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An entropy production approach to the Curzon and Ahlborn cycle

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Abstract. We study the entropy production surface of the Curzon and Ahlborn cycle under usual restrictions: no friction, endoreversible and free of inertial effects. We find that the Curzon and Ahlborn formula for the efficiency, $\eta = 1 - \sqrt{\frac{T_2}{T_1}}$ ($T_2 < T_1$, reservoir temperatures) is not exclusive for the maximum power regime. We obtain this formula by using the mean value theorem for derivatives and also by means of a Legendre transformation applied over a hyperbola associated to the directional maxima of the entropy production surface.

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1. Introduction

A great deal of attention has received the Curzon and Ahlborn (C–A) paper [1], where they shown that a Carnot-type cycle with heat transfer in the isothermal branches given by Newton's Cooling Law at maximum power output performance, has an efficiency given by $\eta_{\text{CA}} = 1 - \sqrt{\frac{T_2}{T_1}}$, where T_2 and T_1 are the temperatures of cold and hot reservoirs respectively. From this work several authors [1–9] have studied the optimal configuration of operation for heat engines at finite time, using some criteria of merit such as maximum power, maximum efficiency and effectiveness, minimum entropy production and others. The C-A efficiency formula has been obtained by different maximum power analyses [2–6]. Recently, Torres [7] published a work about the CA cycle analyzed by means of minimal rate of entropy production as a criterion of merit for the best mode of operation of such a cycle. He found the expected result that entropy production per cycle $\frac{\Delta S}{\tau}$(τ, cycle period) has a minimum for $X = Y = 0$; *i.e.* in the quasi-static limit, where X and Y are as in the C-A paper, the differences of temperature between heat reservoirs and the working substance: $X = T_1 - T_{1W}$ and $Y = T_{2W} - T_2$, with T_{1W} and T_{2W} the corresponding temperatures of the superior and inferior isothermal branches (see Fig. 1). Then, it is clear that by a minimal entropy production criterion the C-A efficiency can not be recuperated.

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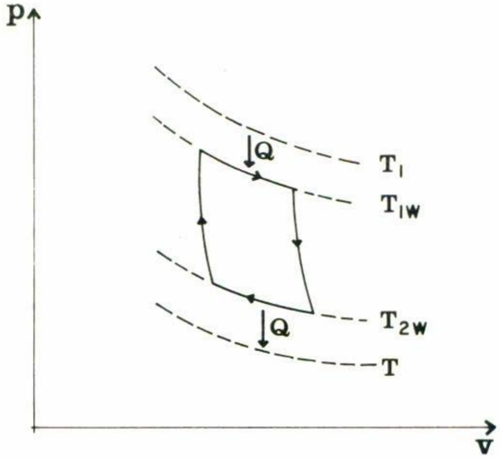

FIGURE 1. Pressure-volume diagram for Curzon and Ahlborn cycle.

In this work we study the C-A cycle under the usual restrictions [2], [6], [8]; *i.e.* without friction, the working fluid in internal equilibrium (endoreversibility) and free of mechanical inertial effects. We propose an alternative analysis of the entropy production function through its directional derivatives along certain monoparametric-straight-line family with negative slope in X–Y plane. We find that it exists an hyperbola in the X–Y plane which is the locus of all pairs (X–Y) where the entropy production surface has directional maxima along the mentioned family of straight lines. The C-A point, corresponding to maximum power belongs to that curve and it is found by using the mean value theorem for derivatives [10] and also by means of the Legendre transformation [11] for such an hyperbola. So, the C-A formula for efficiency is reached by an unusual entropy production procedure. We show also that the C-A efficiency formula is not exclusive for the maximum power regime.

