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Virasoro algebra and geometry

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Abstract. We review the appearance of the Virasoro algebra in the context of classical and quantum bosonic string theory. The classical algebra is closely related to the Lie algebra of vector fields on the circle as a differential manifold, which suggests a deep and beautiful connection between physical and geometrical objects.

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1. Introduction

The Virasoro algebra [1] plays an important role in string theory since it arises as the algebra of the conformal group in one and two dimensions [2] where, unlike three or more dimensions, it is infinite dimensional (in fact in two dimensions the algebra consists in two commuting Virasoro algebras), and at the classical level conformal transformations are the residual symmetries of the gauge-fixed metric $\eta_{\alpha\beta}$ =diag $(1,-1)$ on the world-sheet after use of reparametrization and Weyl invariances. At the quantum level, effects due to normal ordering of operators break these symmetries producing an anomaly which is the origin of the central (or c-number) term in the algebra. In the path integral formulation of the quantum theory, functional integration over ghost (and antighost) fields is introduced to represent the Fadeev-Popov determinants, which appears as a consequence of gauge fixing. The energy-momentum tensor receives a contribution from the ghosts which through Fourier transformation lead to an additive modification of the Virasoro generators. The algebra generated by the modified operators is the conformal algebra *i.e.* the effect of the anomaly cancels) only at the dimensionality of space-time $D = 26$. These results correspond to the bosonic string. In the supersymmetric case the theory is anomaly-free at $D = 10$. In the following we restrict our discussion to the bosonic case.

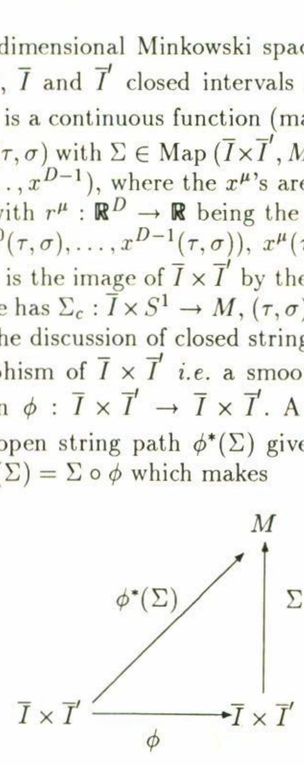
It is the purpose of the present article to review the appearance of the Virasoro algebra in the context of the classical and quantum bosonic string theory (Part I) and to connect the anomaly-free algebra with geometrical objects (Part II). In Secs. 1–3 (Part I) we discuss the Nambu and Polyakov actions, and the classical (without anomaly) and quantum (anomalous) Virasoro algebras. In Secs. 4–9 (Part II) we derive the Lie algebra of the vector fields on the circle which is the unique (up to diffeomorphisms) real 1-dimensional closed (compact without boundary) connected differential manifold. It is the complexification (Sec. 9) of the algebra which coincides with the anomaly-free Virasoro algebra. We do not intend an explanation of the (possibly deep) reason of the above connection. Some discussion is given in Refs. [2,3]. General introductions to the subject of string and superstring theories can be found in Ref. [4].

Part I

1. Nambu action and classical Virasoro algebra

Let $M = (\mathbb{R}^D, g_L)$ be D-dimensional Minkowski space (g_L is the Lorentz metric g_L =diag $(1, -1, \ldots, -1)$), $\overline{I}$ and $\overline{I}'$ closed intervals of the real line and S^1 the circle. An *open string path* is a continuous function (map) Σ from $\overline{I} \times \overline{I}'$ into M *i.e.* $\Sigma : \overline{I} \times \overline{I}' \to M, (\tau, \sigma) \mapsto \Sigma(\tau, \sigma)$ with $\Sigma \in$ Map $(\overline{I} \times \overline{I}', M)$. For Σ we have the ordered set of functions $\Sigma = (x^0, \ldots, x^{D-1})$, where the x^μ's are the coordinate functions of M given by $x^\mu = r^\mu \circ \Sigma$ with $r^\mu : \mathbb{R}^D \to \mathbb{R}$ being the μ-th coordinate function on $\mathbb{R}^D$. We have $(\tau, \sigma) \mapsto (x^0(\tau, \sigma), \ldots, x^{D-1}(\tau, \sigma))$, $x^\mu(\tau, \sigma) = r^\mu(\Sigma(\tau, \sigma)) \in \mathbb{R}$. *An open string world-sheet* Γ_Σ is the image of $\overline{I} \times \overline{I}'$ by the path Σ *i.e.* $\Gamma_\Sigma = \Sigma(\overline{I} \times \overline{I}')$. For a closed string path one has $\Sigma_c : \overline{I} \times S^1 \to M$, $(\tau, \sigma) \mapsto \Sigma_c(\tau, \sigma)$. In what follows we disçuss open strings. The discussion of closed strings is analogous.

Let ϕ be a diffeomorphism of $\overline{I} \times \overline{I}'$ *i.e.* a smooth (C^∞ continuously differentiable) bijective function $\phi : \overline{I} \times \overline{I}' \to \overline{I} \times \overline{I}'$. A *reparametrization of* Σ by $\phi \in$ Diff $(\overline{I} \times \overline{I}')^*$ is the open string path $\phi^*(\Sigma)$ given by $\phi^*(\Sigma) : \overline{I} \times \overline{I}' \to M$, $(\tau, \sigma) \mapsto \Sigma(\phi(\tau, \sigma))$ *i.e.* $\phi^*(\Sigma) = \Sigma \circ \phi$ which makes



a commutative diagram. Since $\phi \in$ Diff $(\overline{I} \times \overline{I}')$, $\phi(\overline{I} \times \overline{I}') = \overline{I} \times \overline{I}'$ and the world sheet remains invariant under reparametrizations:

$$\Gamma_{\phi^*(\Sigma)} = \phi^*(\Sigma)(\overline{I} \times \overline{I}') = (\Sigma \circ \phi)(\overline{I} \times \overline{I}') = \Sigma(\overline{I} \times \overline{I}') = \Gamma_\Sigma.$$

*For any smooth manifold M, Diff (M) is a group, subgroup of Aut (M): group of automorphisms ($f : M \to M$ bicontinuous) of M considered as a topological space. Diff (M) is a topological group, though not closed in Aut (M).

