An exact solution for a decaying symmetric vortex

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(Recibido el 22 de agosto de 1990; aceptado el 20 de septiembre de 1990)

Abstract. A set of closed analytical solutions of the Navier-Stokes equations are obtained for an incompressible, axisymmetrical, time-dependent flow, having only the azimuthal velocity component. The resulting linear diffusion-like equation is solved using standard methods. In spite of its extreme simplicity, it leads to some interesting possible flows that suggest several experiments. The solutions are natural extentions of the Lamb and Taylor vortices. Their main features and potential use are briefly discussed.

PACS: 47.10.+g

1. Introduction

The mathematical structure of the basic equations of fluid dynamics, the Navier-Stokes equations, has been a major difficulty towards a comprehensive understanding of the behavior of fluids. It is not a surprise that exact solutions of the equations are few and dearly appreciated [1]. Much of our present knowledge of fluid dynamics stems from them, and they are still a vital part of continuing research. The present work was motivated by the search of such solutions.

The relevance of vortex motion [2] in many fluid phenomena is evident by the vast literature that deals with idealized systems of vortices (2-dimensional and inviscid) [3]. These studies are due to the lack of precise results from the general theory of viscous fluids. Hence, modeling dynamics of vortices is a common procedure for attempting to understand available data and qualitative observations. Studying these flows was the second motivation.

Here, the problem of finding solutions that can be expressed in terms of well known functions is considered. The guiding idea is to make use of physical symmetries and similarity arguments [4] that lead directly to soluble or linear equations. Most of the well known solutions belong to this class and the rationale is to add a simple item. Computers are changing the difference between exact and numerical solutions; solutions should probably be referred to as being either simple or complex.

In this letter we address the former type of solutions of the Navier-Stokes equations. Actually, the following must be among the simplest of them all.

2. Formulation

We begin with the Navier-Stokes equations and assume that the viscosity (ν) and the density (ρ) are constant. Seeking solutions that can describe symmetrical vortices, cylindrical coordinates are used, and we address the simplest possible case: azimuthal flow. The fields are the pressure p (the factor $-1/\rho$ is included) and the velocity $\vec{u} = \hat{e}_\varphi v$. Other quantities are: vorticity $\vec{\omega}$, kinetic energy E, energy dissipation Φ, angular momentum $\mathbf{M}$ and circulation Γ. A variable appearing as a subindex implies a partial derivative (*e.g.* $v_r \equiv \partial v/\partial r$). In this letter we report on the special case where the only independent variables are r and t. As a consequence, the continuity equation is satisfied identically.

The Navier-Stokes equations are given explicitly by

$$-\frac{v^2}{r} = p_r, \tag{1}$$

$$v_t = \nu \left(\frac{(rv)_r}{r} \right)_r, \tag{2}$$

$$0 = p_z, \tag{3}$$

while the z-component of the vorticity is given by

$$\omega^{(z)} = \frac{(rv)_r}{r}. \tag{4}$$

Equations (1) and (3) define the pressure field. Equation (2) is a heat or diffusion type equation, and well known procedures can be used to find solutions.

3. Results

Dependence on a single variable gives rise to a very simple and widely used result; for r the solution is $v = ar + b/r$, which is either a solid body rotation (when $b = 0$) or the potential vortex (when $a = 0$); a combination of both with piecewise constants is the famous Rankine vortex [5].

For the two variables (r, t) things get a little more interesting, and the equation can be solved by a variety of alternative methods. The boundary conditions determine the solution. The general solution of Eq. (2) can, of course, be expressed in terms of the corresponding Green's function, the fundamental solution. The disadvantage of this solution is that for certain problems (particular classes of initial and boundary conditions) the integral is extremely difficult to evaluate without making use of numerical methods.

For r and t dependence we have:

a) A separable solution

$$v = \int d\mu \left(A(\mu) J_1 \left(\frac{r}{\mu} \right) + B(\mu) Y_1 \left(\frac{r}{\mu} \right) \right) \exp(-\mu^2 \nu t), \tag{7}$$

where J_1 and Y_1 are Bessel functions of order one of the first and second kinds [6]. A special case is when μ can only take discrete values, due to restrictions imposed by the boundary conditions, and the integral is replaced by an infinite sum. For some problems, like fitting a prescribed velocity profile, this solution is far more easy to handle than that expressed in terms of a Green's function.