2. Entropy production surface

We consider the C-A cycle (Fig. 1), with heat transfer in the isothermal branches given by Newton's Cooling Law

$$\begin{aligned} \frac{dQ_1}{dt} &= \alpha(T_1 - T_{2W}) = \alpha X, \\ \frac{dQ_2}{dt} &= \beta(T_{2W} - T_2) = \beta Y, \end{aligned} \tag{1}$$

where α and β are constants depending on the thickness and thermal conductivities of the walls between reservoirs and working substance and t is time. Due to irreversibilities which are present in the coupling between heat reservoirs and working fluid S_u, according to the second Law of thermodynamics, must be a positive definite quantity,

$$S_u = \frac{dS_u}{dt} \geq 0, \tag{2}$$

S_u is the entropy of the thermodynamical universe *i.e.* heat reservoirs plus working substance. Because the working substance undergoes an endoreversible cycle, the entropy production per cycle for a pair (X, Y) is [7]

$$\sigma_u \doteq \frac{\Delta S_u}{\tau} = \frac{\alpha\beta}{T_1 T_2} \frac{X^2 Y T_2 + X Y^2 T_1}{\alpha T_2 X + \beta T_1 Y + (\alpha - \beta) X Y}, \tag{3}$$

this equation is obtained considering only the contribution in σ_u due to entropy production of heat reservoirs, because of the change in entropy of the working substance, ΔS_w is null and its entropy production per cycle is also null. Then we have

$$\frac{Q_1}{T_{1W}} - \frac{Q_2}{T_{2W}} = 0. \tag{4}$$

Thus, the entropy production per cycle for heat reservoirs plus working substance is,

$$\frac{\Delta S_u}{\tau} = \frac{1}{\tau}\left[-\frac{Q_1}{T_1} + \frac{Q_2}{T_2}\right] \tag{5}$$

From Eqs. (4) and (5) and using for one cycle period,

$$\tau = t_1 + t_2 = \frac{Q_1}{\alpha X} + \frac{Q_2}{\beta Y}, \tag{6}$$

we obtain Eq. (3). Here, as it is usual, the adiabatic branches are considered instantaneous [2], [8].

The surface corresponding to function σ_u (Eq. 3) is an increasing surface for increasing X, Y in the same domain $\mathcal{D} = \{X, Y \mid X \in [0, T_1 - T_2], Y = T_1 - T_2 - X\}$ in which the power of the C-A cycle P, is non negative. The expression for P in terms of X, Y and for instantaneous adiabatic branches is [1]

$$P = \frac{\alpha\beta X Y (T_1 - T_2 - X - Y)}{\alpha T_2 X + \beta T_1 Y + (\alpha - \beta) X Y} \tag{7}$$

From Eq. (7) we see that domain $\mathcal{D}$ for $P \geq 0$ is as we indicated in the previous paragraph (see Fig. 5). Power function P (see Fig. 2) corresponds to a convex surface with a single maximum in $\mathcal{D}$, and σ_u is a surface which has zero values along the

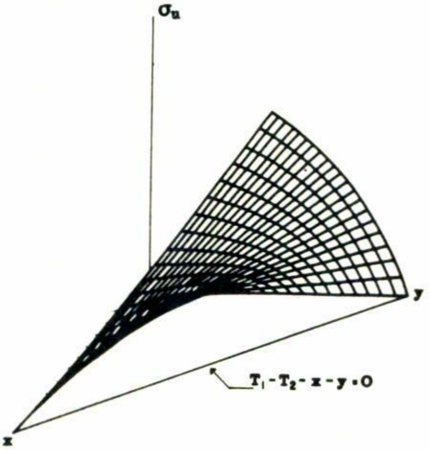

FIGURE 2. Power output surface for C-A cycle with $T_1 = 1000$ K, $T_2 = 500$ K and $\alpha = \beta = 100$ J/K-s.

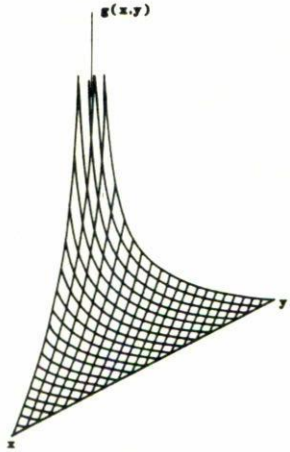

FIGURE 3. Total entropy production surface with $T_1 = 1000$ K, $T_2 = 500$ K and $\alpha = 100$ J/K-s.