An *action* S for a path Σ is a function $S : \mathrm{Map}\,(\overline{I} \times \overline{I}', M) \to \mathbb{R}$, $\Sigma \mapsto S(\Sigma)$ which is invariant under reparametrizations: $S(\phi^*(\Sigma)) = S(\Sigma)$. It is easy to see that this requirement is satisfied by the Nambu action [5]

$$S_{\mathrm{N}}(\Sigma) = \text{const.} \times \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}, \tag{1}$$

where $\dot{x} = (\partial_\tau x^\mu)$, $x' = (\partial_\sigma x^\mu)$ and $AB = BA$ with $A, B = \dot{x}, x'$ is the scalar product in M with the Lorentz metric, $AB = \langle A, B\rangle_{g_L} = \sum_{\mu=0}^{D-1} A^\mu B^\nu (g_L)_{\mu\nu} = A^\mu B_\mu$. We chose $\overline{I} = [\tau_i, \tau_f]$ and $\overline{I}' = [0, \pi]$. The usual value for the constant in Eq. (1) is $-T_0/c$ where T_0 is the string tension and c the velocity of light. It is common to choose units such that $c = 1$ and $T_0 = 1/\pi$. The equations of motion and the edge conditions are obtained by making a variation of the string path, $\Sigma \to \Sigma' + \delta\Sigma$, subject to the condition $\delta\Sigma(\tau_1, \sigma) = \delta\Sigma(\tau_f, \sigma) = 0$. One obtains

$$\dot{p}_\tau + p'_\sigma = 0$$

with

$$\pi p_\tau = \frac{(\dot{x}x')x' - x'^2\dot{x}}{((\dot{x}x')^2 - \dot{x}^2x'^2)^{1/2}},$$

$$\pi p_\sigma = \frac{(\dot{x}x')\dot{x} - \dot{x}^2x'}{((\dot{x}x')^2 - \dot{x}x'^2)^{1/2}}$$

and

$$p_\sigma(\tau, 0) = p_\sigma(\tau, \pi) = 0.$$

The expressions for p_τ and p_σ lead to the identities (constraints)

$$p_\tau x' = p_\sigma \dot{x} = p_\tau^2 + \frac{x'^2}{\pi^2} = p_\sigma^2 + \frac{\dot{x}^2}{\pi^2} = 0.$$

The reparametrization invariance allows to impose two conditions on the string path Σ. The "conformal gauge" is defined by the conditions

$$\dot{x}x' = \dot{x}^2 + x'^2 = 0,$$

which lead to equations of motion, edge conditions and constraints respectively given by

$$\ddot{x} - x'' = 0, \quad x'(\tau, \pi) = x'(\tau, 0) = 0 \quad \text{and} \quad (\dot{x} \pm x')^2 = 0.$$

For p_τ and p_σ one obtains $\pi p_\tau = \dot{x}$ and $\pi p_\sigma = -x'$. The world-sheet solutions Γ_Σ are

$$x^\mu(\tau,\sigma) = q^\mu + p^\mu(\tau-\tau_i) - 2\sum_{n=1}^{\infty}\frac{\cos n\sigma}{\sqrt{n}}\,\mathrm{im}(a_n^\mu e^{-in(\tau-\tau_i)}),$$

with $q^\mu, p^\mu \in \mathbb{R}$, $a_n^\mu \in \mathbb{C}$ integration constants, and the constraint equations are given by $0 = -2\sum_{n\in\mathbb{Z}} e^{-in(\tau\pm\sigma)}L_n$ implying $L_m = 0$, $m \in \mathbb{Z}$. The Virasoro generators were defined as

$$L_m = -\frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \alpha_{m-n\mu},$$

with $\alpha_0^\mu = p^\mu$ and $\alpha_n^\mu = \sqrt{n}a_n^\mu = \alpha_{-n}^{\mu^*}$ for $n \in \mathbb{Z}^+$. As for any mechanical system one introduces a *Lie algebra of Poisson brackets structure* in the phase space of the system, p_τ and x are conjugate maps and then it is natural to set

$$\{p_\tau^\mu(\tau,\sigma), x^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma'),$$
$$\{p_\tau^\mu(\tau,\sigma), p_\tau^\nu(\tau,\sigma')\} = \{x^\mu(\tau,\sigma), x^\nu(\tau,\sigma')\} = 0,$$

$(\eta^{\mu\nu} = (g_L)^{\mu\nu})$ which lead to

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \mathrm{im}\,\delta_{m,-n}\eta^{\mu\nu} \text{ for } m,n \in \mathbb{Z}-\{0\} \text{ and } \{p^\mu, x^\nu\} = \eta^{\mu\nu}.$$

A straightforward calculation using the familiar properties of Poisson brackets $\{AB,C\} = A\{B,C\} + \{A,C\}B$ and $\{A,B\} = -\{B,A\}$, leads to the classical Virasoro (Lie) algebra

$$\{L_m, l_n\} = -i(m-n)L_{m+n}, \qquad m,n \in \mathbb{Z}.$$

A final remark is the following. It is natural to define the extension $\tilde{\Sigma} : [\tau_i, \tau_f] \times [-\pi,\pi] \to M$ of the string path Σ through the equations $\tilde{\Sigma}(\tau,-\sigma) = \tilde{\Sigma}(\tau,\sigma) = \Sigma(\tau,\sigma)$, $\dot{\tilde{\Sigma}}(\tau,-\sigma) = \dot{\tilde{\Sigma}}(\tau,\sigma) = \dot{\Sigma}(\tau,\sigma)$ and $\tilde{\Sigma}'(\tau,-\sigma) = -\tilde{\Sigma}'(\tau,\sigma) = -\Sigma'(\tau,\sigma)$, where $\sigma \in [0,\pi]$, $\dot{\Sigma} = \partial_\tau\Sigma$, $\Sigma' = \partial_\sigma\Sigma$, $\tilde{\Sigma}' = \partial_\sigma\tilde{\Sigma}$ and $\dot{\tilde{\Sigma}} = \partial_\tau\tilde{\Sigma}$. Then the two constraints are expressed by the unique equation $(\dot{\tilde{x}} + \tilde{x}')^2 = 0$ (or $(\dot{\tilde{x}} - \tilde{x}')^2 = 0$), and the Virasoro generators are the Fourier transforms

$$L_n = -\frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{-i\pi(\tau+\sigma)}(\dot{\tilde{x}} + \tilde{x}')^2.$$

2. Polyakov action

The introduction of the Polyakov action [6] (see also Ref. [7]) and its associated method of quantization involves a deep change in the description of the dynamics of the string. While in the Nambu theory the world-sheet Γ_Σ is the image in M of the string path Σ, in the Polyakov approach the world-sheet is considered a one-dimensional complex-analytic manifold (a Riemann surface) and the theory describes fields x^μ ($\mu = 0, 1, \dots, D-1$), which are 0–forms "interacting" with a metric $h_{\alpha\beta}$ ($\alpha, \beta = 0, 1$) with Lorentz signature defining a Riemannian structure on the manifold. In this context, $(\tau, \sigma) \equiv (\xi^0, \xi^1)$ are coordinate functions on the manifold *i.e.* elements of local charts, and the "reparametrization group" is not a group but a pseudogroup of transformations [8] (general covariance). In this section we briefly review the classical equations of the approach and show that the Virasoro generators are the Fourier transforms of the energy-momentum tensor associated with the "matter fields" x^μ.