This solution describes a decaying vortex. The coefficients A and B must be such that the solution is regular everywhere. Hence, B must vanish around the origin, as Y_1 diverges when $r \to 0$; within some suitably defined core we set $B = 0$. It is a decaying Rankine-like vortex.

b) A similarity solution

A very general solution of this type can be obtained by analyzing the group of invariant transformations of the equation [4,7]. A very simple case, that illustrates the essentials, is provided by scale transformations; put $r' = r\exp(\epsilon_1)$, $t' = t\exp(\epsilon_2)$ and $v' = v\exp(\epsilon_3)$ into the original equation and force the stretching constants ϵ_1, ϵ_2 and ϵ_3 to leave the equation unchanged. In the present case the result is $\epsilon_1 = 2\epsilon_2$ and arbitrary ϵ_3. This shows that the combination r^2/t is invariant under these scaling transformations: we shall denote this similarity variable by $s(= r^2/4\nu t)$ in what follows. The factors 4 and ν in the denominator are introduced for convenience and to make the variable dimensionless. The number of free parameters (ϵ) characterizes the type of group to which these transformations belong; in this instance, to a biparametric Lie group [4,7].

Assuming that $v = r^\alpha t^\beta F(r^2/4\nu t)$ we find an ordinary differential equation for F; the exponents α and β will help to cast the resulting equation into one whose solutions are well known. Furthermore, if one lets $F(s) = f(s)\exp(-s)$, to extract the controlling factor, one finds the following equation for f,

$$4s^2 f'' + 4(1 + \alpha - s)sf' + (\alpha^2 - 1 - 4(\alpha + \beta + 1)s)f = 0,$$

with arbitrary α and β. This symmetry can easily be exploited. For example, if $\alpha = 1$ and $\beta = -(n+1)$ $(n = 1, 2, \ldots)$, then the $f's$ are the associated Laguerre polynomials [6], L_n^1 (here, Sneddon's convention has been used). If $\beta = 0$ the solution that vanishes at the origin is given by $f_0 = s^{-1}(e^s - 1)$. Since the solutions can be

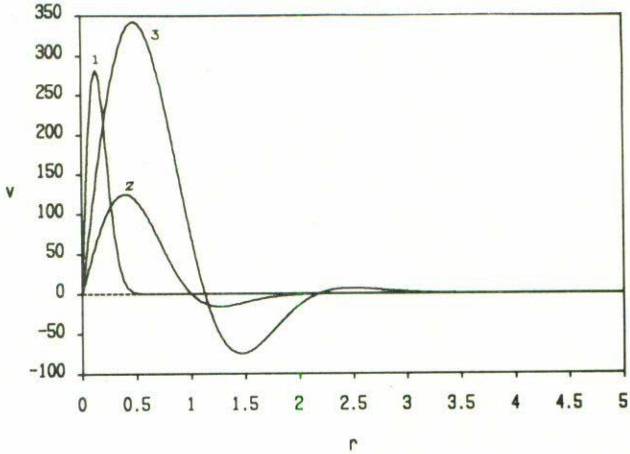

FIGURE 1. Azimuthal velocity field (v) as a function of r, for fixed time; labels 1, 2 and 3 correspond to m in Eq. (8). The times in each case are $4\nu t = 1/30$, $1/2$ and 1. The first is Taylor's result. Note the changes in sign of the velocity for $n > 1$.

superposed we have

$$v = \frac{r}{t}\exp(-s)\sum_{m=0}^{\infty} c_m(4\nu t)^{-m} f_m(s), \tag{8}$$

where the c_m's are arbitrary constants, and $f_n(s) = L_n^1(s)$, $n \geq 1$. The first term is the Lamb vortex [8] ($c_0 = \Gamma/2\pi$, Γ being the circulation),

$$v = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right).$$

The second term corresponds to the Taylor vortex [9] (c_1 is proportional to the total angular momentum),

$$v = c_1 r t^{-2} e^{-r^2/4\nu t}.$$

It appears that the remaining terms have not been considered previously and, in some ways, show a similar behavior to the Lamb and Taylor expressions; their amplitudes remain bounded throughout the fluid domain at all times. The third term, for example, is

$$v = A_3 r t^{-3}\left(1 - \frac{r^2}{8\nu t}\right) e^{-r^2/4\nu t}.$$

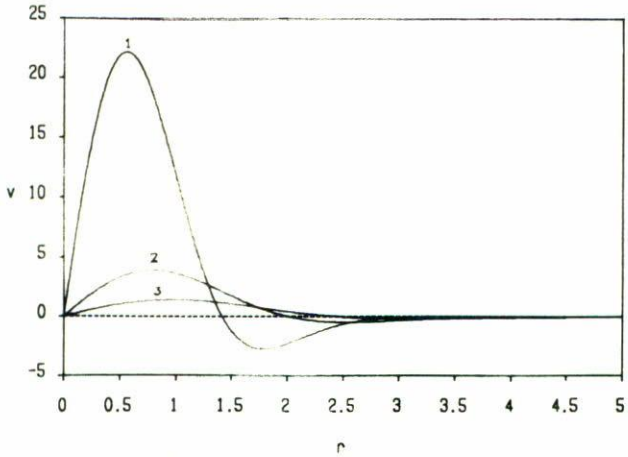

FIGURE 2. Azimuthal velocity field (v) as a function of r, for different times ($4\nu t = 1, 2, 3$; labels correspond to these times), as given by the $m = 2$ term in Eq. (8). Initially the vortex is concentrated at the origin and with time it decays and spreads out.