X, Y axes (see Fig. 3). From the shape of σ_u, we observe that along straight lines with negative slope in domain $\mathcal{D}$, it has directional maxima.

From Eqs. (3) and (7), it follows that

$$P(X,Y) = g(X,Y)\sigma_u(X,Y) \tag{8}$$

where

$$g(X,Y) = T_1 T_2 \frac{T_1 - T_2 - X - Y}{T_2 X + T_{1Y}}, \tag{9}$$

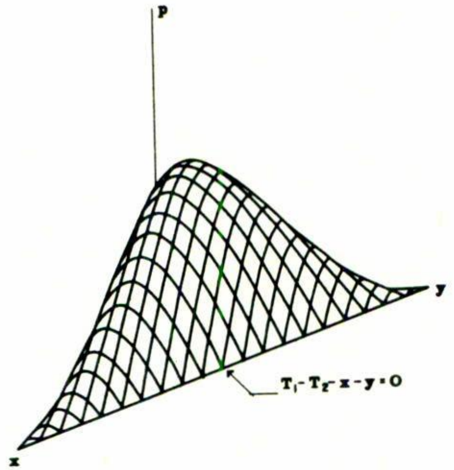

FIGURE 4. $g(x,y)$ surface with $T_1 = 1000$ K and $T_2 = 500$ K.

The graph of function $g(X,Y)$ in domain $\mathcal{D}$ is shown in Fig. 4. We know [1] that $P(X,Y)$ has a single maximum in

$$X^* = T_1 \left[\frac{1 - \sqrt{\frac{T_2}{T_1}}}{1 + \sqrt{\frac{\alpha}{\beta}}} \right], \tag{10}$$

$$Y^* = T_2 \left[\frac{\sqrt{\frac{T_1}{T_2}} - 1}{1 + \sqrt{\frac{\beta}{\alpha}}} \right], \tag{11}$$

and consequently it follows using Eq. (8) that in the point (X^*, Y^*),

$$\begin{aligned} \left(\frac{\partial \sigma_u}{\partial X}\right)_{Y^*} &= -\frac{\sigma_u}{g}\left(\frac{\partial g}{\partial X}\right)_{Y^*}, \\ \left(\frac{\partial \sigma_u}{\partial Y}\right)_{X^*} &= -\frac{\sigma_u}{g}\left(\frac{\partial g}{\partial Y}\right)_{X^*}. \end{aligned} \tag{12}$$

Due to the shape of the surface σ_u (Fig. 3) the directional maxima are such that,

$$\frac{d\sigma_u}{d\lambda} = \left(\frac{\partial \sigma_u}{\partial X}\right)_{Y^*} \frac{dX}{d\lambda} + \left(\frac{\partial \sigma_u}{\partial Y}\right)_{X^*} \frac{dY}{dX}\frac{dX}{d\lambda} = 0, \tag{13}$$

where $d\lambda$ is the magnitude of $d\boldsymbol{\lambda}$, which is the infinitesimal displacement in the

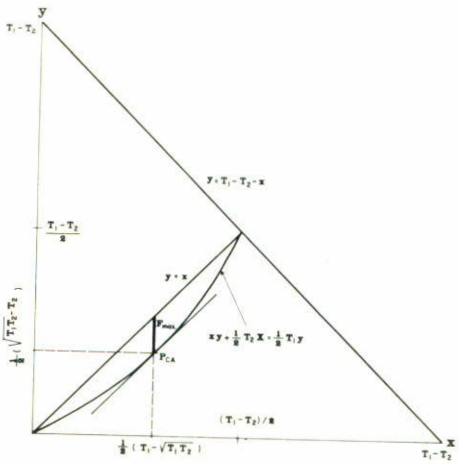

FIGURE 5. Hyperbola corresponding to Eq. (22) in domain $\mathcal{D}$.