The Polyakov action is

$$S_{\mathrm{P}} = -\frac{1}{2\pi} \int d^2\xi \, \sqrt{h(\xi)} h^{\alpha\beta}(\xi) \, \partial_\alpha X^\mu(\xi) \, \partial_\beta x^\nu(\xi) \, \eta_{\mu\nu},$$

where $h(\xi) = -\det h_{\alpha\beta}(\xi)$ and $h^{\alpha\beta} h_{\beta\gamma} = \delta^\alpha{}_\gamma$. Since no derivatives of $h_{\alpha\beta}$ appear in the action, the equations of motion for the metric coincide with the definition of the energy-momentum tensor $T_{\alpha\beta}$ of the matter fields, and lead to the vanishing of this tensor

$$T_{\alpha\beta}(\xi) = -\frac{2\pi}{\sqrt{h}} \frac{\delta}{\delta h^{\alpha\beta}(\xi)} S_{\mathrm{P}}(h_{\alpha\beta}, x^\mu) = 0,$$

where $\delta/\delta h^{\alpha\beta}$ is the functional derivative with respect to $h^{\alpha\beta}$.

One obtains

$$T_{\alpha\beta} = -\tfrac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x_\mu + \partial_\alpha x^\mu \partial_\beta x_\mu$$

with

$$T^\alpha{}_\alpha = h^{\alpha\beta} T_{\alpha\beta} = -h^{\alpha\beta} x^\mu{}_{,\alpha} x_{\mu,\beta} + h^{\alpha\beta} x^\mu{}_{,\alpha} x_{\mu\beta} \equiv 0.$$

For the solution $\overline{h}_{\alpha\beta}$ of $T_{\alpha\beta} = 0$ we have $\overline{h}_{\alpha\beta} = 2\gamma_{\alpha\beta} / \overline{h}^{\gamma\delta} \gamma_{\gamma\delta}$ with $\gamma_{\alpha\beta} = x^\mu{}_{,\alpha} x_{\mu,\beta}$, and then $S_{\mathrm{P}}(\overline{h}_{\alpha\beta}, x^\mu) = S_{\mathrm{N}}(x^\mu)$. We want to emphasize however, that though the Nambu action has been recovered, this is a pure formal result, since both approaches have different geometrical basis. General covariance allows to set $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$ with $\eta_{\alpha\beta} = \mathrm{diag}\,(1, -1)$. Then $T_{00} = T_{11} = \frac{1}{2}(\dot{x}^2 + x'^2)$ and $T_{01} = T_{10} = \dot{x}x'$ which lead to the constraint equations in the Nambu theory. According to the remark at the end of Sec. 1, we see that the generators L_n of the Virasoro algebra are the Fourier transforms of the linear combination $\frac{1}{2}(T_{00}+T_{11})$ of components of the tensor $T_{\alpha\beta}$ associated with the extended fields $\tilde{x}$.

3. Quantum Virasoro algebra

To go from the classical to the quantum theory in the operator formalism we have to make the replacement of Poisson brackets of canonical variables by commutators of the associated operators in the Hilbert space V of state vectors, *i.e.*, $\{\ ,\ \}\to[\ ,\ ]/i\hbar$. One obtains $[\alpha^\mu_m,\alpha^\nu_n]=-m\delta_{m,-n}\eta^{\mu\nu}\hbar\mathbf{1}$, $[p^\mu,x^\nu]=i\hbar\eta^{\mu\nu}\mathbf{1}$, where $\mathbf{1}$ is the identity in V. From the classical expressions for the generators L_m we see that for $m\neq 0$ no order ambiguity exists $\alpha_n\alpha_{m-n}=\alpha_{m-n}\alpha_n$ and the quantum L_m operators have the same dependence on the α_n-operators as the classical L_m's have on the $\alpha_n, s\in\mathbf{C}$. For $m=0$ there exists however an ambiguity, since classically

$$L_0=\frac{1}{2}\alpha_0^2+\frac{1}{2}\sum_{n\in\mathbf{Z}^+}(\alpha_n^*\alpha_n+\alpha_n\alpha_n^*)$$

$$=\frac{1}{2}\alpha_0^2+\frac{1}{2}\sum_{n\in\mathbf{Z}^+}(x\alpha_n^*\alpha_n+y\alpha_n\alpha_n^*)=L_0(x,y),$$

with $x,y\in\mathbf{C}$, $x+y=2$, and $\alpha_n^{\mu *}$ does not commute with α_n^ν when the c-numbers are replaced by operators. The normal-ordered operator $:L_0:=\frac{1}{2}\alpha_0^2+\sum_{n\in\mathbf{Z}^+}\alpha_n^{\mu\dagger}\alpha_{n\mu}$ differs from the naive quantization of $L_0(x,y)$ in an infinite constant times the identity *i.e.*, $L_0(x,y)|_{\text{quant.}}=:L_0:+\frac{1}{2}yD\hbar(\sum_{n\in\mathbf{Z}^+}n)\mathbf{1}$ ($D=\dim M$ appears here since $\alpha_n^\mu\alpha_{n\mu}^\dagger=\alpha_n^{\mu\dagger}\alpha_{n\mu}-Dn\hbar\mathbf{1}$). However, there is no prescription in the theory by which $L_0|_{\text{quant.}}$ should be obtained from its classical counterpart in this way. The usual procedure is to make the *assumption* that in any expression involving L_0 one can write for it the normal-ordered $:L_0:$ plus a *finite constant* (which in each case has to be determined by physical conditions) times the identity (for simplicity of notation, we write in the following L_0 for $:L_0:$). We apply these arguments to the evaluation of $[L_m,L_n]$, following the analysis in Ref. [3].

