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Multipole expansion of the gravitational field in the linearized Einstein theory

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Abstract. By using the method of adjoint operators, the complete solution to the Einstein vacuum field equations linearized about the Minkowski metric is expressed in terms of scalar potentials that satisfy the wave equation. The multipole fields are then obtained from the separable solutions of the wave equation in spherical coordinates and the amplitude of each multipole is related to the energy-momentum tensor of the sources. The gauge-invariant components of the multipole fields are written in terms of the spin-weighted spherical harmonics and the power radiated per unit solid angle by each multipole is also obtained.

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1. Introduction

The Einstein vacuum field equations linearized about the Minkowski metric possess wavelike solutions, interpretable as gravitational radiation, analogous to the electromagnetic waves. The linearized Einstein equations are obtained from Einstein's equations by writing the metric of the space-time $g_{\alpha\beta}$ as the Minkowski metric $\eta_{\alpha\beta}$ plus a small perturbation $h_{\alpha\beta}$, keeping only the first-order terms in the perturbation $h_{\alpha\beta}$. In the standard approach [1,2], by imposing suitable gauge conditions, the linearized vacuum field equations are reduced to wave equations for each cartesian component of the perturbation $h_{\alpha\beta}$; however, the solutions to these wave equations are not independent since they have to satisfy the imposed gauge conditions. Nevertheless, the main difficulty usually encountered in obtaining a multipole expansion for the gravitational radiation comes from the use of tensor spherical harmonics, for which there exists a variety of rather cumbersome definitions and notations [3].

The aim of this paper is to give a very simple derivation of the multipole expansion of the gravitational field, restricted to the linearized Einstein theory, in which no gauge conditions are imposed and where all the expressions for the metric perturbations arise in a natural way, without having to propose expansions in tensor spherical harmonics for the fields. In fact, most of this paper requires only the elementary tensor notation using cartesian coordinates. The derivation presented here is based on the method of adjoint operators introduced by Wald [4] (see also [5]) which is applicable to systems of homogeneous linear equations. A similar derivation for the case of the electromagnetic field has been given in Ref. [6].

In Sec. 2 the method of adjoint operators is briefly summarized and some identities are obtained from the linearized Einstein theory which are used to find the solutions to the linearized vacuum field equations. Two types of multipole fields are then constructed which are analogous to the electric and magnetic multipoles found in electromagnetism. In Sec. 3 the amplitudes of the multipoles generated from localized sources are given by means of integrals of the energy-momentum tensor of the sources; the resulting expressions are equivalent to those found in Ref. [7] (see also Ref. [3]). In Sec. 4 the components of the curvature perturbation are written in terms of the spin-weighted spherical harmonics and the power radiated per unit solid angle by the gravitational waves is expressed in terms of the amplitudes of the multipoles. Throughout this article Greek indices run from 0 to 3 and Latin indices from 1 to 3 and on each repeated index the summation convention applies. Indices are raised and lowered by means of the Minkowski metric.

2. Solution of the linearized vacuum field equations

Following Ref. [4] the Einstein equations linearized about the Minkowski metric will be written in the form

$$[\mathcal{E}(h_{\gamma\delta})]_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{1}$$

with the linear partial differential operator $\mathcal{E}$ given in cartesian coordinates by

$$\begin{aligned}\left[\mathcal{E}(h_{\gamma\delta})\right]_{\alpha\beta} \equiv \tfrac{1}{2}\Big\{&\partial^\gamma\partial_\gamma h_{\alpha\beta} - \partial_\alpha\partial^\gamma h_{\gamma\beta} - \partial_\beta\partial^\gamma h_{\gamma\alpha} + \partial_\alpha\partial_\beta(\eta^{\gamma\delta}h_{\gamma\delta}) \\ &+\eta_{\alpha\beta}\partial^\gamma\partial^\delta h_{\gamma\delta} - \eta_{\alpha\beta}\partial^\gamma\partial_\gamma(\eta^{\delta\epsilon}h_{\delta\epsilon})\Big\},\end{aligned} \tag{2}$$

where

$$h_{\alpha\beta} \equiv g_{\alpha\beta} - \eta_{\alpha\beta} \tag{3}$$

represents a small deviation of the metric $g_{\alpha\beta}$ from the Minkowski metric

$$(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1), \tag{4}$$

$\partial_\alpha \equiv \partial/\partial x^\alpha$, $\partial^\alpha \equiv \eta^{\alpha\beta}\partial_\beta$ and $T_{\alpha\beta}$ is the energy-momentum tensor of the sources to first order in the perturbation $h_{\alpha\beta}$.

In order to solve the linearized vacuum field equations:

$$[\mathcal{E}(h_{\gamma\delta})]_{\alpha\beta} = 0, \tag{5}$$

by combining linearly the equations in (5) and their partial derivatives, we shall derive decoupled scalar equations for certain linear combinations, χ, of $h_{\alpha\beta}$ and its derivatives: $\chi = \mathcal{T}(h_{\alpha\beta})$, where $\mathcal{T}$ is a linear partial differential operator that maps two-index tensor fields into scalar fields. The equation satisfied by χ will be written in the form

$$\mathcal{O}(\chi) = 0, \tag{6}$$

where $\mathcal{O}$ is a linear differential operator that maps scalar fields into themselves. The fact that the decoupled equation (6) is a consequence of the original system (5) means that there exists a linear operator $\mathcal{S}$ such that $\mathcal{O}(\chi) = \mathcal{S}\mathcal{E}(h_{\gamma\delta})$; therefore, the linear operators $\mathcal{E}$, $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{S}$ satisfy the operator equation

