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The role of the slip coefficient on the flow past the sphere

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Abstract. Under certain circumstances, some fluids flow in such a way that the usual sticking boundary condition, commonly explained invoking surface roughness, must be relaxed to allow for some amount of slip. This paper argues that the surface tension at a fluid-solid interface should play a role in defining the appropriate hydrodynamic boundary condition; the effect is incorporated through the slip coefficient. Furthermore, an experiment is proposed, based on a careful study of the motion of settling spheres, to demonstrate the previous contention. The required theoretical results, needed to interpret the outcome of the experiment, are provided.

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1. The background

When Navier [1] formulated the dynamical equations for a non-ideal fluid in 1823, he faced the analytical problem of determining the appropriate boundary conditions. For impervious solid boundaries he assumed [2], following the Bernoullis and Euler [3], that the normal component of the relative velocity is equal to zero. The new equations, of a higher order, required an extra assumption on the velocity or its derivatives at the boundaries. Consistent with his "statistical mechanical" procedure to derive the equations, he deduced that there is slipping at a solid boundary. Finding that slipping is opposed by a force proportional to the relative velocity, he then introduced a slip coefficient to equate the viscous tangential stresses to the tangential component of the relative velocity. Different values of this coefficient were used to reproduce previously proposed hypotheses [2] on the relative motion of fluids and solid boundaries: zero tangential stress corresponding to the slipping boundary condition and absence of relative motion between fluid and solid corresponding to the sticking boundary condition. Confusion on the issue was thus born.

Two decades later, Stokes [4] derived independently the same equations from a phenomenological point of view. By avoiding altogether molecular assumptions he explored some of the consequences of the sticking boundary condition. After five years, he established that they agreed better with the experimental results of Coulomb, Hagen and Poiseuille, among others [2]. A long time elapsed before some general agreement was reached on this point. This was accomplished primarily for practical reasons since inquiries into its origin remained unanswered [3].

A quantitative analysis of a plausible explanation of the sticking boundary condition was given somewhat recently by Richardson [5]. Arguing that all surfaces are, in practice, rough on a mesoscopic scale he proposed a very specialized model for the wall. He then cleverly showed that, as the fluid passes over and around the irregularities, the energy lost due to viscous dissipation is enough to bring the fluid to rest on a macroscopic scale, regardless of whether slip or stick is assumed on the mesoscopic scale of the surface. Deviations from the macroscopic sticking boundary condition being of the order of the asperities of the wall. A few years latter, Zwanzig [6] extended the analysis to a finite body and carefully discussed the relevant length scales.

From the microscopic point of view, Maxwell [7] was probably the first to consider the problem using the kinetic theory of gases. When there is no temperature difference between the fluid and the solid, he showed that Navier's slip coefficient is proportional to the inverse of the mean free path. Thus, under normal conditions, the effect of slipping would be negligible. After a century the issue was again reconsidered, among others, by Cercignani [8], van Beijeren and Dorfman [9] and by Oppenheim and van Kampen [10]. Yet, better understood, the basic problem remains open.

On a different line of reasoning, the pioneering numerical studies of Alder and Wainwright [11] suggested the existence of hydrodynamic behavior on a molecular level. The work of Zwanzig and Bixon [12] provided some of the early clues on such surprising results. The use of hydrodynamical models for molecular motions in liquids became successful when stick was replaced by slip in the boundary conditions [13]. Again, Zwanzig [14] contributed to clarify some points on purely hydrodynamic grounds, giving some insights for the stick to slip transition. Using rough sphere kinetic theory, Hynes, Kapral and Weinberg [15] and Montgomery and Berne [16] gave a microscopic counterpart.

2. Surface tension and the slip coefficient

There is an overwhelming amount of experimental evidence that supports the sticking boundary condition, suggesting a common universal behavior. However, there are some cases where zero shear stress must be necessarily assumed or where partial slip is observed. For example, rarefied gases can, under certain circumstances, be treated as Newtonian continua with a perfect slip condition at solid boundaries [17]; slip velocities have been observed in structured fluids under some flow conditions, such as polymer melts during an extrusion process [18]. Although the rough surface argument is convincing and provides a sound basis in most cases, there are the exceptions to consider and some unanswered questions that seem worth dwelling on: Which boundary condition applies on the scale where the surface looks flat, if it is not so everywhere? How universal is the stick hypothesis? Or, are the exceptions predictable? Furthermore, are there any other macroscopic properties that evolve from microscopic aspects of the interface, such as the fluid-fluid and fluid-solid molecular interaction ratio, that could have some bearing on whether stick or some amount of slip is the appropriate boundary condition?