In the classical theory one obtains

$$\{L_m,L_n\}=\frac{1}{2}\sum_{k\in\mathbf{Z}}(k(\alpha_{m-k}\alpha_{k+n})+(m-k)(\alpha_k\alpha_{m-k+n})).$$

In each term of the two infinite sums in the r.h.s. the factors $\alpha\alpha$ commute at the quantum level unless $m=-n$. For $m\neq-n$ one changes $k+n=k'$ in the first sum and obtains 1st. sum $=\sum_{k'\in\mathbf{Z}}(k'-n)\alpha_{m-k'+n}\alpha_{k'}=\sum_{k\in\mathbf{Z}}(k-n)\alpha_k\alpha_{m-k+n}$; then 1st. sum + 2nd. sum $=(m-n)\sum_{k\in\mathbf{Z}}\alpha_k\alpha_{m+n-k}=-2(m-n)L_{m+n}$ and the replacement $\{\ ,\ \}\to[\ ,\ ]/i\hbar$ leads to $[L_m,L_n]=\hbar(m-n)L_{m+n}$. For $m=-n$ one obtains in the r.h.s. L_0, which is determined up to a constant. Then, for arbitrary

$m, n \in \mathbb{Z}$,

$$[L_m, L_n] = \hbar(m-n)L_{m+n} + A(m)\delta_{m,-n}\mathbf{1}.$$

Once $A(m) \in \mathbb{R}$ is fixed, this algebra is known as the *central extension* of the Virasoro algebra, and the constant term is known as the *anomalous term.* Let $m = 0$; for $n = 0$, $0 = A(0)\mathbf{1}$, then $A(0) = 0$. From $[L_m, L_{-m}] = -[L_{-m}, L_m]$ it follows $A(-m) = -A(m)$. Then $A(m)$ has to be fixed only for $m > 0$ (or $m < 0$). From Jacobi identity $[L_k,[L_m,L_n]] + [L_m,[L_n,L_k]] + [L_n,[L_k,L_m]] = 0$ and for $k+m+n = 0$ one finds $A(k)(m-n) + A(m)(n-k) + A(n)(k-m) = 0$ (a). Putting $k = 1$ and for $n \geq 2$ one obtains $A(n+1) = ((2+n)A(n) - (2n+1)A(1))/(n-1)$. Then the subset $\{A(1), A(2)\}$ fixes the set $\{A(n)\}_{n\in\mathbb{Z}^+}$ *i.e.*, we have to determine only two constants. It is easily verified that for $k+m+n = 0$, $A(\ell) = c_1\ell + c_3\ell^3$ satisfies (a) for any c_1 and c_3. To fix these constants one takes vacuum expectation values $\langle\ \rangle_0$ for the commutators $[L_1, L_{-1}] = 2\hbar L_0 + A(1)$ and $[L_2, L_{-2}] = 4\hbar L_0 + A(2)$, where the vacuum state $|0\rangle$ satisfies $p|0\rangle = 0$, $\alpha_n^\mu|0\rangle = 0$ and $\langle 0|0\rangle = 1$. From the expansions of the $L_{m's}$ we have $\langle[L_j, L_{-j}]\rangle_0 = A(j)$ (since $\langle L_0\rangle_0 = 0$) and a straightforward calculation leads to $A(1) = 0$, $A(2) = D\hbar^2/2$. Solving for the c_1's gives $c_1 = -c_3 = D\hbar^2/12$. Then

$$[L_m, L_n] = \hbar(m-n)L_{m+n} + \frac{D\hbar^2}{12}m(m^2-1)\delta_{n,-m}\mathbf{1}.$$

We notice that the anomaly is a quantum correction ($O(\hbar^2)$) to the "leading term" proportional to $\hbar$. The replacement $[\ ,\ ] \to i\hbar\{\ ,\ \}$ and the subsequent limit $\hbar \to 0$ lead to the classical algebra.

Part II

4. S^1 as a differential manifold

We start by considering S^1 as the subset of $\mathbb{R}^2$ given by the points

$$S^1 = \{(x,y)|x^2 + y^2 = 1\}.$$

In this way, S^1 can be seen to be a real connected Hausdorff one dimensional compact topological manifold (S^1, τ_{S^1}) without boundary (τ_{S^1} is the topology of S^1). In fact, S^1 inherits the (metric) topology from $\mathbb{R}^2$, the open sets of S^1 being the intersections of S^1 with the open sets of $\mathbb{R}^2$ (arbitrary unions of open balls $B_\epsilon(x_0, y_0) = \{(x,y)|(x-x_0)^2 + (y-y_0)^2 < \epsilon\}$ of $\mathbb{R}^2$). S^1 is connected since it is not the disjoint union of two non-empty open sets of its topology, Housdorff since any two different points of S^1 have non-intersecting open neighbourhoods, and compact

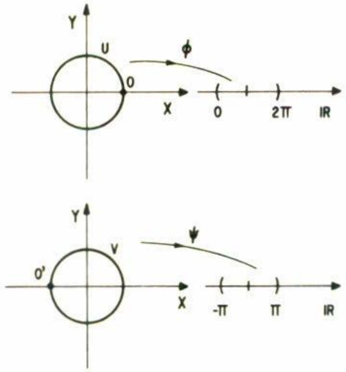


FIGURE 1. Minimum atlas on S^1.

since any open cover (collection of sets of the topology such that its union contains S^1) contains a finite subcollection which is itself a cover.

We define an atlas (not maximum) on S^1 as consisting of the two charts (U, ϕ) and (V, Ψ) with $U = S^1 - \{0\}$, $V = S^1 - \{0'\}$ elements of the topology (Fig. 1) and ϕ^{-1}, Ψ^{-1} given by

$$\phi^{-1} : (0, 2\pi) \to U, \qquad t \mapsto (\cos t, \sin t),$$

$$\Psi^{-1} : (-\pi, \pi) \to V, \qquad t \mapsto (\cos t, \sin t).$$

Both ϕ^{-1} and Ψ^{-1} are continuous functions from open sets of the real line to open sets of the circle. Their inverses are also continuous. Therefore S^1 is locally homeomorphic to $\mathbb{R}^1$. The smooth structure of the manifold is seen as follows. The transition function

$$\Psi \circ \phi^{-1} : \phi(S^1 - \{0, 0'\}) \to \Psi(S^1 - \{0, 0'\}),$$

$$t \mapsto \Psi \circ \phi^{-1}(t) = \begin{cases} t, & 0 < t < \pi \\ t - 2\pi, & \pi < t < 2\pi \end{cases}$$

(Fig. 2) is given by a power series (only the first linear term of the series exists, the others being identically zero). Therefore S^1 is a real-analytic (C^∞) manifold. For

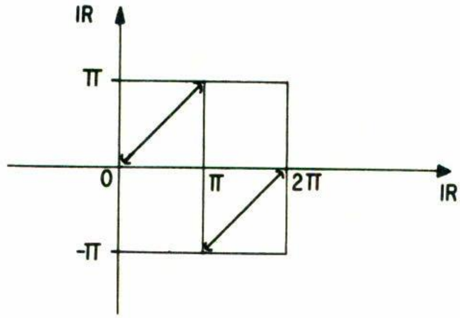


FIGURE 2. Plot of the transition function $\psi \circ \phi^{-1}$.

the derivatives of the transition function we have

$$\frac{d^n}{dt^n}(\Psi \circ \phi^{-1}(t)) = \delta_{n,1}, \qquad 1 \leq n \in \mathbb{Z}, \quad t \in (0,\pi) \cup (\pi, 2\pi).$$

The first derivative does not change sign which means that the manifold is orientable, the existence (though vanishing) of the other derivatives means that the manifold is smooth.

