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Lanczos potential for algebraically special space-times

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Abstract. An explicit expression for the Lanczos potential, which generates the conformal curvature tensor, is obtained for all the space-times that admit a shear-free congruence of null geodesics, provided that the Ricci tensor is suitably restricted. The most general local expression of the Lanczos potential of a conformally flat space-time is also obtained.

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1. Introduction

The Lanczos potential of a given space-time is a tensor field, $H_{\mu\nu\rho}$, that leads, by covariant differentiation, to the conformal curvature tensor [1,2]. The Lanczos potential corresponding to a given metric has to satisfy a set of coupled equations whose integration seems to be a highly involved task, especially when the equations are written in tensor form.

The aim of this paper is to give an explicit expression for the Lanczos potential of any space-time that admits a shear-free congruence of null geodesics and whose Ricci tensor is suitably aligned to this congruence. This class of space-times include all the algebraically special solutions of the Einstein vacuum field equations. The most general local expression of the Lanczos potential corresponding to a conformally flat space-time is also given. The expressions for the Lanczos potentials are obtained here by considering the complex extension of the space-time and using the spinor formalism; this approach has been developed in the study of exact solutions of the Einstein field equations [3–6] and has been applied to the integration of the equations governing various fields (see, for example, Refs. [4–8] and the references cited therein).

In Sect. 2 the equations satisfied by the Lanczos potential are written in spinor form and, in Sect. 3, a solution to these equations is given for any algebraically special space-time belonging to the class mentioned above. In Sect. 4, the local expression for all the Lanczos potentials corresponding to conformally flat space-times is obtained, without imposing explicit restrictions on the Ricci tensor. Throughout this article, all the spinor indices are raised and lowered according to the conventions $\psi_A = \epsilon_{AB}\psi^B$, $\psi^B = \psi_A\epsilon^{AB}$, and similarly for dotted indices.

2. Spinor form of the Lanczos potential

The tensor field $H_{\mu\nu\rho}$ is a Lanczos potential of a given space-time if the conformal curvature tensor can be expressed in the form [1,2]

$$\begin{aligned} C_{\mu\nu\rho\sigma} = {} & H_{\mu\nu\rho;\sigma} - H_{\mu\nu\sigma;\rho} + H_{\rho\sigma\mu;\nu} - H_{\rho\sigma\nu;\mu} \\ & + H_{(\nu\rho)}g_{\mu\sigma} - H_{(\mu\rho)}g_{\nu\sigma} + H_{(\mu\sigma)}g_{\nu\rho} - H_{(\nu\sigma)}g_{\mu\rho} \end{aligned} \tag{1}$$

where

$$H_{\mu\nu} \equiv H_{\mu}{}^{\rho}{}_{\nu;\rho} \tag{2}$$

and the parentheses denote symmetrization on the indices enclosed (e.g., $H_{(\mu\nu)} = \frac{1}{2}(H_{\mu\nu} + H_{\nu\mu})$). The Lanczos potential also satisfies the algebraic conditions

$$H_{\mu\nu\rho} = -H_{\nu\mu\rho} \tag{3a}$$

$$H_{\mu\nu}{}^{\nu} = 0 \tag{3b}$$

$$H_{\mu\nu\rho} + H_{\nu\rho\mu} + H_{\rho\mu\nu} = 0. \tag{3c}$$

If the further condition

$$H_{\mu\nu}{}^{\rho}{}_{;\rho} = 0 \tag{4}$$

is imposed [1], then as a consequence of Eqs. (3*a*) and (3*c*), $H_{\mu\nu}$ is symmetric.