A case in point might be surface tension. The purpose of this paper is to argue in support of this contention.

It is very common to invoke the various intermolecular forces in action at an interface to justify the sticking boundary condition. Adsorption of fluid molecules on a solid surface or the occurrence of different types of chemical bonds between fluid and solid particles most certainly take place continuously. It would be as hard to argue that they are the dominant processes at the interface as it would be difficult to ignore their presence. The wetting properties of different liquids on a given surface are a clear example. Non-wetting fluids (angle of contact close to 180°) represent situations where the forces between liquid molecules may be orders of magnitude greater than those across the interface. One would expect under these circumstance, and a flat surface, that the appropriate condition would be that of zero shear stress. For perfect-wetting liquids the reverse situation is to be expected. Intermolecular forces can therefore be called upon to support a boundary condition between the extremes of slip and stick, depending on their relative magnitudes. Surface tension follows the same trends; for non-wetting liquids it is known to be much greater than for those in which there is perfect wetting. Hence, a dependence between slip and surface tension, through suitably defined coefficients, seems worth looking for. In what follows we assume that surface tension is apparent through the slip coefficient, as defined in section IV.

Provided a relation between the coefficients of slip and surface tension can be established, on both theoretical and, above all, experimental grounds, the previously raised questions would appear to have a common, simple and straight answer (the theory and the experiments might be everything but simple!): The precise boundary condition or the amount of slip on a (locally flat) surface would depend on the local surface tension. In view of the accumulated evidence supporting the sticking boundary condition, the assumed dependence of slip on surface tension should be rather weak in most cases, while explaining those in which slipping occurs and "universality" breaks down. As for the last question, an answer is implicit in the foregoing argument; the direct dependence between the surface tension coefficient and the intermolecular forces is well known for the equilibrium liquid-gas interface [19]. Something analogous is to be expected from a convincing non-equilibrium fluid-solid interface theory.

To convincingly establish the point one should either produce a mathematical proof, based on physical ideas, and then perform an experiment to corroborate the prediction or carry out an experiment with an unambiguous interpretation. Seldom are these procedures simple. Assuming that the latter is easier in the present case, a feasible experiment is sketched and its main features discussed in the next sections [20].

3. Sketch of an experiment

A standard method for measuring the surface tension coefficient, for a given fluid-solid system, consists in determining the contact angle that forms when a drop of the fluid sits on the (flat) surface of the solid; it is assumed that surface roughness is irrelevant. Suppose that in a particular case the measured contact angle is close to zero; the liquid wets the solid. Next, assume that the solid's surface can be treated in such a way that the contact angle is now close to 180°, drastically changing the wetting properties, while leaving the geometrical characteristics of the surface (its roughness) unchanged; like a layer of snow on a landscape. For example, a fine coating of silicone on glass would produce such an effect when the fluid is water, say. With this preliminary experiment in mind, the possible dynamic consequences can be explored.

From a hydrodynamical point of view, few problems have received more attention, both theoretically and experimentally, than that of a solid sphere moving in a viscous fluid; the motivation being its relative mathematical simplicity, such as the known solution of the linear case, and its enormous practical relevance. Though the general problem has yet to be solved, many of its features are well understood and appear in most textbooks [21]. It is then only natural to put to work hydrodynamics' show horse. Furthermore, it is used as a representative case (workhorse) of a system with linear friction in classical mechanics.

The basic idea for the experiment is the following: register the fall of a solid sphere in a liquid and repeat the process keeping everything fixed except for the wetting properties at the interface; the surface tension, as understood in the preliminary experiment discussed above. All things kept constant, differences can only be attributed to changes in the shear stresses on the sphere's surface.

Can the obvious complications that arise after a moment's thought be surmounted? Perhaps.