In what follows, and only for easy of calculations, we shall consider S^1 as a topological subspace of $\mathbb{C}^1$ *i.e.* $S^1 = \{z | |z| = 1\}$. We shall then parametrize the points of S^1 by the complex numbers $e^{i2\pi t}$ with $t \in \mathbb{R}$. Also, $C^\infty(A,B)(C^\omega(A,B))$ will denote the set of smooth (analytic) functions from the manifold A to the manifold B. In particular, with pointwise sum and product, $C^\infty(\mathbb{R}^1, \mathbb{R}^1)$ and $C^\infty(S^1, \mathbb{R}^1)$ are commutative rings with unit e: $A \to \mathbb{R}^1, p \mapsto 1$ ($A = \mathbb{R}^1$ or S^1), and if multiplication by numbers ($\mathbb{R}^1$) is included, the structures are infinite dimensional real associative algebras. In the present context, $\mathbb{R}^1$ is given the usual smooth manifold structure through the minimum atlas $(\mathbb{R}^1, \mathrm{id}|_{\mathbb{R}^1})$.

5. *Projection of* $\mathbb{R}^1$ *onto* S^1

Define the function*

$$\pi : \mathbb{R}^1 \to S^1, \qquad t \mapsto e^{i2\pi t}.$$

*This is a particular case of the covering of the n-torus ($n \in \mathbb{Z}^+$) by $\mathbb{R}^n$ through the projection $\pi_n : \mathbb{R}^n \to T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n$, $(t_1, \ldots, t_n) \mapsto (e^{i2\pi t_1}, \ldots, e^{i2\pi t_n})$. For $n = 1$ we have the 1-torus $T^1 = 1$ – sphere S^1.

We have $\pi(t+n) = \pi(t)$, $n \in \mathbb{Z}$ *i.e.*, π has period 1 (later we shall also use $\hat{\pi} : \mathbb{R}^1 \to S^1$, $t \mapsto e^{it}$ which has period 2π). π is called a covering of S^1 by $\mathbb{R}^1$. Notice that $\pi((n, n+1)) = S^1 - \{1\}$, π is many-to-one, onto and $\pi \in C^\omega(\mathbb{R}^1, S^1)$. For any $p \in S^1$, $\pi^{-1}(\{p\})$ is called the fiber of p and is denoted by F_p^ξ. Also notice that $\mathbb{R}^1 = \bigcup_{p \in S^1} F_p^\xi$ *i.e.*, $\mathbb{R}^1$ can be considered as a bundle of fibers, and each fiber is in a one-to-one correspondence with the integers $\mathbb{Z}$, the fiber of the bundle. We have then the structure of a smooth fiber bundle ξ, which can be represented by the diagram

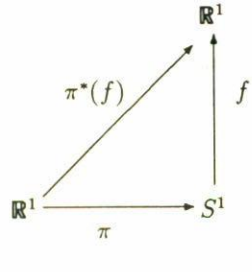
$$\xi : \mathbb{Z} \longrightarrow \mathbb{R}^1 \xrightarrow{\pi} S^1.$$

A function $f \in C^\infty(S^1, \mathbb{R}^1)$ is called a section of the bundle if $\pi \circ f = \mathrm{id}|_{S^1}$.

Let $f \in C^\infty(S^1, \mathbb{R}^1)$. π induces

$$\pi^* : C^\infty(S^1, \mathbb{R}^1) \to C^\infty(\mathbb{R}^1, \mathbb{R}^1),$$
$$f \mapsto \pi^*(f) : \mathbb{R}^1 \to \mathbb{R}^1,$$
$$t \mapsto \pi^*(f)(t) \equiv (f \circ \pi)(t),$$

i.e., one has the diagram



It is easy to see that $\pi^*(f)$ has period 1. In fact, for $n \in \mathbb{Z}$, $\pi^*(f)(t+n) = f(\pi(t+n)) = f(\pi(t)) = \pi^*(f)(t)$. We denote by $C_1^\infty(\mathbb{R}^1, \mathbb{R}^1)$ and $C_{2\pi}^\infty(\mathbb{R}^1, \mathbb{R}^1)$ the rings of smooth functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ with period 1 and 2π respectively, and it is clear that these sets are subrings of the ring $C^\infty(\mathbb{R}^1, \mathbb{R}^1)$. Also, we have $\pi^* : C^\infty(S^1, \mathbb{R}^1) \to C_1^\infty(\mathbb{R}^1, \mathbb{R}^1)$.

6. Definition of the vector field $d/d\theta$

Let $p \in S^1$ and $\lambda \in F_p^\xi$. Define the map

$$\frac{d}{d\theta} : C^\infty(S^1, \mathbb{R}^1) \to C^\infty(S^1, \mathbb{R}^1),$$

$$f \mapsto \frac{d}{d\theta}(f) : S^1 \to \mathbb{R}^1,$$

$$p \mapsto \frac{d}{d\theta}(f)(p) \equiv \frac{d}{dt}(\pi^*(f)(t))|_{t=\lambda} = \frac{d}{dt}(f(e^{2\pi it}))|_{t=\lambda}$$

$$= 2\pi iz\frac{d}{dz}(f(z))|_{z=e^{2\pi i\lambda}}$$

where t is the usual coordinate function on $\mathbb{R}^1$ and $\frac{d}{dt}$ is the base of the tangent space $T_{*t}\mathbb{R}^1$ to $\mathbb{R}^1$ at the point t. It is easy to see that if $\lambda' \in F_p^\xi$ and $\lambda' \neq \lambda$ *i.e.* $\lambda' = \lambda + n$ with $0 \neq n \in \mathbb{Z}$ then

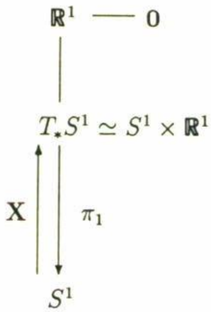
$$\frac{d}{dt}(\pi^*(f)(t))|_{t=\lambda+n} = \frac{d}{dt}(\pi^*(f)(t))|_{t=\lambda}$$

i.e. $\frac{d}{d\theta}$ is well defined. Also, $\pi \in C^\omega(\mathbb{R}^1, S^1)$ and $f \in C^\infty(S^1, \mathbb{R}^1)$ imply $f \circ \pi \in C^\infty(\mathbb{R}^1, \mathbb{R}^1)$ and then $\frac{d}{d\theta}(f) \in C^\infty(S^1, \mathbb{R}^1)$ as it must be. We have the

Proposition: $\frac{d}{d\theta}$ is a derivation *i.e.*, it obeys the Leibnitz rule and it is $\mathbb{R}$-linear.