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}. \tag{7}$$

By defining the adjoint, $\mathcal{A}^{\dagger}$, of a linear operator $\mathcal{A}$ in such a way that $\mathcal{A}^{\dagger}$ is also a linear operator and

$$(\mathcal{A}\mathcal{B})^{\dagger} = \mathcal{B}^{\dagger}\mathcal{A}^{\dagger} \tag{8}$$

for any pair of linear operators $\mathcal{A}$ and $\mathcal{B}$ whose composition is well defined, from Eq. (7) it follows that

$$\mathcal{E}^{\dagger}\mathcal{S}^{\dagger} = \mathcal{T}^{\dagger}\mathcal{O}^{\dagger}. \tag{9}$$

Hence, if ψ satisfies the equation

$$\mathcal{O}^{\dagger}(\psi) = 0, \tag{10}$$

then Eq. (9) implies that $\mathcal{S}^{\dagger}(\psi)$ satisfies $\mathcal{E}^{\dagger}(\mathcal{S}^{\dagger}(\psi)) = 0$; thus, if $\mathcal{E}^{\dagger}$ is proportional to $\mathcal{E}$, then $\mathcal{E}(\mathcal{S}^{\dagger}(\psi)) = 0$, *i.e.*, $\mathcal{S}^{\dagger}(\psi)$ is a solution of Eq. (5).

If the adjoint of a linear partial differential operator $\mathcal{A}$ that maps m-index tensor fields into n-index tensor fields is defined as that linear partial differential operator, $\mathcal{A}^{\dagger}$, which maps n-index tensor fields into m-index tensor fields such that [4]

$$g^{\mu\nu\ldots}[\mathcal{A}(f_{\alpha\beta\ldots})]_{\mu\nu\ldots} - [\mathcal{A}^{\dagger}(g^{\mu\nu\ldots})]^{\alpha\beta\ldots} f_{\alpha\beta\ldots} = \partial_{\alpha} s^{\alpha}, \tag{11}$$

where s^{α} is some vector field, then one finds that Eq. (8) holds and that the operator $\mathcal{E}$ defined in (2) is self-adjoint: $\mathcal{E}^{\dagger} = \mathcal{E}$. Therefore, by solving the scalar equation (10) one obtains a complete solution to Eq. (5), given by $\mathcal{S}^{\dagger}(\psi)$.

The linearized equations (5) are invariant under the gauge transformations induced by "infinitesimal" coordinate transformations:

$$h_{\alpha\beta} \to h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha, \tag{12}$$

where ξ_α are the components of an arbitrary vector field, in the sense that if $h_{\alpha\beta}$ satisfies Eq. (5), so does $h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$, which can be seen from the fact that $\mathcal{E}(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha)$ is identically zero. A gauge-invariant description of the gravitational field is given by the tensor field

$$K_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\left\{\partial_\alpha \partial_\gamma h_{\delta\beta} - \partial_\beta \partial_\gamma h_{\delta\alpha} + \partial_\beta \partial_\delta h_{\gamma\alpha} - \partial_\alpha \partial_\delta h_{\gamma\beta}\right\}, \tag{13}$$

which is the Riemann tensor corresponding to the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ to first order in $h_{\alpha\beta}$. From the definition (13) it follows that $K_{\alpha\beta\gamma\delta}$ has the symmetries of the Riemann tensor:

$$K_{\alpha\beta\gamma\delta} = -K_{\beta\alpha\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\gamma\delta\alpha\beta} \tag{14a}$$

$$K_{\alpha\beta\gamma\delta} + K_{\alpha\delta\beta\gamma} + K_{\alpha\gamma\delta\beta} = 0, \tag{14b}$$

and it satisfies

$$\partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\epsilon K_{\beta\gamma\alpha\delta} + \partial_\delta K_{\beta\gamma\epsilon\alpha} = 0, \tag{15}$$

which are analogous to the Bianchi identities.

In terms of the tensor field

$$K_{\alpha\beta} \equiv K_{\alpha\gamma\beta}{}^{\gamma}, \tag{16}$$

which is symmetric as a consequence of (14a), the linearized Einstein equations (1) are

$$K_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} K_\gamma{}^\gamma = -\frac{8\pi G}{c^4} T_{\alpha\beta}$$

or, equivalently,

$$K_{\alpha\beta} = -\frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} T_\gamma{}^\gamma\right). \tag{17}$$

Equations (14–16) imply the identities

$$\partial^\alpha K_{\beta\alpha\delta\epsilon} = -\partial_\epsilon K_{\beta\alpha}{}^{\alpha}{}_{\delta} - \partial_\delta K_{\beta\alpha\epsilon}{}^{\alpha} = \partial_\epsilon K_{\beta\delta} - \partial_\delta K_{\beta\epsilon}, \tag{18}$$

and, from Eqs. (14), (15) and (18) we obtain

$$
\begin{aligned}
0 &= \partial^\alpha \partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial^\alpha \partial_\epsilon K_{\beta\gamma\alpha\delta} + \partial^\alpha \partial_\delta K_{\beta\gamma\epsilon\alpha} \\
&= \partial^\alpha \partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\epsilon(\partial_\beta K_{\gamma\delta} - \partial_\gamma K_{\beta\delta}) + \partial_\delta(\partial_\gamma K_{\beta\epsilon} - \partial_\beta K_{\gamma\epsilon}) \\
&= \partial^\alpha \partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\beta \partial_\epsilon K_{\gamma\delta} + \partial_\gamma \partial_\delta K_{\beta\epsilon} - \partial_\gamma \partial_\epsilon K_{\beta\delta} - \partial_\beta \partial_\delta K_{\gamma\epsilon},
\end{aligned} \tag{19}
$$

which shows that outside the sources (*i.e.*, in a region where $T_{\alpha\beta} = 0$) each cartesian component $K_{\alpha\beta\gamma\delta}$ satisfies the wave equation.