First, the falling object must be as spherical as possible and the fluid's properties must remain constant (density ρ and shear viscosity coefficient η); a controlled heat bath would keep ρ and η fixed. Second, the spheres must be released systematically avoiding rotation and ensuring a reproducible path. To register the motion, a video film can be made with a high speed shutter camera and suitably chosen images digitalized to be further analyzed.

The crucial point is clearly the sphere's surface "transmutation": The sphere's mass, radius and surface quality must remain fixed, while its wetting response to the fluid is to change as much as possible. This can be accomplished following the schematic procedure outlined in the preliminary experiment. Assume, for the sake of argument, that the working fluid is water and that the spheres are 2 mm diameter glass balls (silicates). With a clean surface the angle of contact is practically zero. A thin (few monolayers thick), stable (covalent bonded) and spherical coating of water repellent material, such as a silicone fluid, can be manufactured so as to produce the desired effect. The resulting contact angle is very close to 180° and the changes in mass and radius negligible.

Leaving aside some other points that should be taken care of, like careful handling of the spheres to avoid contaminants that could mask the effect completely [22], one last critical issue deserves attention: The uncertainties must be small enough to allow the effect to be measurable. In order to be able to carry a thorough treatment of error propagation and accumulation, from which the required accuracy on each of the measured quantities can be estimated *a priori*, the theory underlying the experiment should be developed. An outline of the former is given in the next section.

4. Some theory

Consider a solid sphere of radius a and mass m, released from rest in a quiescent infinite viscous fluid with density ρ and shear viscosity coefficient η, under the action of a uniform gravitational field characterized by g.

The equation of motion for the center of mass of the sphere, assuming it is one-dimensional (along the z axis), is given by

$$m\ddot{z} = mg - m'g - F_0, \tag{1}$$

where each dot represents a time derivative and m' is the fluid's mass displaced by the sphere. The forces on the right hand side are the weight, the buoyancy and the hydrodynamic friction. In order to solve this equation, which is formally exact, an explicit expression for F_0 in terms of z, $\dot{z}$, $\ddot{z}$ and t is required.

The derivation begins with the solution of the Navier-Stokes equations for an incompressible fluid [21]

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\rho\partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \eta\nabla^2\mathbf{u}, \tag{3}$$

where $\mathbf{u}$ and p are the velocity and pressure fields, both depending on position and time; incompressibility applies equally well to liquids and gases provided all velocities are small compared with that of sound. Next, some initial and boundary conditions must be given. The sphere moves with velocity $\mathbf{U}(t)$, released from rest. Hence, initially, fluid and sphere are at rest. One boundary condition is that the fluid remains at rest at infinity. The other boundary condition concerns the behavior of the fluid at the surface of the sphere and requires separate treatment of the normal and tangential components of the velocity field. First, the normal component of the relative velocity of fluid and sphere vanishes on the surface,

$$(\mathbf{u} - \mathbf{U}) \cdot \hat{e}_r = 0, \tag{4}$$

where $\hat{e}_r$ is a unit vector normal to the surface of the sphere. This kinematic condition ensures that the fluid does not penetrate the sphere. Second, the amount of slip is established. Following Navier, the assumption is that the tangential component of the relative velocity of fluid and sphere is proportional to the tangential component of the force acting on the sphere. That is

$$(\hat{e}_r \cdot \Pi) \times \hat{e}_r = \beta(\mathbf{u} - \mathbf{U}) \times \hat{e}_r, \tag{5}$$

where β is a coefficient giving the degree of slip. When $\beta = 0$ there are no tangential stresses, corresponding to perfect slip, and when $\beta = \infty$ the fluid adheres to the sphere. The stress tensor Π is given by

$$\Pi = -p\mathbb{I} + \eta(\nabla\mathbf{u} + \mathbf{u}\nabla). \tag{6}$$

Here, $\mathbb{I}$ is the unit second rank tensor. The force on the sphere is

$$\mathbf{F}_0 = \oint(\Pi \cdot \hat{e}_r)\, d\Omega. \tag{7}$$

As posed, the problem remains unsolved.

Several approximations can be made in order to find the velocity of the sphere as a function of time, $z(t)$. This is done in two stages. First, assumptions are introduced to find an expression for F_0, which are basically of two kinds: linearization of the Navier-Stokes equations or nonlinear steady state calculations. Second, given the hydrodynamic force, Eq. (1) is integrated and the expressions for $\dot{z}(t)$, resulting from the different approximations used, are compared.

