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Physical interpretation of the Weert superpotential

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Abstract. Weert [1] obtained a potential for the bounded part of the Maxwell Tensor which is associated to the Liénard-Wiechert field. We show that this potential can be interpretated as an intrinsic Angular Momentum Density for the corresponding Electromagnetic Field.

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1. Introduction

In this paper we deal with electromagnetic field which is produced by a point particle charge in the Minkowski space. This charge gives the Liénard-Wiechert field (LW) from which the Maxwell tensor T_{jc} can be obtained. The last can be separated in two parts: $\underset{\mathrm{B}}{T}_{jc}$ the bounded one, and $\underset{\mathrm{R}}{T}_{jc}$ the radiative part (in the sense of Teitelboim [2]).

Sec. 2 is devoted to a brief exposition of the LW field. Weert [1] constructed a potential $\underset{\mathrm{B}}{K}_{jbc}$ for $\underset{\mathrm{B}}{T}_{jc}$, this will serve us to propose, in Sec. 3, a physical interpretation of this potential using the Bhabha [4]-Synge [5] region; moreover, if we decompose $\underset{\mathrm{B}}{K}_{jbc}$ in two parts which satisfy the symmetry of the Lanczos Spintensor [6], we obtain the rupture of $\underset{\mathrm{B}}{T}_{jc}$ proposed by López [9], which is very important in the study of the angular momentum of the LW field. In Sec. 4 we construct a non-local Superpotential (depending of the past history of the charge) for the radiative part; and we show the terms of $\underset{\mathrm{R}}{T}_{jc}$ which do not participate in the flux of energy and momentum through the Bhabha-Synge tube.

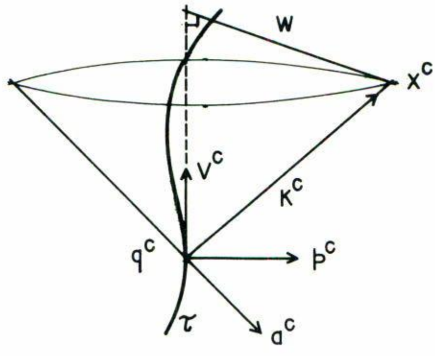

FIGURE 1. Kinematics of the world-line.

2. Point charge in arbitrary motion

A charged particle moving in an arbitrary motion in the Minkowskian space, produced the electromagnetic LW field with the 4–potential and Faraday tensor given by

$$A^c = qw^{-1}v^c, \qquad F_{bc} = qw^{-2}(U_bK_c - U_cK_b), \tag{1.a}$$

and the Maxwell Tensor

$$T_{bc} = q^2w^{-4}\Big[K_bU_c + K_cU_b + (a^2 - B^2)K_bK_c + \tfrac{1}{2}g_{bc}\Big], \tag{1.b}$$

where (Fig. 1):

$$(x^c) = (x, y, z, t), \qquad (g_{bc}) = \mathrm{Diag}\,(1, 1, 1, -1)$$

τ is the proper timer, $q^c(T)$ will be the retarded point, v^c is the 4-velocity, a^c the 4–acceleration

$$K^c = x^c - q^c \tag{1.c}$$

$w = -K^cv_c$ which is the so called retarded distance

$$W = -K^ca_c, \qquad a^2 = a^ca_c$$

$B = w^{-1}(1 - W)$ known as the Plebański [10] invariant

$$U_c = Bv_c + a_c, \qquad p^c = w^{-1}K^c - v^c.$$

Teitelboim [2] proved that Eq. (1.b) can be written as the sum of a bounded part $\underset{\mathrm{B}}{T}_{bc}$ and a radiative one $\underset{\mathrm{R}}{T}_{bc}$, namely,

$$T_{bc} = \underset{\mathrm{B}}{T}_{bc} + \underset{\mathrm{R}}{T}_{bc}, \tag{2.a}$$

such that

$$\underset{\mathrm{B}}{T}_{bc} = q^2 w^{-4}\left[\tfrac{1}{2}g_{bc} + (K_b a_c + K_c a_b) + B(K_b v_c + K_c v_b) - w^{-2}(1-2W)K_b K_c\right], \tag{2.b}$$

and

$$\underset{\mathrm{R}}{T}_{bc} = q^2 w^{-2}(a^2 - w^{-2}W^2)K_b K_c. \tag{2.c}$$

Each tensor possesses the following differential properties

$$\underset{\mathrm{B}}{T}_{b}{}^{c}{}_{,c} = 0, \tag{2.d}$$

$$\underset{\mathrm{R}}{T}_{b}{}^{c}{}_{,c} = 0, \tag{2.e}$$

respectively, when it is valued out of the world-line.