Due to the symmetries (14), only 20 components $K_{\alpha\beta\gamma\delta}$ can be independent and in the case of vacuum (where, according to Eq. (17), $K_{\alpha\beta} = 0$) this number is reduced to 10. In this latter case, the independent components of $K_{\alpha\beta\gamma\delta}$ can be given by K_{0i0j}, which, by virtue of (14*a*), is symmetric in the indices i and j and has vanishing trace: $K_{0i0}{}^{i} = K_{0\alpha0}{}^{\alpha} = K_{00} = 0$, and by K^*_{0i0j}, where

$$
K^*_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\epsilon_{\gamma\delta\rho\sigma} K_{\alpha\beta}{}^{\rho\sigma} \tag{20}
$$

and $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. The components K^*_{0i0j} are also symmetric in the indices i and j since its antisymmetric part on these indices vanishes:

$$
\epsilon_{0ijk} K^{*i\ j}_{0\ 0} = \frac{1}{2}\epsilon_{0ijk}\epsilon_0{}^{j\rho\sigma} K_0{}^{i}{}_{\rho\sigma} = K_0{}^{i}{}_{ki} = K_0{}^{\alpha}{}_{k\alpha} = K_{0k} = 0,
$$

and from Eq. (14*b*) it follows that the trace $K^*_{0i0}{}^{i}$ is equal to zero: $K^*_{0i0}{}^{i} = \frac{1}{2}\epsilon_0{}^{i\rho\sigma} K_{0i\rho\sigma} = 0$.

We shall now obtain identities of the form (7) starting from the decoupled equations satisfied by certain gauge-invariant quantities made out of the metric perturbation. Making use of Eqs. (14*b*), (15), (18) and (20) we find that

$$
\partial^\alpha \partial_\alpha (K^*_{0i0j} x^i x^j) = \tfrac{1}{2} x^i x^j \epsilon_{0jkm} \partial^\alpha \partial_\alpha K_{0i}{}^{km} - 2x^i \epsilon_{0ikm} \partial^k K_0{}^m,
$$

hence, from Eqs. (19) and (17),

$$
\begin{aligned}
\partial^\alpha \partial_\alpha (K^*_{0i0j} x^i x^j) &= x^i x^j \epsilon_{0jkm} \partial_0 \partial^k K_i{}^m - x^j \epsilon_{0jkm}(x^i \partial_i + 2)\partial^k K_0{}^m \\
&= -\frac{8\pi G}{c^4}\Big\{ x^i x^j \epsilon_{0jkm} \partial_0 \partial^k T_i{}^m \\
&\qquad - x^j \epsilon_{0jkm}(x^i \partial_i + 2)\partial^k T_0{}^m \Big\}.
\end{aligned} \tag{21}
$$

Thus, if the linearized vacuum field equations are satisfied then the scalar field

$K^*_{0i0j}x^ix^j$ satisfies the wave equation, which has the form (6) with

$$\mathcal{O} = \partial^\alpha \partial_\alpha = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{22}$$

and $\chi = K^*_{0i0j}x^ix^j = \frac{1}{2}x^ix^j\epsilon_{0j}{}^{km}(\partial_0\partial_k h_{mi} - \partial_i\partial_k h_{m0})$; this means that, in this case, $\mathcal{T}(h_{\alpha\beta}) = \frac{1}{2}x^ix^j\epsilon_{0j}{}^{km}(\partial_0\partial_k h_{mi} - \partial_i\partial_k h_{m0})$.

By using Eq. (1) the right-hand side of Eq. (21) can be written as

$$\left\{ix^j L_m \partial_0 \delta^\alpha_j \eta^{\beta m} - i(x^j\partial_j + 1)L_m \delta^\alpha_0 \eta^{\beta m}\right\}\left[\mathcal{E}(h_{\gamma\delta})\right]_{\alpha\beta},$$

where

$$L_m \equiv -i\epsilon_{0mjk}x^j\partial^k, \tag{23}$$

which are the components of the orbital angular momentum operator divided by $\hbar$. Therefore, Eq. (21) takes the form (7) with

$$\mathcal{S}(b_{\alpha\beta}) \equiv \left\{ix^j L_m \partial_0 \delta^{(\alpha}_j \eta^{\beta)m} - i(x^j\partial_j + 1)L_m \delta^{(\alpha}_0 \eta^{\beta)m}\right\} b_{\alpha\beta}, \tag{24}$$

where the parenthesis denote symmetrization on the indices enclosed. The adjoint operators can be easily obtained using Eq. (8) and the relations $(\mathcal{A}+\mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger$, $\partial^\dagger_\alpha = -\partial_\alpha$ and $f^\dagger = f$, for an arbitrary function f, which follow from (11); hence, $L^\dagger_m = -L_m$, $\mathcal{O}^\dagger = \partial^\alpha\partial_\alpha$ and $[\mathcal{S}^\dagger(\psi)]^{\alpha\beta} = i\delta^{(\alpha}_j\eta^{\beta)m}\partial_0 L_m x^j\psi - i\delta^{(\alpha}_0\eta^{\beta)m}L_m(\partial_j x^j - 1)\psi = i\delta^{(\alpha}_j\eta^{\beta)m}x^j L_m\partial_0\psi - i\delta^{(\alpha}_0\eta^{\beta)m}L_m(x^j\partial_j + 2)\psi$. Thus, if ψ is a solution of the wave equation

$$\mathcal{O}^\dagger(\psi) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{25}$$

then the tensor field $h_{\alpha\beta} = [\mathcal{S}^\dagger(\psi)]_{\alpha\beta}$ satisfies the linearized Einstein vacuum field equations. The nonvanishing components of $h_{\alpha\beta}$ are given by

$$\begin{aligned} h_{0m} &= \tfrac{i}{2}L_m(x^j\partial_j + 2)\psi = \tfrac{i}{2}L_m\frac{1}{r}\frac{\partial}{\partial r}r^2\psi \\ h_{jm} &= ix_{(j}L_{m)}\partial_0\psi, \end{aligned} \tag{26}$$

where r is the usual radial spherical coordinate.