The basic result is that the dominating feature in the sphere's motion is the boundary condition used; whichever of the hydrodynamic approximations here taken, the terminal velocity of the sphere cannot exceed a certain value unless there is some amount of slip at its surface.

Hydrodynamic force

We begin with the linear approximation [21]. Writing Eq. (3) in terms of dimensionless variables, using a characteristic length a and a typical velocity v_0, the nonlinear term comes out multiplied by the Reynolds number ($R \equiv a\rho v_0/\eta$). Provided R is small the nonlinear term can then be omitted (see below). With this approximation, an exact expression for F_0 can be found. Extending well known results due to Stokes, Basset and Boussinesq, Zwanzig and Bixon [12] obtained the drag force on a sphere, including compressibility and viscoelastic effects (through Maxwell's approach of frequency-dependent viscosity coefficients). Their result, extended for arbitrary slip and specialized for an incompressible and constant viscosity fluid, is

$$\mathbf{F}_0(\omega) = -\zeta(\omega)\mathbf{U}(\omega), \tag{8}$$

where the friction coefficient ζ, as a function of frecuency (ω), is given by

$$\zeta(\omega) = 4\pi\eta a\left(1+\frac{\xi}{2}\right) - 2\pi a^3\rho i\omega/3 - \frac{2\pi a^2(2+\xi)^2 i(i\omega\eta\rho)^{1/2}}{3-(1-\xi)ia(i\omega\rho/\eta)^{1/2}}. \tag{9}$$

The slip coefficient ξ is defined by

$$\xi = \left(1+\frac{3\eta}{\beta a}\right)^{-1} \tag{10}$$

Stick corresponds to $\xi = 1$ and perfect slip to $\xi = 0$. The first term is Stokes's well known zero- frequency result. The second term is related to the effective or virtual mass. The last term, giving rise to a memory type time dependence, originates from the viscous unsteady flow around the sphere; $\delta = (2\eta/\rho\omega)^{1/2}$, the viscous penetration depth, being its signature. In the two extreme cases of slip and stick, Eq. (7) reproduces the Zwanzig- Bixon results (except for the sign in the denominator when $\xi = 0$). The corresponding expression for the time-dependent force $F_0(t)$ is discussed below.

An altogether different starting point is the assumption of steady motion. In this case, the time dependence of the hydrodynamic force comes in solely through its dependence on the instantaneous velocity of the sphere. Dropping the time derivative in Eq. (3) still leaves an insoluble problem. The usual approach is to look for a solution in powers of the Reynolds number. The zeroth order approximation leads directly to Stokes's law, as extended by Basset; omitting the known details, the result is

$$F_0(\dot{z}) = 4\pi\eta a\left(1+\frac{\xi}{2}\right)\dot{z}. \tag{11}$$

As expected, it is the zero-frequency limit of equations (8) and (9) combined. The velocity of the sphere $\mathbf{U}$ has been written as $\dot{z}$. That the higher order approximations fail to exist is known as Whitehead's paradox. Oseen [23] was the first one to point out that at a great distance from the sphere the neglected nonlinear terms become more important than the retained viscous terms, and suggested a new approximation. Later, Goldstein [24] solved the resulting equations in powers of the Reynolds number. However, his result was shown to be valid only up to linear terms in R; using the sticking boundary condition, Eq. (5) with $\beta = \infty$, the resulting expression for the drag force is

$$F_0(\dot{z}) = 6\pi\eta a\left(1+\frac{3R\dot{z}}{8v_0}\right)\dot{z}. \tag{12}$$

Within Oseen's theory, Shanks [25] approximated the drag force using the Goldstein series up to and including terms of order R^5 by a rational fraction (2-2 Padé approximant). He obtained

$$F_0 = 6\pi\eta a \frac{Av_0^2 + BRv_0\dot{z} + CR^2\dot{z}^2}{Av_0^2 + DRv_0\dot{z} + ER^2\dot{z}^2}\dot{z}, \tag{13}$$

where A, B, C, D and E are real numbers (73920, 66600, 10880, 38880, 689). This result works well for values of R as high as 10, whereas Eq. (12) is useless above $R = 1$.