Weert [1] proved that Ec. (2.d) comes out from the existence of the superpotential

$$\underset{\mathrm{B}}{K}_{jbc} = -q^2/4w^{-4}\left[w^{-1}(3-4W)(v_j \times K_b)K_c + 4(a_j \times K_b)K_c + g_{cj}K_b - g_{cb}K_j\right], \tag{3.a}$$

where we have used the Lowry [3] notation

$$A_j \times B_c = A_j B_c - A_c B_j, \tag{3.b}$$

such that

$$\underset{\mathrm{B}}{T}_{bc} = \underset{\mathrm{B}}{K}_{b}{}^{j}{}_{c,j}. \tag{3.c}$$

From here, (2.d) follows immediately.

In the next section we will study (3.a) in order to give a physical interpretation of this superpotential. In Sec. 4 we will do a brief analysis of Eqs. (2.c-e).

3. Weert superpotential

Our point charge in arbitrary motion gives an electromagnetic field which possesses an intrinsic angular momentum (IAM). Here we will prove that $\underset{\mathrm{B}}{K}_{jbc}$ behaves like a

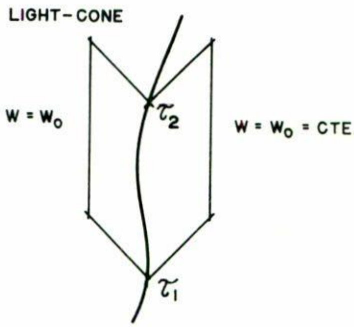

FIGURE 2. Bhabha-Synge tube.

density for IAM when the corresponding fluxes are calculated through a Bhabha [4]-Synge [5] tube.

The superpotential (3.a) has the following properties

$$
\begin{aligned}
\underset{\mathrm{B}}{K}_{jbc} &= -\underset{\mathrm{B}}{K}_{bjc} && \text{antisymmetry,}\\
\underset{\mathrm{B}}{K}_{j\ c}^{\ c} &= 0 && \text{null trace,}\\
\underset{\mathrm{B}}{K}_{jbc} + \underset{\mathrm{B}}{K}_{bcj} + \underset{\mathrm{B}}{K}_{cjb} &= 0 && \text{cyclic,}\\
\underset{\mathrm{B}}{K}_{jb\ ,c}^{\ \ c} &= 0 && \text{null divergence.}
\end{aligned}
\tag{4}
$$

These properties agree with those of the Lanczos [6] Generator K_{jbc} for the Weyl tensor of the space-time. Lanczos calculated K_{jbc} for weak gravitation fields and in his analysis, the Dirac equation, for spin $\frac{1}{2}$, appeared. For this reason he called the potential K_{jbc} spin-tensor. In our case, $\underset{\mathrm{B}}{K}_{jbc}$ will be associated with the IAM of the LW field.

Consider the Bhabha-Synge tube (Fig. 2), which is composed by the light cones with the tops in $\tau = \tau_1$ and $\tau = \tau_2$, and a surface of constant retarded distance. First, let us calculate the $\underset{\mathrm{B}}{K}_{jbc}$ flux through a light cone; the expression is given by Synge [5]

$$
\dot{M}_{jb} = \int_{\tau=\mathrm{const}} \underset{\mathrm{B}}{K}_{jbc}\, d\sigma^c = -\int_0^{w_0} w\, dw \int_{\tau=\mathrm{const}} d\Omega\, \underset{\mathrm{B}}{K}_{jbc} K^c, \tag{5.a}
$$

where $d\Omega$ is the element of solid angle.

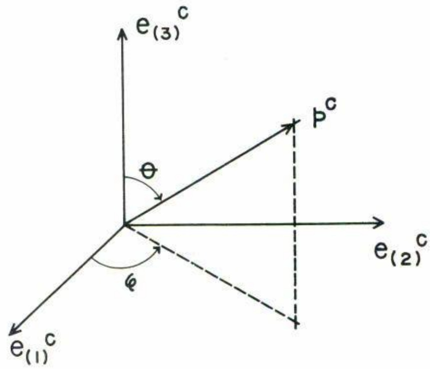

FIGURE 3. Fermi tetrad.

The unitary space-like vector p^c has been defined in Fig. 1

$$p^c = w^{-1}k^c - v^c, \qquad p^c p_c = 1, \qquad p^c v_c = 0. \tag{5.b}$$

Now for any event in the line-universe, p^c can be written in terms of a Fermi tetrad $e_{(\gamma)c}$ with $\gamma = 1,2,3$ (Fig. 3).

$$\begin{aligned} \frac{de_{(\gamma)}{}^c}{d\tau} &= a_{(\sigma)}v^c = (a^r e_{(\sigma)r})v^c, \\ p^c &= \sin\theta\cos\phi\, e_{(1)}{}^c + \sin\theta\sin\phi\, e_{(2)}{}^c + \cos\theta\, e_{(3)}{}^c \\ &= P_{(\sigma)}e_{(\sigma)}{}^c = (p^r e_{(\sigma)r})e_{(\sigma)}{}^c \end{aligned} \tag{5.c}$$

and $d\Omega = \sin\theta\, d\theta\, d\phi$.