considered direction. From Eq. (13), we obtain

$$\frac{dY}{dX} = -\frac{\left(\dfrac{\partial g}{\partial X}\right)_{Y^*}}{\left(\dfrac{\partial g}{\partial Y}\right)_{X^*}}. \tag{14}$$

After substitution of Eq. (9) in Eq. (14) we get,

$$\frac{dY}{dX} = -\frac{T_{2W}}{T_{1W}{}^*} = -\frac{Y + T_2}{T_1 - X} = -m^* \tag{15}$$

which gives us the slope $-m^*$ of the straight line which is parallel to the directional derivate in (X^*, Y^*). From Eq. (15) we obtain

$$Y = m^* T_1 - T_2 - m^* X, \tag{16a}$$

which is a particular case of the monoparametric-straight-line family given by,

$$Y = mT_1 - T_2 - mX \tag{16b}$$

with $m = \frac{T_{2W}}{T_{1W}}$. As we shall see below, along the straight-line family given by Eq. (16*b*), the σ_u surface has directional maxima. If we use Eq. (16*b*) in the expresion for endoreversible efficiency [1],

$$\eta = 1 - \frac{T_2 + Y}{T_1 - X}.$$

We are lead to the result,

$$\eta = 1 - m. \tag{17}$$

Eq. (17) means that efficiency for engine configurations (X, Y) which belong to straight line (16*b*) have all of them the same efficiency $\eta(m)$. Thus, the C-A formula for efficiency at maximum power regime is not exclusive for the point (X^*, Y^*). All points (X, Y) on the straight line (for $\alpha = \beta$),

$$Y = \sqrt{T_1 T_2} - T_2 - \sqrt{\frac{T_2}{T_1}} X \tag{18}$$

correspond to cycle configurations with $\eta = 1 - \sqrt{\frac{T_2}{T_1}}$.

Straight lines related with Eqs. (16*b*) and (17) are displayed, in other context, in Fig. 1 of Ref. [9].

3. C-A point and entropy production surface

In this section we find the C-A point of maximum power output by an entropy production surface analysis. In order to manipulate algebraically Eq. (3), for simplicity we take the case $\alpha = \beta$, as an example of the procedure. For this case

$$\sigma_u = \frac{\alpha}{T_1 T_2} XY, \tag{19}$$

this function has directional maxima along straight lines given by Eq. (16*b*). By substitution of Eq. (16*b*) in Eq. (19), we obtain

$$\frac{T_1 T_2}{\alpha} \sigma_u = (mT_1 - T_2)X - mX^2 \tag{20}$$

The point (X^*, Y^*) where Eq. (20) attains its directional maximum is

$$X^* = \frac{1}{2} T_1 \left[1 - \frac{1}{m} \frac{T_2}{T_1} \right], \qquad Y^* = \frac{1}{2} T_2 \left[m \frac{T_1}{T_2} - 1 \right], \tag{21}$$

and by elimination of m, it follows that

$$X^* Y^* + \tfrac{1}{2} T_2 X^* = \tfrac{1}{2} T_1 Y^*. \tag{22}$$

Eq. (22) corresponds to an hyperbola (see Fig. 5), which is the locus of all points

(X, Y) in domain $\mathcal{D}$, which maximize entropy production along straight lines given by Eq. (16b), as can be seen from Fig. 5, this curve intersect the domain $\mathcal{D}$ boundaries at the origin and at the point $\left(\frac{T_1-T_2}{2}, \frac{T_1-T_2}{2}\right)$.