The antisymmetry of $H_{\mu\nu\rho}$ in the first two indices [Eq. (3*a*)] implies that the spinor equivalent of $H_{\mu\nu\rho}$ has the form $H_{ABC\dot{C}}\epsilon_{\dot{A}\dot{B}} + H_{\dot{A}\dot{B}\dot{C}C}\epsilon_{AB}$, where $H_{ABC\dot{C}}$ is symmetric in the first two indices and $H_{\dot{A}\dot{B}\dot{C}C} = \overline{H_{ABC\dot{C}}}$ if $H_{\mu\nu\rho}$ is real. (This expression is analogous to that of the electromagnetic field tensor in spinor form.) Substituting into Eqs. (3*b*–*c*), using the identity

$$\psi_{AC\cdots} - \psi_{CA\cdots} = \psi^{R}{}_{R\cdots}\epsilon_{AC}, \tag{5}$$

one finds that Eqs. (3*b*–*c*) are equivalent to the relations

$$H_{ABC\dot{C}} = H_{(ABC)\dot{C}}, \tag{6}$$

which means that $H_{ABC\dot{C}}$ is totally symmetric in the undotted indices. Hence, $H_{ABC\dot{C}}$ has eight (complex) independent components and therefore, if $H_{\mu\nu\rho}$ is real, Eqs. (3) imply that $H_{\mu\nu\rho}$ has 16 real independent components, which is not so easily obtained using the tensor formalism.

Using repeatedly the identity (5) one finds that Eq. (1) is equivalent to the simple relation

$$C_{ABCD} = \nabla_{\dot{R}(A} H_{BCD)}{}^{\dot{R}}, \tag{7}$$

where C_{ABCD} is the Weyl spinor. For a given metric the Lanczos potential is not unique: by adding to $H_{ABC\dot{C}}$ any symmetric spinor field $J_{ABC\dot{C}}$ such that $\nabla_{\dot{R}(A} J_{BCD)}{}^{\dot{R}} = 0$, one obtains another Lanczos potential for the same metric. However, this does not mean that $J_{ABC\dot{C}}$ is necessarily a Lanczos potential for a conformally flat space-time, since the covariant derivatives appearing in these equations correspond to the original metric, which may not be conformally flat (compare Ref. [9]).

3. Lanczos potential for algebraically special space-times

Following Refs. [3,4], in Ref. [5] it was shown that the metric of a space-time that admits a shear-free congruence of null geodesics, defined by a spinor field ℓ_A, can be written locally in the form

$$ds^2 = 2\phi^{-2} dq^{\dot{A}} \left(dp_{\dot{A}} + Q_{\dot{A}\dot{B}} dq^{\dot{B}} \right) \tag{8}$$

where ϕ is a (complex) function defined by

$$\ell^A \nabla_{B\dot{C}} \ell_A = \ell_B \ell^A \partial_{A\dot{C}} \ln \phi, \tag{9}$$

$q^{\dot{A}}$ and $p^{\dot{A}}$ are complex coordinates and $Q_{\dot{A}\dot{B}} = Q_{\dot{B}\dot{A}}$ are some functions, provided that the trace-free part of the Ricci tensor satisfies

$$\ell^A \ell^B C_{AB\dot{C}\dot{D}} = 0. \tag{10}$$

Under these conditions, ℓ_A is a multiple principal spinor of C_{ABCD} which is, therefore, algebraically special [10]. (In terms of the Newman-Penrose notation, taking ℓ_A as an element of the spin-frame, Eqs. (9) and (10) amount to $\kappa = \sigma = 0$, $\rho = D \ln \phi$, $\tau = \delta \ln \phi$ and $\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$, respectively and, therefore, $\Psi_0 = \Psi_1 = 0$.)

Using the components of the connection relative to the null tetrad

$$\partial_{1\dot{A}} = \sqrt{2} \frac{\partial}{\partial p^{\dot{A}}}, \qquad \partial_{2\dot{A}} = \sqrt{2} \phi^2 \left(\frac{\partial}{\partial q^{\dot{A}}} - Q_{\dot{A}}{}^{\dot{B}} \frac{\partial}{\partial p^{\dot{B}}} \right), \tag{11}$$

given in Ref. [5], one finds that

$$\begin{aligned}\nabla_{R(\dot{A}}H_{\dot{B}\dot{C}\dot{D})}{}^{R} = \sqrt{2}\phi \Big\{ D_{(\dot{A}}\phi H_{\dot{B}\dot{C}\dot{D})1} - 3\left(\partial^{\dot{R}} Q_{(\dot{A}\dot{B}}\right) \phi H_{\dot{C}\dot{D})\dot{R}1} \\ + 2\left(\partial^{\dot{R}} Q_{\dot{R}(\dot{A}}\right) \phi H_{\dot{B}\dot{C}\dot{D})1}\Big\} - \sqrt{2}\phi\partial_{(\dot{A}}\phi^{-1} H_{\dot{B}\dot{C}\dot{D})2}\end{aligned} \quad (12)$$

where

$$\partial_{\dot{A}} \equiv \frac{\partial}{\partial p^{\dot{A}}}, \qquad D_{\dot{A}} \equiv \frac{\partial}{\partial q^{\dot{A}}} + Q_{\dot{A}\dot{B}}\partial^{\dot{B}}. \quad (13)$$