From the definition (13) and the expressions (26) one gets:

$$K_{0i0j} = \tfrac{i}{2}\left\{x_{(i}L_{j)}\partial_0\partial_0 - \partial_{(i}L_{j)}(x^k\partial_k + 2)\right\}\partial_0\psi, \tag{27}$$

and from Eq. (20), using the fact that K^{*}_{0i0j} must be symmetric in the indices i and j,

$$K^{*}_{0i0j} = \tfrac{i}{4}\Big\{[\epsilon_{0km(i}x_{j)}\partial^{k}L^{m} - iL_{(i}L_{j)}]\partial_0\partial_0\psi - \epsilon_{0km(i}\partial_{j)}\partial^{k}L^{m}(x^{n}\partial_n + 2)\psi\Big\}. \qquad (28)$$

According to (27), the "electric" components K_{0i0j} corresponding to (26) satisfy

$$K_{0i0j}x^{i}x^{j} = 0 \qquad (29)$$

As it is well known, in spherical coordinates the scalar wave equation (25) is separable with solutions of the form

$$\psi = e^{-i\omega t}f_{\ell}(kr)Y_{\ell m}(\theta,\phi), \qquad (30)$$

where $Y_{\ell m}$ is a (scalar) spherical harmonic and $f_{\ell}(kr)$ obeys the differential equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] f_{\ell}(kr) = 0, \qquad (31)$$

with $k \equiv \omega/c$; hence, f_{ℓ} can be written as a linear combination of the spherical Bessel functions j_{ℓ} and n_{ℓ} or of the spherical Hankel functions $h_{\ell}^{(1)}$ and $h_{\ell}^{(2)}$. By analogy with the electromagnetic case, in view of (29), the field generated by the potential (30) by means of Eqs. (26) can be called *magnetic or transverse electric multipole of order* (ℓ, m). From Eq. (28) one finds that

$$K^{*}_{0i0j}x^{i}x^{j} = \tfrac{1}{4}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} - 2 - \frac{r^2}{c^2}\frac{\partial^2}{\partial t^2}\right)L_m L^m\psi \qquad (32)$$

therefore, in the case of a magnetic multipole of order (ℓ, m) with ψ given by (30), using Eq. (31) one obtains

$$K^{*}_{0i0j}x^{i}x^{j} = \tfrac{1}{4}(\ell-1)\ell(\ell+1)(\ell+2)\psi \qquad (33)$$

which vanishes when $\ell = 0$ or $\ell = 1$. In fact, one can see from the explicit expressions given in Sec. 4 [Eqs. (60) and (61)] that the curvature perturbations corresponding to (magnetic or electric) multipoles with $\ell = 0$ or $\ell = 1$ are equal to zero; this means that these metric perturbations can be reduced to zero by using the gauge transformations (12).

Not every solution of the linearized Einstein vacuum field equations is of the form (26) or can be brought to this form by means of the gauge transformations (12). In order to obtain the general solution to Eqs. (5) it is necessary to introduce a second (real) scalar potential. (It may be pointed out that the metric perturbations (26) are real if ψ is a real function; it is understood that one takes the real part of expressions like (30) and (33).) From Eqs. (16–19) we get

$$\begin{aligned}\partial^\alpha\partial_\alpha(K_{0i0j}x^ix^j) &= (x^ix^j\partial_i\partial_j + 4x^i\partial_i + 2)K_{00} \\ &\quad - (2x^ix^j\partial_j\partial_0 + 4x^i\partial_0)K_{0i} + x^ix^j\partial_0\partial_0 K_{ij} \\ &= -\frac{8\pi G}{c^4}\Big\{(x^ix^j\partial_i\partial_j + 4x^i\partial_i + 2)\Big(T_{00} + \tfrac{1}{2}T_\gamma{}^\gamma\Big) \\ &\quad - (2x^ix^j\partial_j\partial_0 + 4x^i\partial_0)T_{0i} \\ &\quad + x^ix^j\partial_0\partial_0\Big(T_{ij} - \tfrac{1}{2}\delta_{ij}T_\gamma{}^\gamma\Big)\Big\}, \end{aligned} \tag{34}$$

which gives another identity of the form (7), with $\mathcal{O} = \partial^\alpha\partial_\alpha$, $\mathcal{T}(h_{\alpha\beta}) = K_{0i0j}x^ix^j$ and

$$\begin{aligned}\mathcal{S}(b_{\alpha\beta}) \equiv \Big\{&(x^ix^j\partial_i\partial_j + 4x^i\partial_i + 2)(\delta_0^\alpha\delta_0^\beta + \tfrac{1}{2}\eta^{\alpha\beta}) \\ &- (2x^ix^j\partial_j\partial_0 + 4x^i\partial_0)\delta_0^{(\alpha}\delta_i^{\beta)} \\ &+ x^ix^j\partial_0\partial_0(\delta_i^\alpha\delta_j^\beta - \tfrac{1}{2}\delta_{ij}\eta^{\alpha\beta})\Big\}b_{\alpha\beta}.\end{aligned}$$