From (3.a), it is clear that

$$\underset{B}{K}_{jbc}K^c = 0. \tag{5.d}$$

Therefore (5.a) implies $\tilde{M}_{jb} = 0$, that is, $\underset{B}{K}_{jbc}$ flux vanishes through a light cone.

Now we will calculate the $\underset{B}{K}_{jbc}$ flux through the 3-space $w = w_0 = \text{const}$; the expression is given in Synge [5]

$$\underset{I}{M}_{jb} = \int_{w=w_0} \underset{B}{K}_{jbc}\, d\sigma^c = w^2 \int_{\tau_1}^{\tau_2} d\tau \int \underset{B}{K}_{jb}{}^c w_{,c}\, d\Omega, \tag{6.a}$$

where

$$w_{,c} = \text{Gradient of } w = -v_c + BK_c \tag{6.b}$$

Therefore, Ecs. (3.a, 5.c, 6.a, b) imply that

$$\underset{\text{I}}{M}_{jb} = \frac{8\pi}{3} q^2 \int_{\tau_1}^{\tau_2} (v_j \times a_b)\, d\tau. \tag{6.c}$$

This last result agrees with the intrinsic angular momentum of the LW Field [7]. Therefore, $\underset{\text{B}}{K}_{jbc}$ behaves as a density for such an angular momentum.

The superpotential (3.a) accepts the following rupture

$$\underset{\text{B}}{K}_{jbc} = \underset{\text{B}}{\tilde{K}}_{jbc} + \underset{\text{B}}{\bar{K}}_{jbc}, \tag{7.a}$$

where

$$\underset{\text{B}}{\tilde{K}}_{jbc} = q^2 w^{-4} \Big[(-a_j + w^{-1} W v_j) \times K_b\Big] K_c \tag{7.b}$$

and

$$\underset{\text{B}}{\bar{K}}_{jbc} = -w^{-4} \Big[g_{cj} K_b - g_{cb} K_j + 3w^{-1}(v_j \times K_b) K_c\Big]. \tag{7.c}$$

The potentials in Eqs. (7.b, c) satisfy all the properties in (4), as the Lanczos Spin-tensor does.

It is simple to prove that

$$\int_{\tau=\text{const}} \underset{\text{B}}{\bar{K}}_{jbc}\, d\sigma^c = \int_{w=\text{const}} \underset{\text{B}}{\bar{K}}_{jbc} d\sigma^c = 0, \tag{8.a}$$

and

$$\underset{\text{I}}{M}_{jb} = \int_{w=\text{const}} \underset{\text{B}}{\tilde{K}}_{jbc}\, d\sigma^c \quad \text{and} \quad \int_{\tau=\text{const}} \underset{\text{B}}{\tilde{K}}_{jbc}\, d\sigma^c = 0. \tag{8.b}$$

That is, $\underset{\text{B}}{\bar{K}}_{jbc}$ doesn't contribute to the IAM of the electromagnetic field; this means that $\underset{\text{B}}{\tilde{K}}_{jbc}$ is the active part of part of $\underset{\text{B}}{K}_{jbc}$.

The result can be obtained by using Stokes Theorem and the Rowe [8] identity

$$\underset{\text{B}}{\bar{K}}_{jbc} = \left(\frac{q^2}{4} w^{-4} D_{jbc}{}^{r}\right)_{,r}, \tag{8.c}$$

where D_{jbcr} is a tensor used by Synge [5] in another context

$$D_{sarb} = (g_{sr}K_b - g_{sb}K_r)K_a + (g_{ab}K_r - g_{ar}K_b)K_s. \tag{8.d}$$

Finally using (7.a) in (3.c), the following decomposition is obtained

$$\underset{\mathrm{B}}{T}_{bc} = \underset{\mathrm{B}}{\tilde{T}}_{bc} + \underset{\mathrm{B}}{\bar{T}}_{bc}, \tag{9}$$

with $\underset{\mathrm{B}}{\tilde{T}}_{bc} = \underset{\mathrm{B}}{\tilde{K}}_{b\ c,j}^{\ j}$ and $\underset{\mathrm{B}}{\bar{T}}_{bc} = \underset{\mathrm{B}}{\bar{K}}_{b\ c,j}^{\ j}$.

Eq. (9) is important at the moment we relate it to the angular moment radiated by the charge [9].

4. On a rupture for $\underset{\mathrm{R}}{T}_{ab}$

Here we show how the radiative part of T_{ab} can be written as the sum of two terms; one of them doesn't participate in the energy and momentum flux through the Bhabha-Synge tube. Moreover, we will give a potential for $\underset{\mathrm{R}}{T}_{bc}$.