If we apply the mean value theorem for derivatives [10] to the function $Y^*(X^*)$ obtained from Eq. (22) and given by

$$Y^*(X^*) = \frac{T_2 X^*}{T_1 - 2X^*} \tag{23}$$

we obtain

$$\frac{dY(X^*)}{dX} = \frac{\left(\frac{T_1 - T_2}{2}\right) - Y(0)}{\frac{T_1 - T_2}{2} - 0} = 1. \tag{24}$$

Taking the derivative of $Y^*(X^*)$ and substituting in Eq. (24), it follows that

$$\frac{T_1 T_2}{(T_1 - 2X^*)^2} = 1, \tag{25}$$

From here, we obtain

$$X^* = \frac{1}{2}\left(T_1 - \sqrt{T_1 T_2}\right), \tag{26}$$

and substituting this expression in Eq. (23), we get

$$Y^* = \frac{1}{2}\left(\sqrt{T_1 T_2} - T_2\right). \tag{27}$$

Eqs. (26) and (27) give us the C-A point for $\alpha = \beta$ [see Eqs. (10) and (11)]. An alternative way to find the C-A point from the hyperbola given by Eq. (22) is by using the Legendre transformation [11] of that curve. The Legendre transformation F is obtained by the vertical distance between the straight line $Y = X$ and the curve $Y = \frac{T_2 X}{T_1 - 2Y}$. It is given by

$$F = X - \frac{T_2 X}{T_1 - 2X}. \tag{28}$$

Distance $F(X)$ necessarily has a maximum in the interval $\left[0, \frac{T_1-T_2}{2}\right]$; *i.e.*

$$\frac{dF}{dX} = 1 - \frac{T_1 T_2}{(T_1 - 2X)^2} = 0 \tag{29}$$

Eq. (29) is the same that Eq. (25) and consequently, we obtain the same pair (X^*, Y^*) given by Eqs. (26) and (27). So, the point where the tangent at the curve $Y = Y(X)$ is parallel to straight line $Y = X$ is also the point where the vertical distance between the straight line and the curve is maximum. In this way, we have shown that the C-A point has a very special place in the entropy production surface and corresponds to a directional maximum in that surface.

For the case $\alpha \neq \beta$ the algebra is more complicated but the equivalent results are obtained using the equation for the hyperbola given by

$$\left[2 + \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}}\right] XY + \left[1 + \sqrt{\frac{\alpha}{\beta}}\right] T_2 X = \left[1 + \sqrt{\frac{\beta}{\alpha}}\right] T_1 Y, \qquad (30)$$

which reduces to Eq. (22) for the case $\alpha = \beta$. Eq. (30) follows by using the fact that the C-A point [Eqs. (10) and (11)] belongs to the hyperbola. This procedure seems to be a tautology, but it is not. Our aim is to show that the C-A point has a conspicuous place in this curve and in the entropy production surface also.

4. Conclusions

Until now only minimal entropy production approaches [2], [7], [8] have been applied to the C-A endoreversible cycle. In our work, we make a different entropy production approach to such a cycle. We find a monoparametric-straight-line family along which σ_u has directional maxima. The efficiency results a constant for pairs (X, Y) belonging to each of these lines. Thus, the famous formula $\eta_{CA} = 1 - \sqrt{\frac{T_2}{T_1}}$, is not exclusive for maximum power regime. We show that the C-A point (X^*, Y^*) has a conspicuous place in the σ_u surface and we meet it by using both the mean value theorem and a Legendre transformation over a hyperbola obtained from directional maxima of σ_u. The mean value theorem for derivatives and the maximum of the Legendre transformation result equivalent for convex curves.

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Resumen. Se estudia la superficie de producción de entropía del ciclo de Curzon y Ahlborn bajo restricciones usuales: sin fricción, endorreversible y libre de efectos inerciales. Se encuentra que la fórmula de Curzon y Ahlborn para la eficiencia, $\eta = 1 - \sqrt{\frac{T_2}{T_1}}$, no es exclusiva del régimen de potencia máxima. Se obtiene esa fórmula usando, tanto el teorema del valor medio para derivadas como una transformación de Legendre, aplicados sobre una hipérbola asociada a los máximos direccionales de la superficie de producción de entropía.