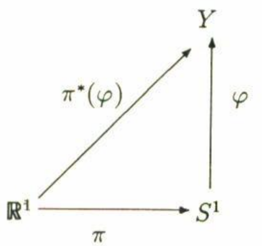
Proof: i) $z\frac{d}{dz}(f(z)g(z)) = (zf')g + f(zg')$ *i.e.* $\frac{d}{d\theta}(fg) = \frac{d}{d\theta}(f)g + f\frac{d}{d\theta}(g)$; ii) $\frac{d}{dz}(\alpha f(z) + \beta g(z)) = \alpha f'(z) + \beta g'(z)$ *i.e.* $\frac{d}{d\theta}(\alpha f + \beta g) = \alpha\frac{d}{d\theta}(f) + \beta\frac{d}{d\theta}(g)$, for all $\alpha, \beta \in \mathbb{R}^1$. QED.

Let $\kappa(\mathbb{R}^1)$ be the modulo of vector fields on $\mathbb{R}^1$ over the ring $C^\infty(\mathbb{R}^1, \mathbb{R}^1)$. For any $V \in \kappa(\mathbb{R}^1)$ we have $V = \varphi\frac{d}{dt}$ with $\varphi \in C^\infty(\mathbb{R}^1, \mathbb{R}^1)$ and $V : \mathbb{R}^1 \to T_*\mathbb{R}^1$, $t \mapsto (t, \varphi(t)\frac{d}{dt}) \in \{t\} \times T_{*t}\mathbb{R}^1$ where $T_*\mathbb{R}^1$ is the tangent bundle of $\mathbb{R}^1$. Call $\kappa(S^1)$ the modulo of vector fields on S^1 over the ring $C^\infty(S^1, \mathbb{R}^1)$. From the above analysis it is clear that $\frac{d}{d\theta} \in \kappa(S^1)$ and for any $\mathbf{X} \in \kappa(S^1)$ we have $\mathbf{X} = \psi_\mathbf{X}\frac{d}{d\theta}$ with $\psi_\mathbf{X} \in C^\infty(S^1, \mathbb{R}^1)$ and $\mathbf{X} : S^1 \to T_*S^1$, $p \mapsto (p, \psi_\mathbf{X}(p)\frac{d}{d\theta}(p)) \in \{p\} \times T_{*p}S^1$, where T_*S^1 is the union of all tangent spaces to S^1 and $T_{*p}S^1$ is the tangent space at the point p. T_*S^1 together with the left projection $\pi_1 : T_*S^1 \to S^1$, $\pi_1^{-1}(\{p\}) = \{p\} \times T_{*p}S^1 = F_p$ has the structure of a smooth vector bundle, $\tau(S^1)$: tangent bundle of the circle, with fiber $\mathbb{R}^1$ and structure group $\mathbf{0} = (\{1\}, \cdot)$. In this context $\kappa(S^1)$ is nothing but the space of smooth sections of the bundle since $\mathbf{X} \in \kappa(S^1)$ is smooth and $\pi_1 \circ \mathbf{X} = \mathrm{id}|_{S^1}$. We write $\kappa(S^1) = C^\infty(T_*S^1)$. $\tau(S^1)$ is a product (trivial) bundle, since T_*S^1 is topologically homeomorphic to $S^1 \times \mathbb{R}^1$. These facts are represented by the diagram



7. A theorem on $C^\infty(S^1, \mathbb{R}^1)$

Let $\pi : \mathbb{R}^1 \to S^1$, $t \mapsto e^{i2\pi t}$ as before and let $\varphi : S^1 \to Y$ with Y arbitrary set. We consider the diagram



i.e., $\pi^*(\varphi) : \mathbb{R}^1 \to Y$, $t \mapsto (\varphi \circ \pi)(t) = \varphi(e^{i2\pi t})$. For $n \in \mathbb{Z}$, $\pi^*(\varphi)(t+n) = \pi^*(\varphi)(t)$ *i.e.* $\pi^*(\varphi)$ has period 1. Denoting $F(S^1, Y)$ $(F_1(\mathbb{R}^1, Y))$ the set of functions from $S^1(\mathbb{R}^1)$ to Y (with period 1) we have $\pi^* : F(S^1, Y) \to F_1(\mathbb{R}^1, Y)$, $\varphi \mapsto \pi^*(\varphi)$. Consider now the map $\alpha : F_1(\mathbb{R}^1, Y) \to F(S^1, Y)$, $\psi \mapsto \alpha(\psi) : S^1 \to Y$, $p \mapsto \alpha(\psi)(p) \equiv \psi(\lambda)|_{\lambda \in F_p^\xi}$. We have the

Proposition: $\pi^* \circ \alpha = \mathrm{id}|_{F_1(\mathbb{R}^1, Y)}$ and $\alpha \circ \pi^* = \mathrm{id}|_{F(S^1, Y)}$ *i.e.* $\pi^* = \alpha^{-1}$ provides a set isomorphism between $F(S^1, Y)$ and $F_1(\mathbb{R}^1, Y)$.

Proof: i) $\pi^* \circ \alpha : F_1(\mathbb{R}^1, Y) \to F_1(\mathbb{R}^1, Y)$, $\psi \mapsto \pi^*(\alpha(\psi)) = \alpha(\psi) \circ \pi : \mathbb{R}^1 \to Y$, $t \mapsto (\alpha(\psi) \circ \pi)(t) = \alpha(\psi)(p) = \psi(t')|_{t' \in F_p^\xi} = \psi(t+n)|_{n \in \mathbb{Z}} = \psi(t)$; the result holds for all $t \in \mathbb{R}^1$, then $(\pi^* \circ \alpha)(\psi) = \psi$. ii) $\alpha \circ \pi^* : F(S^1, Y) \to F(S^1, Y)$, $\varphi \mapsto \alpha(\pi^*(\varphi)) = \alpha(\varphi \circ \pi) : S^1 \to Y$, $p \to \alpha(\varphi \circ \pi)(p) = (\varphi \circ \pi)(\lambda)|_{\lambda \in F_p^\xi} = \varphi(p)$; the result holds for all $p \in S^1$, then $(\alpha \circ \pi^*)(\varphi) = \varphi$. QED.

In the above proposition, let $Y = \mathbb{R}^1$ and $F = C^\infty$. Then we have

Theorem A: Let $\pi : \mathbb{R}^1 \to S^1$, $t \mapsto e^{i2\pi t}$. Then $\pi^* : C^\infty(S^1, \mathbb{R}^1) \to C_1^\infty(\mathbb{R}^1, \mathbb{R}^1)$, $\varphi \mapsto \pi^*(\varphi) = \varphi \circ \pi$ is a set isomorphism.

Similarly, we have

Theorem B: Let $\hat{\pi} : \mathbb{R}^1 \to S^1$, $t \mapsto e^{it}$. Then $\hat{\pi}^* : C^\infty(S^1, \mathbb{R}^1) \to C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$, $\varphi \mapsto \hat{\pi}^*(\varphi) = \varphi \circ \hat{\pi}$ is a set isomorphism.