The next improvement, due to Kaplun and Lagerstrom [26] and Proudman and Pearson [27], required a specially suited development of the method of matched asymptotic expansions [28]. Their analysis provides a systematic procedure to calculate higher order approximations for the drag force obtaining a series in powers and logarithms of the Reynolds number. Although well founded, the results are of limited use due to the fact that the applicability is restricted to values of R smaller than 0.4 and the computations are of enormous complexity. For the present purpose Eq. (13) will suffice.

Settling velocity

Given the hydrodynamic friction acting on the sphere the solution of its equation of motion can be pursued. Beginning with some preliminary remarks, two separate calculations are performed: With fixed (sticking) boundary conditions, the motion of the sphere is analyzed when different hydrodynamic approximations for F_0 are introduced. Then, within the hydrodynamic approximation that led to Eq. (9), the effect of changing the boundary conditions is considered.

To begin, the previous results for F_0 are specialized for sticking boundary conditions ($\xi = 1$). Equation (9), in this case [29], can be Fourier inverted to obtain $F_0(t)$ and the result is

$$F_0 = 6\pi\eta a\dot{z} + \frac{m'\ddot{z}}{2} + \frac{9m'}{2a}\left(\frac{\eta}{\pi\rho}\right)^{1/2}\int_0^1 \frac{d^2z}{d\tau^2}\frac{d\tau}{(t-\tau)^{1/2}}. \tag{14}$$

As mentioned above, the first term is Stokes's law, the second term is related to the virtual mass and the last term is connected to the viscous penetration depth and gives rise to a memory effect on the motion; when Eq. (14) is substituted into Eq. (1) and the second term is taken to the left hand side of the equation, the acceleration is multiplied by a renormalized mass $m_0 = m + m'/2$. This is an inherently time-dependent effect. Since the approximations leading to Eqs. (11), (12) and (13) ignore this point, they can all be marginally improved by simply putting in m_0 into Eq. (1). Next, dimensionless variables are introduced using the following definitions:

$$T = \gamma t \qquad \text{and} \qquad u = \frac{z}{v_0}, \tag{15}$$

where

$$\gamma = \frac{6\pi\eta a}{m_0} \qquad \text{and} \qquad v_0 = \frac{(m - m')g}{6\pi\eta a}. \tag{16}$$

The equation of motion now reads

$$u' = 1 - uf(u, T), \tag{17}$$

where $u' = du/dT$ and $f(u, T)$ is the dimensionless friction coefficient. The initial condition is, in all cases, that the sphere is released from rest at the origin. Proceeding in order of complexity, Eq. (17) is solved using Eqs. (11), (12), (13) and (14).

For Stokes's law one has $f(u, T) = 1$. The solution is

$$u = 1 - \exp(-T), \tag{18}$$

where, obviously, $(u')_{T=0} = 1$ and the terminal velocity is 1.

Next, Oseen's extension of Stokes's law, Eq. (12), corresponds to $f(u, T) = 1 + \frac{3}{8}Ru$. The resulting nonlinear (quadratic) equation can easily be integrated. The solution is

$$u = 2(1 + \chi \coth \chi T/2)^{-1}, \tag{19}$$

where $\chi^2 = 1 + \frac{3}{2}R$; $(u')_{T=0} = 1$ and the terminal velocity is $2(1 + \chi)^{-1} \leq 1$. When $R \to 0$ Eq.(18) is, of course, recovered.

The solution of the quadrature obtained when the Padé approximation, Eq. (13), is introduced into Eq. (17) is

$$T = X \ln\left(1 - \frac{u}{u_1}\right) + V \ln\left(1 + \frac{u}{u_2}\right) + W \ln\left(1 + \frac{u}{u_3}\right), \tag{20}$$

where the u_i's ($i = 1, 2, 3$) and X, V and W depend on the constants A, B, C, D, E and R, given after Eq. (13). The u_i's are positive real numbers; u_1, $-u_2$ and $-u_3$ are the roots of

$$u^3 + \frac{(B - ER)u^2}{CR} + \frac{(A - DR)u}{CR^2} + \frac{A}{CR^2} = 0.$$

Also, one can show that $(u')_{T=0} = 1$ and that the terminal velocity is $u_1 \leq 1$, depending on the value of R which must be smaller than 10. For small R, $u_1 = 1 - (B - D)R/A + \cdots$. In Fig. 1 the behavior of the terminal velocity (u_1) as a function of R is shown.

