The n–dimensional classical Kepler's problem "without integration"

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Abstract. The classical Kepler's problem is generalized to n–dimensional Euclidean or pseudo-Euclidean (real or complex) space and a simple solution of this generalization is given for any $n \geq 2$.

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1. Introduction

From the old works of Laplace, Runge and Lenz we know that the classical (*i.e.*, 3–dimensional, Newtonian) Kepler's problem can be solved almost without integration [1,2].

The aim of this note is:

i) to generalize the solution on n–dimensional ($n \geq 2$) Euclidean or pseudo–Euclidean space.

ii) to show that the formal complex extension of the problem also holds.

The generalization of the Kepler problem n–dimensional Euclidean space with $n \geq 2$ is self-evident. The cases of pseudo-Euclidean space or complex space are less obvious but as we show in our note they can be also considered, and the solution of the Kepler's problem in these cases can be found in the analogous way as for an Euclidean space (although some aspects of the complex case require further analysis).

We hope that our paper elucidate the role of constants of motion and especially the role of the Laplace-Runge-Lenz vector in the Kepler's problem [1–4].

2. Solution of the problem

Let x^a be coordinates, p_a–the canonically conjugated linear momentum; $a, b, \ldots$ run through $1, 2, \ldots, n$; the summational convention will apply. Let g_{ab} be a non-singular, constant metric tensor with the inverse g^{ab}. The latin indices will be ma-

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nipulated by this metric in the usual manner. We intend to study the motion of a mechanical system of the Hamiltonian

$$H := \frac{1}{2\mu} p_a p^a - \lambda \cdot (x_a x^a)^{-\frac{1}{2}}, \tag{1}$$

where μ and λ are some fixed constants. In other words, we would like to find the solutions of the following differential equations

$$\frac{d}{dt} x^a = \frac{\partial H}{\partial p_a} = \frac{1}{\mu} p^a; \qquad \frac{d}{dt} p_a = -\frac{\partial H}{\partial q^a} = -\lambda \cdot (x_b x^b)^{-\frac{3}{2}} x_a. \tag{2}$$

(Remark: In the real case we assume $x_a x^a > 0$).

We will now review the basic first integrals of Eqs. (2). First of all we have the energy integral

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \frac{d}{dt} H = 0 \Rightarrow H = \text{const.} \tag{3}$$

Then defining the angular momentum tensor

$$J_{ab} := p_a x_b - p_b x_a, \tag{4}$$

one has

$$\frac{d}{dt} J_{ab} = 0 \Rightarrow J_{ab} = \text{const.} \tag{5}$$

The next basic integral is the Laplace-Runge-Lenz vector [1–4]

$$L_a := J_{ab} p^b + \lambda\mu \cdot (x_b x^b)^{-\frac{1}{2}} x_a. \tag{6}$$

One easily shows that with (2) assumed

$$\frac{d}{dt} L_a = 0 \Rightarrow L_a = \text{const.} \tag{7}$$

Finally it is convenient to introduce the vector

$$K_a := J_{ab} L^b, \qquad K_a = \text{const.} \tag{8}$$

We can find some important relations between the integrals discussed above,

$$L_a L^a = (\lambda\mu)^2 + 2\mu H J^2, \tag{9a}$$

$$K_a K^a = J^2 L_a L^a, \tag{9b}$$

$$L_a K^a = 0, \tag{9c}$$

$$K_a L_b - L_a K_b = L_c L^c J_{ab}, \tag{9d}$$

where $J^2 := \frac{1}{2} J_{ab} J^{ab}$.

Now the following relations

$$\frac{1}{2\mu} p^2 - \lambda r^{-1} = H = \text{const.}; \qquad p^2 r^2 - s^2 = J^2 = \text{const.}, \tag{10}$$

where

$$p^2 := p_a p^a, \qquad r^2 := x_a x^a, \qquad s := p_a x^a, \tag{11}$$

imply that the scalars (11) can be always parameterized (along the trajectory) by a single parameter.

