restricted type Eq. (20), where the role of the functional is played by the entropy-like change minus its production summed over all physical space. The restricted variations take effect only on nonconserved variables leaving fixed time and spatial derivatives and the conserved local variables. So, the "Euler-Lagrange" derivatives of a generalized Lagrangian Eq. (16) are the time evolution equations for the fast variables.

Our principle has a visible resemblance with that of Onsager [8]. However, we do not include the surface term (outflow) in the functional because our variation on it vanished. That term seems to have been introduced in an *ad hoc* manner in the Onsager's formulation. On the other hand, our variation process and the Onsager's run by parallel ways. Nevertheless, note that our variations become to be justified in the new extended formalism.

It must be pointed out that our principle is general while it does not suppose any explicit form of the generalized lagrangian. On the contrary, two recent works [21,22] present a variational principle in other EIT formalism which has resemblance with Hamilton's action functional, but it is very particular because it contains in a handsome manner special relaxation term. Another difference with previous variational principles [16] lies in that all fast variables are varied simultaneously and only one type of restricted variations is used during the process. Moreover, the form in which the functional I is constructed has a physical significance: the generalized lagrangian is constituted by the conserved part of the entropy-like change.

As it is well known, some authors have argued that the function η must be considered as a generalized entropy function, because it reduces to the entropy function of LIT when the extended thermodynamic space is projected onto the conserved space [24,26,28]. The fact that η is acting out as the outstanding thermodynamic function in our restricted variational principle strengthens that conclusion.

The goal of this work is to have found a restricted variational principle that leads to the time evolution equations of the nonconserved variables in the framework of EIT, in spite that the restriction on spatial and time derivatives impeded us to look out spatial and time unhomogeneities. Nevertheless, our analysis enlightens some physical aspects of the extended space. For instance: that Eqs. (21) and (28) be satisfied means that the conserved part of the η change has remained invariant when the nonconserved subset of the extended thermodynamic space is varied. This fact may be understood as if the description of the evolution of the system given by Eqs. (21) or (28) is near to that given by the corresponding equations in the conserved subset of the extended thermodynamic space. Therefore, these equations are valid in the neighborhood of the local equilibrium state. This conclusion is inherited by the time evolution equations for the fluxes, Eqs. (23) and (31). Hence, EIT is a phenomenological theory describing systems not too far from the local equilibrium state.

By the generality of the variational principle presented in this work, the specific conclusions of the two examined systems are susceptible to be transferred to another systems described with the mexican formalism of EIT, and could be used as a variational method to approximate solutions to transient problems.

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Resumen. Mostramos que no existe un principio variacional clásico para un conductor rígido de calor en TIE. Formulamos un principio variacional restringido que conduce a las ecuaciones de evolución temporal para las variables no conservadas en la versión mexicana de TIE. El principio se ilustra para el caso del conductor rígido de calor y después se aplica al fluido viscoso simple. La estructura del principio nos permite discutir algunos aspectos del espacio termodinámico extendido.