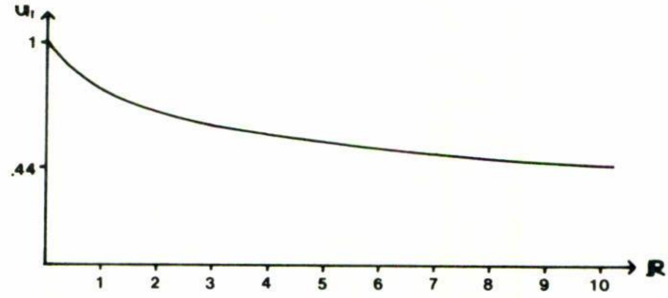

FIGURE 1. The terminal velocity of the sphere (u_1), in reduced units, as a function of the Reynolds number (R) for the Padé approximation, Eq. (20).

The expressions for the remaining quantities in Eq. (20) are

$$X = -\frac{A + DRu_1 + ER^2u_1^2}{CR^2(u_1 + u_3)(u_1 + u_2)} \quad \text{and} \quad V = \frac{A - DRu_2 + ER^2u_2^2}{CR^2(u_3 - u_2)(u_1 + u_2)};$$

For W one interchanges the subindices 2 and 3 in the expression for V. These results are discussed and displayed below.

In the time-dependent case, substituting Eq. (14) into Eq. (17) with the appropriate variables, the equation of motion is

$$u' = 1 - u - 2\left(\frac{\mu}{4\pi}\right)^{1/2} \int_0^T \frac{du}{dS}\frac{dS}{\sqrt{T-S}}, \tag{21}$$

where $\mu = 9m^2/2m_0$. This integro-differential equation can be transformed into a Volterra equation of the second kind and, being linear, solved by standard methods. The final result is

$$u = 1 - \operatorname{Re} w(q) - \frac{\mu}{4-\mu} \operatorname{Im} w(q), \tag{22}$$

where $\operatorname{Re} w$ and $\operatorname{Im} w$ denote the real and imaginary parts of the Error Function with complex argument [30]

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz).$$

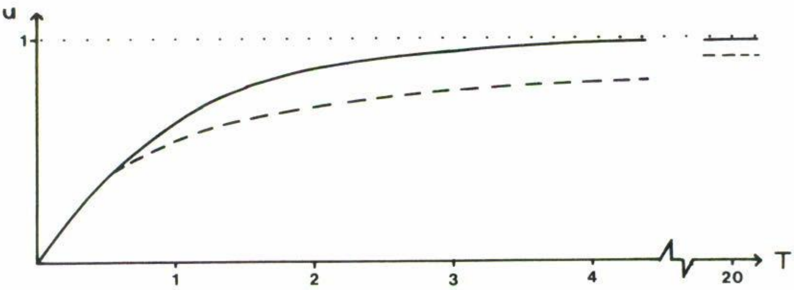

FIGURE 2. Sphere's velocity as a function of time, in reduced units (broken line Eq. (22), continuous line Eq. (18)).

Here

$$q^2 = T\left(\lambda + i\sqrt{\frac{3\mu}{8}}\right)^2 \quad \text{and} \quad \lambda^2 = \frac{3(4-\mu)}{8}.$$

Also, $(u^{\vee})_{\tau=0} = 1$ and the terminal velocity is 1.

In Fig. 2, the velocity of the sphere is shown as a function of time as described by the linear approximations ($R \to 0$), Eqs. (18) and (22). The difference is noticeable at intermediate times and eventually the velocities reach their common asymptotic value. The assumed mass ratio (m'/m) was 0.128, corresponding to iron spheres in water, say; for glass spheres in water $(m'/m = 0.667)$ the differences are even smaller.

When the inertial effects are taken into account, at least in an approximate way, the limiting value of the velocity becomes lower. This is illustrated in Fig. 3, using Eq. (20) that incorporates such an effect; Eq. (19) would give the same result in the range where applicable, $R \leq 1$.