Using the explicit form of J_{ab} in the definitions of L_a, K_a (see (6), (8)) we rewrite these integrals in the form

$$L_a = (-p^2 + \lambda \mu r^{-1}) x_a + s p_a, \tag{12a}$$

$$K_a = (-\lambda \mu r^{-1} s) x_a + (\lambda \mu r - J^2) p_a. \tag{12b}$$

Treating $(12a, b)$ as a pair of linear algebraic equations on x_a, p_a we find their determinant to be

$$(-p^2 + \lambda \mu r^{-1}) \cdot (\lambda \mu r - J^2) - s \cdot (-\lambda \mu r^{-1} s) = (\lambda \mu)^2 + 2\mu H J^2 = L_a L^a. \tag{13}$$

[The first equality follows from Eq. (10) and the second from Eq. $(9a)$].

Thus a special situation arises when $L_a L^a \neq 0$. In this case $(12a, b)$ lead to the formulae

$$x_a = \frac{1}{L_b L^b} \cdot \left[(\lambda \mu r - J^2) L_a - s K_a \right], \tag{14a}$$

$$p_a = \frac{1}{L_b L^b} \cdot \left[(\lambda \mu r^{-1} s) L_a + (-p^2 + \lambda \mu r^{-1}) K_a \right], \tag{14b}$$

Due to the relations (10), the formulae $(14a, b)$ represent already a trajectory in the phase space: L_a, K_a are constant vectors and the coefficients in $(14a, b)$ depend on one variable parameter.

We explore our solution in the case where the initial conditions are such that not only $L_a L^a \neq 0$ but also $J^2 \neq 0$. (This case can be achieved if and only if $n \geq 2$). Then we define the constant vectors

$$\overset{1}{e}_a := -(L_b L^b)^{-\frac{1}{2}} L_a, \qquad \overset{2}{e}_a := -J^{-1} (L_b L^b)^{-\frac{1}{2}} K_a. \tag{15}$$

From $(9a, b, c)$ it follows that these dimensionless vectors satisfy the conditions

$$\overset{1}{e}_a \overset{1a}{e} = 1 = \overset{2}{e}_a \overset{2a}{e}; \qquad \overset{1}{e}_a \overset{2a}{e} = 0. \tag{16}$$

In terms of $\overset{1}{e}_a$, $\overset{2}{e}_a$ $(14a, b)$ read

$$x_a = X_1 \overset{1}{e}_a + X_2 \overset{2}{e}_a, \tag{17a}$$

$$p_a = P_1 \overset{1}{e}_a + P_2 \overset{2}{e}_a, \tag{17b}$$

where

$$X_1 := (L_a L^a)^{-\frac{1}{2}} \cdot (J^2 - \lambda\mu r), \qquad X_2 := (L_a L^a)^{-\frac{1}{2}} J s, \tag{18a}$$

$$P_1 := -(L_a L^a)^{-\frac{1}{2}} \lambda\mu r^{-1} s, \qquad P_2 := (L_a L^a)^{-\frac{1}{2}} J \cdot (p^2 - \lambda\mu r^{-1}). \tag{18b}$$

Now applying $(9a)$ and (10) one finds that the pairs X_1, X_2 and P_1, P_2 fulfil the following quadratic relations

$$\left(-\frac{2H}{\lambda}\right)^2 \cdot \left[X_1 - \frac{1}{2\mu H} \cdot (L_a L^a)^{\frac{1}{2}}\right]^2 - \frac{2\mu H}{J^2} X_2^2 = 1, \tag{19a}$$

$$P_1^2 + \left[P_2 - \frac{1}{J} \cdot (L_a L^a)^{\frac{1}{2}}\right]^2 = \left(\frac{\lambda\mu}{J}\right)^2. \tag{19b}$$

Discussing these relations it is convenient to work with the familiar parameters [1,3]

$$a := \frac{\lambda}{-2H}, \qquad \epsilon := \left[1 + 2\mu H \frac{J^2}{(\lambda\mu)^2}\right]^{\frac{1}{2}} = \frac{1}{\lambda\mu} \cdot (L_a L^a)^{\frac{1}{2}},$$

$$p' := a(1 - \epsilon^2) \frac{J^2}{\lambda\mu}. \tag{20}$$

In terms of these parameters $(19a, b)$ reduce to

$$\frac{(X_1 + \epsilon a)^2}{a^2} + \frac{X_2^2}{a^2(1 - \epsilon^2)} = 1, \tag{21a}$$

$$P_1^2 + \left(P_2 - \frac{\lambda\mu\epsilon}{J}\right)^2 = \left(\frac{\lambda\mu}{J}\right)^2. \tag{21b}$$