Most important, we also have

Theorem C: $\hat{\pi}^* : C^\infty(S^1, \mathbb{R}^1) \to C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$ is a linear space isomorphism.

Proof: $\hat{\pi}^*$ is a bijection, $C^\infty(S^1, \mathbb{R}^1)$ and $C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$ are linear spaces, and for any $\alpha, \beta \in \mathbb{R}^1$, $\hat{\pi}^*(\alpha\varphi + \beta\psi) = \alpha\hat{\pi}^*(\varphi) + \beta\hat{\pi}^*(\psi)$ since $(\alpha\varphi + \beta\psi) \circ \hat{\pi} = \alpha(\varphi \circ \hat{\pi}) + \beta(\psi \circ \hat{\pi})$.QED. (Notice that $\hat{\pi}^*$ does not give an algebra isomorphism *i.e.*, $\hat{\pi}^*$ is not a ring homomorphism since $\hat{\pi}^*(\varphi\psi) \neq \hat{\pi}^*(\varphi)\hat{\pi}^*(\psi)$.)

The isomorphism between smooth functions on the circle and smooth periodic functions on the real numbers allows us to rewrite the vector fields on S^1 entirely in terms of functions belonging to $C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$. In fact, from the theory of Fourier series, for any $\mathbf{X} \in \kappa(S^1)$ we can write

$$\mathbf{X} = \alpha_{0\mathbf{X}} c_0 + \sum_{n=1}^{\infty} (\alpha_{n\mathbf{X}} c_n + \beta_{n\mathbf{X}} s_n) : C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1) \to C^\infty_{2\pi}(\mathbb{R}^1, \mathbb{R}^1),$$

$$f \mapsto \mathbf{X}(f) = \alpha_{0\mathbf{X}} c_0(f) + \sum_{n=1}^{\infty} (\alpha_{n\mathbf{X}} c_n(f) + \beta_{n\mathbf{X}} s_n(f)) : (\mathbb{R}^1 \to \mathbb{R}^1),$$

$$\lambda \mapsto \alpha_{0\mathbf{X}} c_0(f)(\lambda) + \sum_{n=1}^{\infty} (\alpha_{n\mathbf{X}} c_n(f)(\lambda) + \beta_{n\mathbf{X}} s_n)(f)(\lambda))$$

$$= \alpha_{0\mathbf{X}} + \sum_{n=1}^{\infty} \alpha_{n\mathbf{X}} \cos nt \frac{d}{dt}(f(t))|_{t=\lambda} + \beta_{n\mathbf{X}} \sin nt \frac{d}{dt}(f(t))|_{t=\lambda},$$

with

$$\alpha_{0\mathbf{X}}^2 + \sum_{n=1}^{\infty} (\alpha_{n\mathbf{X}}^2 + \beta_{n\mathbf{X}}^2) < \infty \qquad \text{(Parseval inequality)}.$$

Clearly, the structure $(\kappa(S^1)), +; \mathbb{R}^1, .)$ is a vector space, with the fields

$$\{c_0, c_n, s_n\}_{n=1}^{\infty}$$

acting as a basis. The linear space operations are given by

$$+ : \kappa(S^1) \times \kappa(S^1) \to \kappa(S^1), \quad (X,Y) \mapsto (\alpha_{0\mathbf{X}} + \alpha_{0Y})c_0 + \sum_{n=1}^{\infty}((\alpha_{nX} + \alpha_{nY})c_n$$

$$+ (\beta_{nX} + \beta_{nY})s_n),$$

$$\cdot : \mathbb{R}^1 \times \kappa(S^1) \to \kappa(S^1), \quad (\lambda, X) \mapsto (\lambda\alpha_{0X})c_0 + \sum_{n=1}^{\infty}((\lambda\alpha_{nX})c_n + (\lambda\beta_{nX})s_n).$$

For $\mathbf{X} \in \kappa(S^1)$ we can also write

$$\mathbf{X} = \varphi_{\mathbf{X}}\mu : C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1) \to C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1), \quad f \mapsto (\varphi_{\mathbf{X}}\mu)(f) : \mathbb{R}^1, \to \mathbb{R}^1,$$

$$\lambda \mapsto \varphi_{\mathbf{X}}(t)\frac{d}{dt}(f(t))|_{t=\lambda}$$

with $\varphi_{\mathbf{X}} \in C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$ given by $\varphi_{\mathbf{X}} = \alpha_{0\mathbf{X}}C_0 + \sum_{n=1}^{\infty}(\alpha_{n\mathbf{X}}C_n + \beta_{n\mathbf{X}}S_n)$, $C_0(\lambda) = 1$, $C_n(\lambda) = \cos n\lambda$, $S_n(\lambda) = \sin n\lambda$, $n \geq 1$.

8. Lie algebra of vector fields

On the vector space $(\kappa(S^1), \mathbb{R}^1)$ we define the associative product $* : \kappa(S^1) \times \kappa(S^1) \to \kappa(S^1)$, $(\mathbf{X}, \mathbf{Y}) \mapsto \mathbf{X} * \mathbf{Y} : C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1) \to C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1)$,

$$f \mapsto (\mathbf{X} * \mathbf{Y})(f) : \mathbb{R}^1 \to \mathbb{R}^1,$$

$$\lambda \mapsto \varphi_{\mathbf{X}}(t)\frac{d}{dt}(\varphi_{\mathbf{Y}}(t)\frac{d}{dt}(f(t))|_{t=\lambda}$$

$$= \varphi_{\mathbf{X}*\mathbf{Y}}(t)\frac{d}{dt}(f(t))|_{t=\lambda}$$

with

$$\varphi_{\mathbf{X}*\mathbf{Y}}(t) = \varphi_{\mathbf{X}}(t)\frac{d}{dt}(\varphi_{\mathbf{Y}}(t)) + \varphi_{\mathbf{X}}(t)\varphi_{\mathbf{Y}}(t)\frac{d}{dt},$$

which makes $(\kappa(S^1), +, *)$ a ring; and the non-associative Lie product

$$
\begin{aligned}
&[\ ,\ ] : \kappa(S^1) \times \kappa(S^1) \to \kappa(S^1),\\
&(\mathbf{X}, \mathbf{Y}) \mapsto [\mathbf{X}, \mathbf{Y}] \equiv \mathbf{X} * \mathbf{Y} - \mathbf{Y} * \mathbf{X} : C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1) \to C^{\infty}_{2\pi}(\mathbb{R}^1, \mathbb{R}^1),\\
&f \mapsto [\mathbf{X}, \mathbf{Y}](f) : \mathbb{R}^1, \to \mathbb{R}^1,\\
&\lambda \mapsto [\mathbf{X}, \mathbf{Y}](f)(\lambda) = \varphi_{[\mathbf{X},\mathbf{Y}]}(t) \frac{d}{dt}(f(t))|_{t=\lambda}
\end{aligned}
$$

with

$$\varphi_{[\mathbf{X},\mathbf{Y}]}(t) = \varphi_{\mathbf{X}}(t) \frac{d}{dt}(\varphi_{\mathbf{Y}}(t)) - \frac{d}{dt}(\varphi_{\mathbf{X}}(t)\varphi_Y(t),$$

which makes the structure $(\kappa(S^1), +, [\ ,\ ]; \mathbb{R}^1, \cdot)$ a Lie algebra, *the Lie algebra of the vector fields on* S^1. In fact, as can be easily verified,

i) $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$,
ii) $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$.