Therefore

$$\begin{aligned}h_{\alpha\beta} = [\mathcal{S}^\dagger(\psi)]_{\alpha\beta} = \Big\{&(\eta_{0\alpha}\eta_{0\beta} + \tfrac{1}{2}\eta_{\alpha\beta})(\partial_j\partial_ix^jx^i - 4\partial_ix^i + 2) \\ &- \eta_{0(\alpha}\eta_{\beta)i}(2\partial_0\partial_jx^jx^i - 4\partial_0x^i) \\ &+ (\eta_{i\alpha}\eta_{j\beta} - \tfrac{1}{2}\delta_{ij}\eta_{\alpha\beta})\partial_0\partial_0x^ix^j\Big\}\psi\end{aligned} \tag{35}$$

is a solution of the linearized Einstein vacuum field equations, provided that ψ

satisfies Eq. (25). The solution (35) can be written more explicitly as

$$\begin{aligned} h_{00} &= \tfrac{1}{2}(x^j\partial_j x^i\partial_i + 3x^i\partial_i + 2 + x^i x_i \partial_0\partial_0)\psi = \tfrac{1}{2}\left(\frac{\partial^2}{\partial r^2}r^2 + \frac{r^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi \\ h_{0i} &= (x_i x^j\partial_j + 2x_i)\partial_0\psi = x_i\frac{1}{r}\frac{\partial}{\partial r}r^2\frac{1}{c}\frac{\partial\psi}{\partial t} \\ h_{ij} &= \tfrac{1}{2}\delta_{ij}(x^k\partial_k x^m\partial_m + 3x^k\partial_k + 2 - x^k x_k\partial_0\partial_0)\psi + x_i x_j\partial_0\partial_0\psi \\ &= \tfrac{1}{2}\delta_{ij}\left(\frac{\partial^2}{\partial r^2}r^2 - \frac{r^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi + \frac{1}{c^2}x_i x_j\frac{\partial^2\psi}{\partial t^2}. \end{aligned} \tag{36}$$

The components of $K_{\alpha\beta\gamma\delta}$, obtained from the definitions (13) and (20), corresponding to (36) are determined by

$$\begin{aligned} K_{0i0j} = \tfrac{i}{4}\Big\{&[\epsilon_{0km(i}x_{j)}\partial^k L^m - iL_{(i}L_{j)}]\partial_0\partial_0\psi \\ &- \epsilon_{0km(i}\partial_{j)}\partial^k L^m(x^n\partial_n + 2)\psi\Big\} \end{aligned} \tag{37}$$

and

$$K^*_{0i0j} = -\tfrac{i}{2}\Big\{x_{(i}L_{j)}\partial_0\partial_0 - \partial_{(i}L_{j)}(x^k\partial_k + 2)\Big\}\partial_0\psi \tag{38}$$

[cf. Eqs. (27) and (28)] where we have assumed that ψ obeys Eq. (25). These components satisfy

$$K^*_{0i0j}x^i x^j = 0 \tag{39}$$

and are analogous to those of a transverse magnetic electromagnetic field. Hence, the metric perturbation (36) generated by a potential of the form (30) will be called *electric or transverse magnetic multipole of order* (ℓ, m). In the case of an electric multipole of order (ℓ, m), one has

$$K_{0i0j}x^i x^j = \tfrac{1}{4}(\ell-1)\ell(\ell+1)(\ell+2)\psi. \tag{4}$$

Again, the curvature perturbations corresponding to multipoles with $\ell = 0$ or $\ell = 1$ turn out to be equal to zero.

By combining the two types of fields given by Eqs. (26) and (36) we obtain a solution to Eqs. (5) of the form

$$h_{00} = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2}r^2 + \frac{r^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi_E$$

$$h_{0j} = x_j \frac{1}{r}\frac{\partial}{\partial r} r^2 \frac{1}{c}\frac{\partial \psi_E}{\partial t} + \frac{i}{2} L_j \frac{1}{r}\frac{\partial}{\partial r} r^2 \psi_M$$

$$h_{ij} = \frac{1}{2}\delta_{ij}\left(\frac{\partial^2}{\partial r^2} r^2 - \frac{r^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi_E + \frac{1}{c^2} x_i x_j \frac{\partial^2 \psi_E}{\partial t^2} + i x_{(i} L_{j)} \frac{1}{c}\frac{\partial \psi_M}{\partial t} \qquad (41)$$

with

$$\psi_E = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}} a_E(\ell,m) e^{-i\omega t} f_\ell(kr) Y_{\ell m}(\theta,\phi) \qquad (42)$$

$$\psi_M = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}} a_M(\ell,m) e^{-i\omega t} g_\ell(kr) Y_{\ell m}(\theta,\phi) \qquad (43)$$

where the coefficients $a_E(\ell,m)$ and $a_M(\ell,m)$ determine the amplitude of the electric and magnetic multipoles of order (ℓ,m), respectively, and f_ℓ and g_ℓ are solutions to Eq. (31). The factors $4/\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}$ are introduced for later convenience.

It must be pointed out that one can also consider singular solutions to the wave equation (25) that generate well-behaved perturbations (Sec. 5).

3. Sources of multipole radiation

We shall now derive the relation between the multipole coefficients $a_E(\ell,m)$ and $a_M(\ell,m)$, introduced in Eqs. (42) and (43), and the sources of the gravitational field in the linearized theory. We shall assume that the components of the tensor $T_{\alpha\beta}$ appearing in Eq. (1) and the metric perturbations have a time dependence of the form $e^{-i\omega t}$. Then the solutions of the inhomogeneous wave equations (21) and (34), with the boundary condition of outgoing waves at infinity, are expressed in terms of the corresponding Green function for the Helmholtz equation [8]

$$(K^*_{0i0j} x^i x^j)(\mathbf{x}) = \frac{2G}{c^4}\int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\left\{k x^j L_m T_j{}^m - i\frac{\partial}{\partial r} r L_m T_0{}^m\right\}(\mathbf{x}')\, d^3x' \qquad (44)$$

and

$$(K_{0i0j} x^i x^j)(\mathbf{x}) = \frac{2G}{c^4}\int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\left\{\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} r^2 - k^2 r^2\right) T_{00} + 2ikx^m \frac{1}{r}\frac{\partial}{\partial r} r^2 T_{0m}\right.$$

$$\left. + \left[\frac{1}{2}\delta^{ij}\left(\frac{\partial^2}{\partial r^2} r^2 + k^2 r^2\right) - k^2 x^i x^j\right] T_{ij}\right\}(\mathbf{x}')\, d^3x'. \qquad (45)$$