In Figure 1 the flows described by $m = 1, 2, 3$ in Eq. (8) are compared, as a function of r, at a fixed time. In Figure 2 the time evolution of the third term is shown.

The n-th term ($n > 0$) can also be expressed as

$$v_n = \frac{A}{r} e^{-s} (L_n(s) - n L_{n-1}(s))(4\nu t)^{-n}, \tag{9}$$

where L_n is the Laguerre polynomial of order n and the constants A will depend on n and clearly have different physical interpretations (as well as dimensions). All the relevant quantities to describe the flow can be explicitly calculated. From Eqs. (4) and (8) one finds for the vorticity

$$\omega^{(z)} = -2A(4\nu t)^{-(n+1)} e^{-s} L_n(s),$$

In Figures 3 and 4 the corresponding comparison is shown for the vorticity.

The total angular momentum is

$$\mathbf{M} = \int \mathbf{r} \times \rho \mathbf{u}\, dV = \hat{\mathbf{k}}\, 2\pi\rho \int_0^\infty v r^2\, dr = -\hat{\mathbf{k}}\, 4\pi\rho\nu c_1 \delta_{n1},$$

being different from zero only for the Taylor vortex ($n = 1$) and infinite for the Lamb vortex. All quantities have been calculated per unit length in the z direction. The zeros of each term of the velocity field, in Eq. (9), are given by the roots of

$$L_n(x) - n L_{n-1}(x) = 0;$$

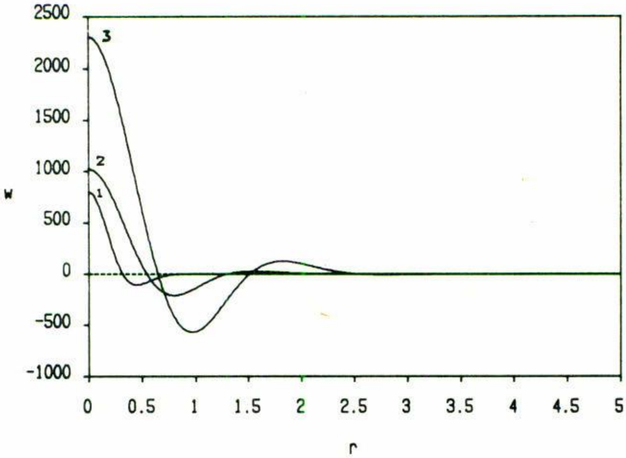

FIGURE 3. Vorticity (ω) as a function of r. Same cases as in Fig. 1, except that for $m = 1$ the time is $4\nu t = 1/10$.

the n-th term having $n - 1$ zeros, while the corresponding maxima and minima follow from the roots of

$$(2x + 1)L_n(x) - nL_{n-1}(x) = 0.$$

For the pressure, the kinetic energy and the dissipation, we have

$$p_n(s) = p_0 + A^2(2n^2(4\nu t)^{2n+1})^{-1} \int_0^s \left(\frac{d}{dx}L_n(x)\right)^2 e^{-2x}\, dx,$$

$$E_n(s) = \frac{1}{2}\pi\rho A^2(n(4\nu t)^n)^{-1} \int_0^s x \left(\frac{d}{dx}L_n(x)\right)^2 e^{-2x}\, dx,$$

$$\Phi_n = -\pi\rho n! A^2((4\nu t)^{2n+1})^{-1} \int_0^\infty x \left(\frac{d}{dx}L_n(x)\right)^2 e^{-2x}\, dx,$$

all of which remain finite as $s \to \infty$. When $t \to \infty$ the pressure approaches a constant while the energy and the dissipation vanish.