The expression (2.c) can be written in the form

$$\underset{\mathrm{R}}{T}_{bc} = \underset{\mathrm{i}}{T}_{bc} + \underset{\mathrm{R}}{\tilde{T}}_{bc}, \tag{10.a}$$

where

$$\underset{\mathrm{i}}{T}_{bc} = q^2 w^{-4}(a^2 - 3w^{-2}W^2)K_bK_c, \qquad \underset{\mathrm{i}}{T}_{b\ .c}^{\ c} = 0 \tag{10.b}$$

$$\underset{\mathrm{R}}{T}_{bc} = 2q^2 w^{-6}W^2K_bK_c, \qquad \underset{\mathrm{R}}{T}_{b\ .c}^{\ c} = 0 \tag{10.c}$$

It is simple to demonstrate that

$$\int_{\substack{\tau=\mathrm{const}\\ \mathrm{or}\\ w=\mathrm{const}}} \underset{\mathrm{i}}{T}_{bc}\, d\sigma^c = 0, \tag{11.a}$$

that is, (10.b) doesn't contribute to the energy and momentum flux through the Bhabha-Synge tube. In a similar way

$$\int_{\substack{w=\mathrm{const}\\ \mathrm{or}\\ \tau=\mathrm{const}}} (x^j \underset{\mathrm{i}}{T}^{bc} - x^b \underset{\mathrm{i}}{T}^{jc})\, d\sigma_c = 0. \tag{11.b}$$

Hence $\underset{\text{i}}{T}_{bc}$ doesn't participate either in the momentum fluxes for such tube. Due to Ecs. $(11.a, b)$ we say that the tensors in $(10.b)$ represent the inactive part of $\underset{\text{R}}{T}_{ab}$ with respect to the Bhabha-Synge region.

The conservation law is immediately deduced from the existence of the superpotential:

$$\underset{\text{i}}{K}_{jbc} = -\frac{q^2}{4}w^{-2}\Big[w^{-2}W^2(g_{cj}K_b - g_{cb}K_j) + w^{-1}W(v_j \times K_b)(a_c - 3w^{-2}WK_c) + (a_j \times K_b)(4w^{-2}WK_c - a_c)\Big], \tag{12.a}$$

such that

$$\underset{\text{i}}{T}_{bc} = \underset{\text{i}}{K}_{b\ c,j}^{\ j}. \tag{12.b}$$

Moreover, the identity $\underset{\text{R}}{T}_{b\ ,c}^{\ c} = 0$ is a consequence of

$$\underset{\text{R}}{\tilde{T}}_{bc} = \underset{\text{R}}{\tilde{K}}_{b\ c,j}^{\ j}, \tag{13.a}$$

where

$$\underset{\text{R}}{\tilde{K}}_{bjc} = -2qF_{bj}p(\sigma)p(\gamma)\left[\int_0^\tau a(\sigma)a(\gamma)v_c\,d\tau + p(\beta)\int_0^\tau a(\sigma)a(\gamma)e_{(\beta)_c}\,d\tau\right] \tag{13.b}$$

in equation $(13.b)$, the sum over $\sigma, \gamma, \beta = 1, 2, 3$ has to be done.

To verify Eq. $(13.a)$, we have to use the following relations

$$\begin{aligned}
\tau_{,j} &= -w^{-1}K_j, && \text{retarded derivative}\\
F_{b\ ,j}^{\ j} &= 0, && \text{Maxwell equations}\\
F_{b\ ,j}^{\ j} &= qw^{-2}K_b, && \text{null eigenvector}\\
F_b^{\ j}p_{(\sigma)j} &= 0, && \text{Fermi tetrad}\\
W &= -wp(\sigma)a(\sigma), &&\\
k^c &= w(v^c + p^c). &&
\end{aligned} \tag{13.c}$$

The existence of integrals in equation $(13.b)$ shows the non-local character of $\underset{\text{R}}{\tilde{K}}_{bjc}$; that means that it depends on the past history of the charge. Therefore

Eqs. (2.*a*, 3.*c*, 10.*a*, 12.*b*, 13.*a*) imply

$$T_{bc} = \left(\underset{\mathrm{B}}{K}{}_{b}{}^{j}{}_{c} + \underset{\mathrm{i}}{K}{}_{b}{}^{j}{}_{c} + \underset{\mathrm{R}}{\tilde{K}}{}_{b}{}^{j}{}_{c} \right)_{,j} . \tag{14}$$

Hence the Maxwell tensor associated to the LW Field is an *exact divergence*.

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Resumen. Weert [1] obtuvo un potencial para la parte acotada del tensor de Maxwell asociado al campo de Liénard-Wiechert. Aquí mostramos que dicho potencial puede interpretarse como una densidad de momento angular intrínseco del correspondiente campo electromagnético.