The formulae $(21a, b)$ suggest a specific parametrization

$$X_1 + \epsilon a = a \cos u, \qquad X_2 = a \cdot (1-\epsilon^2)^{\frac{1}{2}} \sin u \tag{22a}$$

$$P_1 = -\frac{\lambda\mu}{J} \sin\phi, \qquad P_2 - \frac{\lambda\mu\epsilon}{J} = \frac{\lambda\mu}{J}\cos\phi \tag{22b}$$

with u (the "eccentric anomaly") and ϕ (the "true anomaly") being new parameters. Then from $(17a, b)$, $(18a, b)$, (20) and $(22a)$ one finds

$$r = a \cdot (1 - \epsilon \cos u), \quad s = \epsilon \cdot (\lambda\mu a)^{\frac{1}{2}} \sin u, \tag{23}$$

and

$$x_a = a \cdot \left[(\cos u - \epsilon) \overset{1}{e}_a + (1-\epsilon^2)^{\frac{1}{2}} \sin u \overset{2}{e}_a \right], \tag{24a}$$

$$p_a = \left(\frac{\lambda\mu}{a}\right)^{\frac{1}{2}} \cdot \left[-\frac{\sin u}{1-\epsilon\cos u} \overset{1}{e}_a + \frac{(1-\epsilon^2)^{\frac{1}{2}} \cos u}{1-\epsilon\cos u} \overset{2}{e}_a \right]. \tag{24b}$$

Similarly, using $(17a, b)$, $(18a, b)$, (20) and $(22b)$ we have

$$r = \frac{p'}{1+\epsilon\cos\phi}, \qquad s = \epsilon \cdot (\lambda\mu p')^{\frac{1}{2}} \frac{\sin\phi}{1+\epsilon\cos\phi} \tag{25}$$

and

$$x_a = p' \cdot \left[\frac{\cos\phi}{1+\epsilon\cos\phi} \overset{1}{e}_a + \frac{\sin\phi}{1+\epsilon\cos\phi} \overset{2}{e}_a \right], \tag{26a}$$

$$p_a = \left(\frac{\lambda\mu}{p'}\right)^{\frac{1}{2}} \cdot \left[-\sin\phi \overset{1}{e}_a + (\cos\phi + \epsilon) \overset{2}{e}_a \right]. \tag{26b}$$

Of course the coefficients at $\overset{1}{e}_a$, $\overset{2}{e}_a$ in $(24a, b)$ and $(26a, b)$ must agree what leads to four relations between u and ϕ. All these relations are equivalent to the one which follows by comparing expressions for r through u or ϕ (see (23) and (25))

$$1 - \epsilon\cos u = \frac{1-\epsilon^2}{1+\epsilon\cos\phi}. \tag{27}$$

This on its turn can be written more symmetrically in the form

$$(1+\epsilon)^{\frac{1}{2}} \tan\frac{u}{2} = (1-\epsilon)^{\frac{1}{2}} \tan\frac{\phi}{2}. \tag{28}$$

It remains to determine the dependence of u (or ϕ) on t. Substituting $(24a, b)$ into $\frac{d}{dt}x_a = \frac{1}{\mu}p_a$ and comparing coefficients at $\overset{1}{e}_a$ one obtains

$$\frac{du}{dt} = \left(\frac{\lambda}{\mu a^3}\right)^{\frac{1}{2}} \frac{1}{1 - \epsilon \cos u}, \tag{29}$$

which implies the Kepler's equation

$$\left(\frac{\lambda}{\mu a^3}\right)^{\frac{1}{2}} t = u - \epsilon \sin u \tag{30}$$

(the integration constant is so selected that $u = 0 \Rightarrow t = 0$).

References

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Resumen. El problema clásico de Kepler se generaliza a un espacio euclideano o pseudo-euclideano de n dimensiones (real o complejo) y se aporta una solución simple de esta generalización para cualquier $n \geq 2$.