4. Discussion and summary

The solutions that have been described, Eqs. (7) and (8), have some features that are worth pointing out. First, they allow the possibility of fitting a given velocity profile or particular boundary and initial conditions, in contrast to the fundamental solution, and can therefore be used to model particular vortex flows. Second, the

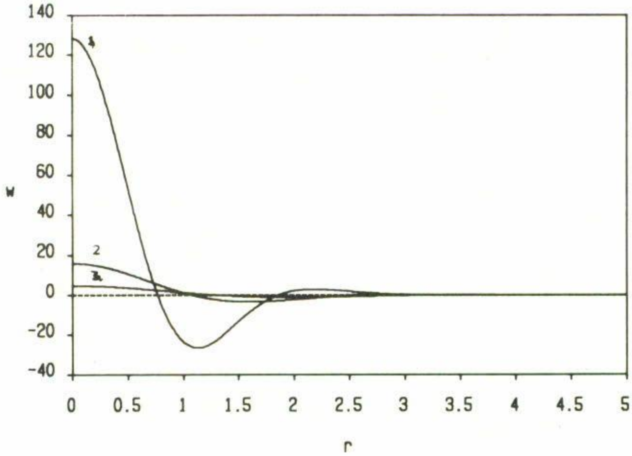

FIGURE 4. Vorticity (ω) as a function of r, for different times. Same cases as in Figure 2.

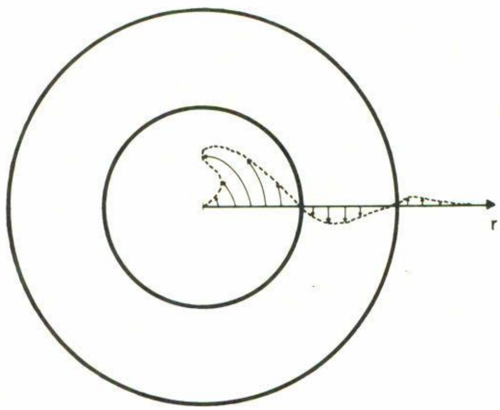

FIGURE 5. Diagram in the x–y plane of the flow described by v_3 from Eq. (9); $4\nu t = 1$ and $A = 1$.

solution in terms of the associated Laguerre polynomials puts the Lamb and Taylor vortex flows in their appropriate context; as the first two terms in a class of scale invariant solutions. The connection between both solutions should be explored, as the relation between separable solutions and symmetries is not yet well understood.

An interesting feature of the results summarized in Eq. (8), is that for $n > 1$ they describe the decay of $n-1$ counter-rotating cylindrical shells of fluid. In Figure 1 the $n = 1, 2$ and 3 modes, as given by Eq.(8), are shown; the time corresponding to each curve is different to make easier a visual comparison. Figure 2 illustrates the time decay and space "diffusion" of a typical flow ($n = 2$ term). Analogous plots for the vorticity follow in Figures 3 and 4. In Figure 5 a sketch shows the flow pattern described by the $n = 3$ term. Within the first circle (cylinder) the fluid turns around the axis in the positive φ direction; inside the next cylindrical shell the fluid rotates in the opposite direction and further out the swirl is again positive; the flow decays exponentially with the radial coordinate. All this at some fixed time. As time lapses, the radius of each rotating region grows, spreading out the influence of the vortex, while the magnitude of the azimuthal flow decreases. The energy is initially concentrated at the origin and it dissipates (decreases monotonically) in time while it diffuses out in space.

Perhaps more relevant, the superposition of the different terms can be made to fit an arbitrary azimuthal velocity profile by properly adjusting the coefficients; the solution can therefore be used to model particular time- dependent symmetrical vortices. The complicated azimuthal velocity profile of a hurricane or a tornado [2,10], in a given horizontal plane, can be modeled by an expression like that of Eq. (8).

Here we report the initial part [11] of an extensive exploration of the separable and similarity solutions [12] of the Navier-Stokes equations. This was done here for the case of cylindrical symmetry, pure azimuthal flow and two independent variables, r and t; it is one of over 40 possible cases. Yet, these simple results seem worth showing, as they can provide a way to model real vortex flows.

Acknowledgements

The authors acknowledge support from UNAM-DGAPA Grant IN-10-28-89, from IBM de México with a PS/50 and from State of Texas Advanced Project Grant 2002-ARP.

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Resumen. Se obtiene un grupo de soluciones cerradas exactas de las ecuaciones de Navier-Stokes. Se consideran únicamente flujos dependientes del tiempo, incompresibles y con simetría axial, en los que el campo de velocidades es puramente azimutal. La ecuación (de difusión) lineal resultante se resuelve usando métodos usuales. A pesar de la simplicidad del problema se obtienen flujos interesantes que sugieren diversos experimentos. Las ecuaciones son la extensión natural de las soluciones de Lamb y de Taylor. Se discuten sus características más relevantes y sus posibles aplicaciones.