Since we are considering outgoing waves at infinity, the radial functions appearing in Eqs. (42) and (43) must be the spherical Hankel functions $h_\ell^{(1)}$; then, taking into account that Eqs. (29), (33), (39) and (40) hold for each multipole, using the orthonormality of the spherical harmonics we have

$$a_M(\ell,m)h_\ell^{(1)}(kr) = \frac{1}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int (K^*_{0i0j}x^i x^j)\overline{Y_{\ell m}}\,d\Omega \tag{46}$$

$$a_E(\ell,m)h_\ell^{(1)}(kr) = \frac{1}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int (K_{0i0j}x^i x^j)\overline{Y_{\ell m}}\,d\Omega \tag{47}$$

where the bar denotes complex conjugation. Making use of the spherical wave expansion for the Green function $e^{ik|\mathbf{x}-\mathbf{x}'|}/|\mathbf{x}-\mathbf{x}'|$ it follows that [8]

$$\int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\overline{Y_{\ell m}}(\theta,\phi)\,d\Omega = 4\pi ik h_\ell^{(1)}(kr)j_\ell(kr')\overline{Y_{\ell m}}(\theta',\phi').$$

Therefore, by combining Eqs. (44–47) we find that the multipole coefficients are given by

$$a_M(\ell,m) = \frac{8\pi ikG}{c^4\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int j_\ell(kr)\overline{Y_{\ell m}}\Big\{kx^j L_n T_j{}^n - i\frac{\partial}{\partial r}rL_nT_0{}^n\Big\}\,d^3x \tag{48}$$

$$a_E(\ell,m) = \frac{8\pi ikG}{c^4\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int j_\ell(kr)\overline{Y_{\ell m}}\left\{\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}r^2 - k^2\right)T_{00} + 2ikx^n\frac{1}{r}\frac{\partial}{\partial r}r^2T_{0n} + \left[\frac{1}{2}\delta^{ij}\left(\frac{\partial^2}{\partial r^2}r^2 + k^2r^2\right) - k^2x^ix^j\right]T_{ij}\right\}\,d^3x. \tag{49}$$

Integrating by parts, these expressions can be rewritten as

$$a_M(\ell,m) = -\frac{8\pi ikG}{c^4\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int (L_n\overline{Y_{\ell m}})\Big\{j_l(kr)kx^jT_j{}^n + \frac{i}{r}\frac{\partial}{\partial r}(r^2j_\ell(kr))T_0{}^n\Big\}\,d^3x \tag{50}$$

$$a_E(\ell,m) = \frac{8\pi ikG}{c^4\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\int \overline{Y_{\ell m}}\left\{\frac{1}{2}T_{00}\left(\frac{\partial^2}{\partial r^2}r^2 - k^2r^2\right)j_\ell(kr) - 2ikT_{0n}\frac{\partial}{\partial r}(rx^nj_\ell(kr)) + T_{ij}\left[\frac{1}{2}\delta^{ij}\left(\frac{\partial^2}{\partial r^2}r^2 + k^2r^2\right) - k^2x^ix^j\right]j_\ell(kr)\right\}\,d^3x. \tag{51}$$

Assuming now that the source dimensions are very small compared to the wavelength of the waves, the Bessel functions can be taken approximately as $j_\ell(kr) \simeq (kr)^\ell/(2\ell+1)!!$ and restricting ourselves to slow-motion sources with negligible internal stresses, keeping only the lowest power in kr, the multipole coefficients are determined by $T_{00} = \rho c^2$ and $T_{0i} = -\rho c v_i$, where ρ is the mass density and v_i are the components of its velocity field, according to

$$a_M(\ell, m) = -\frac{8\pi G k^{\ell+1}}{c^3\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\frac{\ell+2}{(2\ell+1)!!}\int\left(L_n\overline{Y_{\ell m}}\right)r^\ell \rho v^n\, d^3x \quad (52)$$

$$a_E(\ell, m) = \frac{4\pi i G k^{\ell+1}}{c^2\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\frac{(\ell+2)(\ell+1)}{(2\ell+1)!!}\int r^\ell\overline{Y_{\ell m}}\rho\, d^3x \quad (53)$$

[cf. Ref. [7] and Eqs. (5.27) of Ref. [3]]. Thus in the long-wavelength, newtonian limit the electric multipoles are related to the mass density and the magnetic multipoles depend on the angular momentum density of the source.