In Fig. 2 the improved approximation, given by Eq. (22), shows a slower approach to the terminal velocity, the difference never being greater than 20%. In Fig. 3, the asymptotic values for u (1.00, 0.80 and 0.45) depend on the Reynolds number R (0,1 and 10, respectively); the approach to the terminal velocities is faster as R increases.

To conclude this section, Eq. (17) is solved assuming arbitrary slip on the sphere's surface. The simplest approximation is, of course, given by Eq. (11). The result, shown in Fig. 4 for the two extreme cases $\xi = 0$ and $\xi = 1$, is a trivial

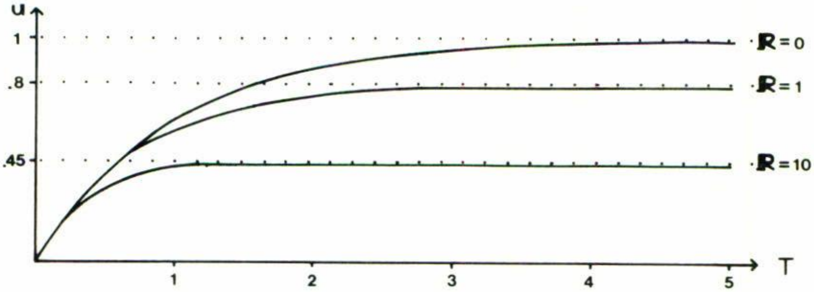

FIGURE 3. Sphere's velocity as a function of time, in reduced units, for different values of R. From Eq. (20).

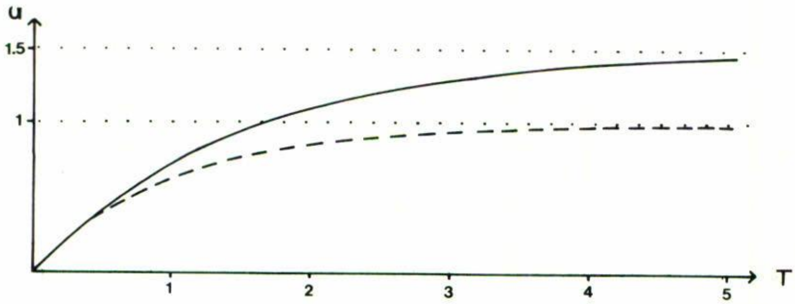

FIGURE 4. The time dependence of the sphere's velocity for stick (broken line) and slip (continuous line) boundary conditions, as given by Eq. (23).

extension of Eq. (18)

$$u = \frac{3}{2+\xi}\left[1 - e^{-(2+\xi)T/3}\right]. \tag{23}$$

A generalization of equation (23) is easier to obtain working in frequency space from the beginning. Hence, one takes the Fourier Transform of Eq. (17); the second term on the right-hand-side corresponds to Eqs. (8) and (9), written in dimensionless variables, while the first term (corresponding to the 1) introduces a singular behavior (a delta function) that must be handled carefully. We label the later by δ_ω and formally carry out the calculations. Solving for u_s, the Fourier Transform of u, gives

$$u_s = \mu \frac{3-(1-\xi)\sqrt{-is}}{\mu(2+\xi)(1-\sqrt{-is}) - is(3-(1-\xi)\sqrt{-is})}\delta_s, \tag{24}$$

where s is a dimensionless frequency ($s = \omega a^2\rho/\eta$) and μ was defined after Eq. (21) ($\mu = \gamma a^2 \rho/\eta = 9m'/2m_0$). It is easy to see that for low and high frequencies the behaviors are

$$u_{\omega\sim 0} = \frac{3}{2+\xi}\delta_\omega + \cdots \quad \text{and} \quad u_{\omega\sim\infty} = \gamma(-i\omega)^{-1}\delta_\omega + \cdots$$