For the basis vector fields one has $c_n(f)(\lambda) = \cos nt(\frac{d}{dt} f(t))|_{t=\lambda}$, $n \geq 0$, $s_n(f)(\lambda) = \sin nt \frac{d}{dt}(f(t))|_{t=\lambda}$, $n \geq 1$, and a straightforward calculation of their Lie products leads to

$$
\begin{aligned}
[c_n, c_m] &= \tfrac{1}{2}(n-m)s_{n+m} + \tfrac{1}{2}(n+m)s_{|n-m|}\mathrm{sg}(n-m),\\
[s_n, s_m] &= -\tfrac{1}{2}(n-m)s_{n+m} + \tfrac{1}{2}(n+m)s_{|n-m|}\mathrm{sg}(n-m),\\
[c_n, s_m] &= -\tfrac{1}{2}(n-m)c_{n+m} + \tfrac{1}{2}(n+m)c_{|n-m|},
\end{aligned}
$$

where

$$|n-m| = \begin{cases} n-m, & n \geq m \\ m-n, & n \geq m \end{cases}, \qquad \mathrm{sg}(n-m) = \begin{cases} +1, & n \geq m \\ -1, & n < m \end{cases}$$

and $s_0 = 0 \in \kappa(S^1)$. This result is known as the Witt algebra of the vector fields on the circle or equivalently *the algebra of smooth sections of the tangent bundle of the circle.* The structure constants can be identified by writing the previous commutators in the form

$$[c_n, c_m] = \sum_{r=0}^{\infty} (\alpha^{cc}_{nmr} c_r + \beta^{cc}_{nmr} s_r),$$

$$[s_n, s_m] = \sum_{r=0}^{\infty} (\alpha^{ss}_{nmr} c_r + \beta^{ss}_{nmr} s_r),$$

$$[c_n, s_m] = \sum_{r=0}^{\infty} (\alpha^{cs}_{nmr} c_r + \beta^{cs}_{nmr} s_r).$$

One obtains

$$\begin{pmatrix} \alpha^{cc} & \beta^{cc} \\ \alpha^{ss} & \beta^{ss} \\ \alpha^{cs} & \beta^{cs} \end{pmatrix}_{nmr}$$
$$= \begin{pmatrix} 0 & \frac{1}{2}(n-m)\delta_{r,n+m} + \frac{1}{2}(n+m)\mathrm{sg}(n-m)\delta_{r,|n-m|} \\ 0 & -\frac{1}{2}(n-m)\delta_{r,n+m} + \frac{1}{2}(n+m)\mathrm{sg}(n-m)\delta_{r,|n-m|} \\ -\frac{1}{2}(n-m)\delta_{r,n+m} + \frac{1}{2}(n+m)\delta_{r,|n-m|} & 0 \end{pmatrix}$$

It is instructive to compare this algebra with the Lie algebra of S^1 as a Lie group. The former is a highly non-trivial infinite dimensional algebra, while the latter is a trivial (all the Lie products vanish) one dimensional algebra, since as a vector space the algebra is the tangent space to the circle at the identity of the group ($\mathbb{C} \ni z = 1$).

9. Complexification

To complexify the previous algebra we define the set $\kappa(S^1)^c \equiv \{\alpha_0 c_0 + \sum_{n=1}^{\infty}(\alpha_n c_n + \beta_n s_n)\}$ with $\{\alpha_0, \alpha_n, \beta_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $|\alpha|_0^2 + \sum_{n=1}^{\infty}(|\alpha|_n^2 + |\beta_n|^2) < \infty$. The usual sum and product by complex numbers make $\kappa(S^1)^c$ a linear space with the same basis as $\kappa(S^1)$, namely the vector fields c_n and s_n; however the elements of $\kappa(S^1)^c$ considered as vector fields can not represent fluxes on S^1 since now the fields have complex coefficients and no complex flux can be carried on the circle which is a real one-dimensional manifold. We go now to the basis $\{\ell_0, \ell_{\pm n}\}_{n=1}^{\infty}$, $\ell_0 \equiv c_0$, $\ell_{\pm n} \equiv c_n \pm is_n$. One has $\ell_{-n} = \ell_n^*$.

It is clear that the structure

$$(\kappa(S^1)^c, +, [\ ,\ ]; \mathbb{C}, .)$$

is a complex infinite dimensional Lie algebra, and for the Lie products of the basis vectors one gets

$$[\ell_m, \ell_n] = -i(m-n)\ell_{m+n}, \qquad m, n \in \mathbb{Z}$$

which is the Virasoro algebra without the central term. (For the linear combinations $\ell_{\pm n}, n \geq 1$, one obtains $\ell_{\pm n}(f)(\lambda) = e^{\pm int}\frac{d}{dt}f(t)|_{t=\lambda}$ which (up to a factor i) coincide with the infinitesimal generators of diffeomorphisms of S^1 given in Eq. (2.1.86) of Ref. [3] in terms of the angular variable θ.)

Conclusions

In the context of bosonic string theory we discussed the appearance of the Virasoro algebra, both in the classical and in the quantum cases. The classical algebra is free of anomalies while the quantum algebra has an extra term (central charge), which appears due to an operator order ambiguity in one of the generators of the algebra (L_0) in the transition from classical to quantum mechanics. The classical algebra is then shown to coincide with the complexification of the Lie algebra of the vector fields on the circle S^1 considered as a differential manifold (Witt algebra). The vector fields on a manifold are the sections of the tangent bundle of the manifold. One thus obtains a relation between physical objects (classical propagating strings) and geometrical ones (a vector bundle structure). Recent developments in the geometric approach to the quantum theory of closed bosonic strings can be found in Ref. [9], where the group of diffeomorphisms of the circle plays a central róle.

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Resumen. Se presenta la derivación del álgebra de Virasoro en el contexto de las teorías clásica y cuántica de cuerdas bosónicas. El álgebra clásica está estrechamente relacionada con el álgebra de Lie de los campos vectoriales sobre el círculo como variedad diferenciable, lo que sugiere una bella y profunda conexión entre objetos físicos y geométricos.