4. Curvature perturbations and energy of multipole radiation

The curvature perturbations given by Eqs. (27), (28), (37) and (38) can be rewritten in a simple and convenient way by calculating the components of K_{0i0j} and K^*_{0i0j} with respect to the orthonormal basis $\{\hat{e}_\theta, \hat{e}_\phi, \hat{e}_r\}$ induced by the spherical coordinates and constructing combinations of these components with a well defined spin-weight [9–11]. From the components K_{0i0j} one can form the five combinations

$$\begin{aligned}
K_{(2)} &\equiv K_{0\theta0\theta} - K_{0\phi0\phi} + 2iK_{0\theta0\phi} = K_{0r0r} + 2(K_{0\theta0\theta} + iK_{0\theta0\phi})\\
K_{(1)} &\equiv -(K_{0r0\theta} + iK_{0r0\phi})\\
K_{(0)} &\equiv K_{0r0r}\\
K_{(-1)} &\equiv K_{0r0\theta} - iK_{0r0\phi}\\
K_{(-2)} &\equiv K_{0\theta0\theta} - K_{0\phi0\phi} - 2iK_{0\theta0\phi} = K_{0r0r} + 2(K_{0\theta0\theta} - iK_{0\theta0\phi})
\end{aligned} \quad (54)$$

which have spin-weight 2, 1, 0, -1 and -2, respectively. Similarly, we will denote by $K^*_{(a)}$ $(a = 2, 1, 0, -1, -2)$ the combinations obtained by substituting $K^*_{\alpha\beta\gamma\delta}$ in place of $K_{\alpha\beta\gamma\delta}$ in (54). A straightforward but somewhat lengthy computation, using Eqs. (27–29) and (32) shows that in the case of a transverse electric field

$$K_{(2)} = -\frac{i}{2c}\frac{\partial}{\partial t}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\eth\eth\psi$$

$$
\begin{aligned}
K_{(1)} &= -\frac{i}{4c}\frac{\partial}{\partial t}r\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\eth\psi \\
K_{(0)} &= 0 \\
K_{(-1)} &= -\frac{i}{4c}\frac{\partial}{\partial t}r\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{\eth}\psi \\
K_{(-2)} &= \frac{i}{2c}\frac{\partial}{\partial t}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\overline{\eth\eth}\psi
\end{aligned} \tag{55}
$$

where, acting on a function η of spin-weight s [9–11],

$$
\begin{aligned}
\eth\eta &\equiv -\operatorname{sen}^s\theta\left(\frac{\partial}{\partial\theta} + \frac{i}{\operatorname{sen}\theta}\frac{\partial}{\partial\phi}\right)(\operatorname{sen}^{-s}\theta)\eta \\
\bar{\eth}\eta &\equiv -\operatorname{sen}^{-s}\theta\left(\frac{\partial}{\partial\theta} - \frac{i}{\operatorname{sen}\theta}\frac{\partial}{\partial\phi}\right)(\operatorname{sen}^{s}\theta)\eta
\end{aligned} \tag{56}
$$

and,

$$
\begin{aligned}
K^*_{(2)} &= \frac{1}{4}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}r^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\eth\eth\psi \\
K^*_{(1)} &= \frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} - \frac{1}{r^2}\eth\bar{\eth} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{r}\frac{\partial}{\partial r}r^2\eth\psi \\
K^*_{(0)} &= -\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{\eth}\eth\psi \\
K^*_{(-1)} &= -\frac{1}{8}\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} - \frac{1}{r^2}\bar{\eth}\eth - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{r}\frac{\partial}{\partial r}r^2\bar{\eth}\psi \\
K^*_{(-2)} &= \frac{1}{4}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}r^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{\eth}\bar{\eth}\psi
\end{aligned} \tag{57}
$$

Hence, in the case of a *magnetic multipole* of order (ℓ, m), taking

$$
\psi = \frac{4}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}e^{-i\omega t}f_\ell(kr)Y_{\ell m}(\theta,\phi) \tag{58}
$$

[cf. Eqs. (42) and (43)], using Eq. (31) and the relations

$$
{}_sY_{\ell m} = \begin{cases} \left[\dfrac{(\ell-s)!}{(\ell+s)!}\right]^{1/2} \eth^s Y_{\ell m}, & 0 \le s \le \ell \\[2ex] (-1)^s \left[\dfrac{(\ell+s)!}{(\ell-s)!}\right]^{1/2} \bar{\eth}^{-s} Y_{\ell m}, & -\ell \le s \le 0 \\[2ex] 0, & |s| > \ell \end{cases} \tag{59}
$$

satisfied by the spin-weighted spherical harmonics [9–11], we get

$$
\begin{aligned}
K_{(\pm 2)} &= \mp 2k e^{-i\omega t} \frac{1}{r^2}\frac{d}{dr} r^2 f_\ell(kr)\, {}_{\pm 2}Y_{\ell m}(\theta,\phi) \\
K_{(\pm 1)} &= \mp \sqrt{(\ell-1)(\ell+2)}\, k e^{-i\omega t} \frac{1}{r} f_\ell(kr)\, {}_{\pm 1}Y_{\ell m}(\theta,\phi) \\
K_{(0)} &= 0
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
K^*_{(\pm 2)} &= e^{-i\omega t}\left(\frac{1}{r^2}\frac{d^2}{dr^2} r^2 - k^2\right) f_\ell(kr)\, {}_{\pm 2}Y_{\ell m}(\theta,\phi) \\
K^*_{(\pm 1)} &= \sqrt{(\ell-1)(\ell+2)}\, e^{-i\omega t} \frac{1}{r^2}\frac{d}{dr} r f_\ell(kr)\, {}_{\pm 1}Y_{\ell m}(\theta,\phi) \\
K^*_{(0)} &= \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}\, e^{-i\omega t} \frac{1}{r^2} f_\ell(kr) Y_{\ell m}(\theta,\phi).
\end{aligned} \tag{61}
$$

In the case of outgoing waves at infinity we can take $f_\ell = h_\ell^{(1)}$, which has the asymptotic form $h_\ell^{(1)}(kr) \to (-i)^{\ell+1} e^{ikr}/kr$, for $kr \gg \ell$. Then, Eqs. (60) and (61) show that in the radiation zone ($kr \gg 1$) the curvature components of spin-weight ± 2 are greater than those of spin-weight ± 1 by a factor kr and greater than the components with spin-weight 0 at least by a factor $(kr)^2$. One finds that, in the radiation zone,

$$
\begin{aligned}
K_{(\pm 2)} &\to \mp 2i(-i)^{\ell+1} \frac{k}{r} e^{i(kr-\omega t)}\, {}_{\pm 2}Y_{\ell m} \\
K^*_{(\pm 2)} &\to -2(-i)^{\ell+1} \frac{k}{r} e^{i(kr-\omega t)}\, {}_{\pm 2}Y_{\ell m}.
\end{aligned} \tag{62}
$$