Consequently, the long and short-time asymptotic behaviors are

$$u = 3(2+\xi)^{-1} + \cdots \ (T \to \infty) \quad \text{and} \quad u = T + \cdots \ (T \to 0); \tag{25}$$

where the initial conditions are such that the Fourier inversion for these asymptotic behaviors are guaranteed. The terminal velocity and the initial acceleration are the same as in the previous case (Eq. (23)). To Fourier invert Eq. (24), in order to find $u(T)$, the denominator can be factorized as $(-(1-\xi)\sqrt{q}+k_1)(\sqrt{q}+k_2)(\sqrt{q}+k_3)$, where $q = -is$, and k_1, k_2 and k_3 are complicated expressions involving μ and ξ. Next, the numerator is divided out and the result is expanded in partial factions, each denominator being a binomial times q; the Fourier Transforms are, in each case, of the form $1 - \exp(b^2 T)\mathrm{erfc}\,(b\sqrt{T})$ times some constant depending on k_1, k_2 and k_3; the delta multiplying each term takes care of itself when using the convolution theorem and the appropriate initial conditions. The b's, being complex, lead to an expression for $u(T)$ similar to Eq. (22). We have not cared to write down the explicit results since the behavior must be very close to that shown in Fig. 4; this is because the initial and final values of the velocity as well as the corresponding values for the slopes are identical (see Eq. (25)).

The main conclusion of the foregoing calculations is that the motion of a settling sphere is controlled by the shear stresses on its surface. That is, the boundary condition used, regardless of the hydrodynamic approximation, can change the terminal velocity by more than 30%, between the extreme cases of perfect slip and no-slip.

5. Discussion

Arguing that slipping at a fluid-solid interface should depend on the surface (or interfacial) tension, an experiment to prove the point was proposed and briefly sketched: To perform a careful determination of the settling velocity of two mechanically identical spheres with different wetting characteristics, for a given viscous fluid. The argument stems from the possibility of relating the slip and surface tension coefficients through the procedure outlined in Sec. 3. Wall effects on the drag experienced by the spheres [31] have been overlooked. The experimental setup must be such that these can be neglected; this is not a crucial point because the issue is to keep all things the same and to compare the motion of the spheres when only the wetting characteristics are changed.

A theoretical basis was then given (schematically) in Sec. 4 to interpret the outcome of the experiment and to establish the amount of slip present: Several calculations were carried through to determine the time-dependence of the velocity of a descending sphere. Using sticking boundary conditions, it was shown that linear approximations lead to a common terminal velocity and that allowing for slip at the sphere's surface could increase this value by as much as 30%, when the fluid's shear stresses vanish at the surface. These included time-dependent effects in the hydrodynamic calculations. Nonlinear corrections giving a lower asymptotic value. A full and correct theoretical treatment should include the coupled time dependent and nonlinear effects; even if feasible, for the present purpose this seems unnecessary.

The lack of a comprehensive microscopic theory to connect the coefficients of slip and surface tension, as understood in this paper, suggests a simple dimensional argument to derive, at least, a formal functional relation. From the whole set of physical parameters that play a role in this problem, it can be shown that there are four independent dimensionless quantities: $\mathcal{M} = m/\rho a^3$ (a mass ratio), $l^2 = \sigma/\rho g a$ (a squared length ratio of the capillary length to the sphere's radius), $r = g a^3 \rho^2/\eta^2$ and ξ (the slip coefficient). In the present case the parameters m and r remain fixed, hence the only possible functional relation is $\xi = G(l)$. Where, for large values of the argument $G \rightarrow 0$ and for small values $G \rightarrow 1$. The explicit form can only be provided by the experiment and understood from a Statistical Mechanical based theory.

Aside from the main idea put forward in this paper, most of the analysis deals with the study of a descending sphere in a viscous fluid. Other experiments might be simpler to conceive and perform in order to prove the working hypothesis; the underlying theory to determine the amount of slip would probably be harder to develop. This is the main reason justifying the present analysis.

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Resumen. Bajo ciertas circunstancias, algunos fluidos fluyen de tal manera que la condición de frontera usual de adherencia, explicada por lo general en términos de la rugosidad de la superficie, debe ser relajada para permitir deslizamiento parcial. En este trabajo se argumenta que la tensión superficial, asociada a la región de contacto entre un fluido y un sólido, debe jugar un papel en la forma que debe tomar la condición de frontera hidrodinámica correspondiente; el efecto es incorporado a través del coeficiente de deslizamiento. Con base en el análisis detallado del movimiento de esferas en un fluido viscoso, se propone un experimento que demostraría la tesis del trabajo. Con objeto de poder interpretar el resultado del experimento propuesto, se dan los resultados teóricos necesarios.