For an *electric multipole* of order (ℓ, m), with ψ given again by Eq. (58), the components $K_{(a)}$ and $K^*_{(a)}$ are exactly of the same form as $K^*_{(a)}$ and $(-K_{(a)})$, respectively, for a magnetic multipole of order (ℓ, m) given in Eqs. (61) and (60). Thus, we see from Eqs. (62) that in the radiation zone the electric and magnetic multipoles of order (ℓ, m) have the same dependence but have polarizations that differ by a rotation through 45° around the radial direction. (A quantity that has spin-weight s transforms under rotations around $\hat{e}_r$ by a multiplicative factor $e^{is\alpha}$.)

The power radiated per unit solid angle can be calculated by means of the expression [1–2]

$$\frac{dE}{dt\, d\Omega} = \lim_{r\to\infty} \frac{c^5 r^2}{16\pi G k^2} \left| K_{(2)} \right|^2$$

therefore, using Eqs. (62), for a superposition of magnetic and electric multipoles given by Eqs. (41–43)

$$\frac{dE}{dt\, d\Omega} = \frac{c^5}{4\pi G} \left| \sum_{\ell,m} [a_E(\ell,m) + i a_M(\ell,m)]\, {}_2Y_{\ell m} \right|^2. \tag{63}$$

Hence, in view of the orthonormality of the spin-weighted spherical harmonics for a fixed spin-weight, the total power radiated is

$$\frac{dE}{dt} = \frac{c^5}{4\pi G} \sum_{\ell,m} |a_E(\ell,m) + i a_M(\ell,m)|^2. \tag{64}$$

The mass quadrupole part of the power radiated is

$$\left(\frac{dE}{dt}\right)_{\text{mass quadrupole}} = \frac{c^5}{4\pi G} \sum_m |a_E(2,m)|^2$$

which, according to Eq. (53), becomes

$$\begin{aligned} \left(\frac{dE}{dt}\right)_{\text{mass quadrupole}} &= \frac{8\pi G c k^6}{75} \left| \int r^2 \overline{Y_{2m}} \rho \, d^3x \right|^2 \\ &= \frac{G c k^6}{45} D_{ij} D^{ij}, \end{aligned}$$

where

$$D_{ij} \equiv \int \rho (3x_i x_j - \delta_{ij} r^2)\, d^3x.$$

5. Concluding remarks

Apart from the metric perturbations generated by the regular solutions (30) of the wave equation, there exist well-behaved solutions to the linearized vacuum field equations corresponding to singular potentials [12]. For example, by expanding the Schwarzschild metric to first order in the mass parameter M, one obtains the static spherically symmetric metric perturbation

$$h_{00} = \frac{2GM}{c^2 r}, \quad h_{ij} = \frac{2GM}{c^2 r^3} x_i x_j, \quad h_{0i} = 0,$$

which, by means of a gauge transformation (12) with $\xi_0 = 0$, $\xi_i = -\frac{GM}{c^2 r} x_i$, is transformed into

$$h_{00} = \frac{2GM}{c^2 r}, \quad h_{ij} = \frac{2GM}{c^2 r} \delta_{ij}, \quad h_{0i} = 0.$$

This metric perturbation is of the form (36) with $\psi = \frac{4GM}{c^2 r}(\ln r \csc\theta - 1)$, which is a solution of Eq. (25) that diverges on the axis $\theta = 0, \pi$.

The invariance of the linearized vacuum field equations (5) under the gauge transformations (12) is related with the identity $\partial^\beta [\mathcal{E}(h_{\gamma\delta})]_{\alpha\beta} = 0$, that follows from the definition (2) [13]. If $\mathcal{G}$ is defined by $\mathcal{G}(b_{\alpha\beta}) \equiv \zeta^{(\alpha}\partial^{\beta)} b_{\alpha\beta}$, where ζ^α is an arbitrary vector field, then $\mathcal{G}\mathcal{E} = 0$; therefore Eq. (7) remains valid if $\mathcal{S}$ is replaced by $\mathcal{S} + \mathcal{G}$. This ambiguity in the choice of $\mathcal{S}$ amounts to the replacement of $h_{\alpha\beta} = [\mathcal{S}^\dagger(\psi)]_{\alpha\beta}$ by $h_{\alpha\beta} = [\mathcal{S}^\dagger(\psi)]_{\alpha\beta} + [\mathcal{G}^\dagger(\psi)]_{\alpha\beta} = [\mathcal{S}^\dagger(\psi)]_{\alpha\beta} - \partial_{(\alpha}\zeta_{\beta)}\psi$, which is precisely of the form (12) with $\xi_\alpha = \frac{1}{2}\psi\zeta_\alpha$.

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Resumen. Usando el método de operadores adjuntos, la solución completa de las ecuaciones de Einstein para el vacío linealizadas alrededor de la métrica de Minkowski se expresa en términos de potenciales escalares que satisfacen la ecuación de ondas. Los campos multipolares se obtienen entonces de las soluciones separables de la ecuación de ondas en coordenadas esféricas y la amplitud de cada multipolo es relacionada con el tensor de energía-impulso de las fuentes. Las componentes invariantes de norma de los campos multipolares se escriben en términos de los armónicos esféricos con peso de espín y se obtiene también la potencia radiada por unidad de ángulo sólido por cada multipolo.