On the other hand, in terms of the tetrad (11), the components of $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ corresponding to the metric (8) are given by [4,5]

$$C_{\dot{A}\dot{B}\dot{C}\dot{D}} = -\phi^2 \partial_{(\dot{A}}\partial_{\dot{B}} Q_{\dot{C}\dot{D})} \quad (14)$$

and Eq. (10) amounts to

$$C_{11\dot{A}\dot{B}} = -\phi^{-1}\partial_{\dot{A}}\partial_{\dot{B}}\phi = 0. \quad (15)$$

Therefore, assuming that Eq. (15) holds, Eq. (14) can be rewritten as

$$C_{\dot{A}\dot{B}\dot{C}\dot{D}} = -\phi\partial_{(\dot{A}}\phi^2\partial_{\dot{B}}\phi^{-1}Q_{\dot{C}\dot{D})} \quad (16)$$

and comparing with Eq. (12) and (the complex conjugate of) Eq. (7) one sees that

$$H_{\dot{A}\dot{B}\dot{C}1} = 0, \qquad H_{\dot{A}\dot{B}\dot{C}2} = \frac{1}{\sqrt{2}}\phi^3\partial_{(\dot{A}}\phi^{-1}Q_{\dot{B}\dot{C})} \quad (17)$$

are the components relative to the tetrad (11) of a Lanczos potential for the dotted Weyl spinor $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$. This last expression shows that $Q_{\dot{A}\dot{B}}$ acts as a potential for the Lanczos potential.

Equation (17) can be written in a covariant form, valid in an arbitrary null tetrad, by noticing that [6]

$$\delta\Gamma_{\dot{A}\dot{B}} \equiv \Gamma_{\dot{A}\dot{B}} - {}^{\circ}\Gamma_{\dot{A}\dot{B}} = \phi\partial_{(\dot{A}}\phi^{-1}Q_{\dot{B}\dot{C})}\,dq^{\dot{C}}, \quad (18)$$

where $\Gamma_{\dot{A}\dot{B}}$ are connection one-forms for the tetrad (11) and ${}^{\circ}\Gamma_{\dot{A}\dot{B}}$ are connection one-forms for the tetrad

$${}^{\circ}\partial_{1\dot{A}} = \sqrt{2}\frac{\partial}{\partial p^{\dot{A}}}, \qquad {}^{\circ}\partial_{2\dot{A}} = \sqrt{2}\frac{\partial}{\partial q^{\dot{A}}}, \quad (19)$$

(obtained by setting $Q_{\dot{A}\dot{B}} = 0$ in Eq. (11)), which corresponds to the conformally flat metric

$$^{\circ}ds^2 = 2\phi^{-2}\, dq^{\dot{A}}\, dp_{\dot{A}}. \tag{20}$$

From Eqs. (11), (17) and (18) one concludes that

$$H_{\dot{A}\dot{B}\dot{C}R} = \frac{1}{2}\delta\Gamma_{(\dot{A}\dot{B}}\left(\partial_{\dot{C})R}\right) \tag{21}$$

or equivalently,

$$H_{\dot{A}\dot{B}\dot{C}R} = \frac{1}{2}\nabla^{S}{}_{(\dot{A}}\phi^{-2}\Omega_{\dot{B}\dot{C})}\ell_R\ell_S \tag{22}$$

(see Eq. (4.10) of Ref. [6]), where $\Omega_{\dot{A}\dot{B}}$ is a symmetric spinor field defined by the relation

$$ds^2 = {}^{\circ}ds^2 + \phi^{-2}\Omega_{\dot{A}\dot{B}}\ell_C\ell_D g^{C\dot{A}}g^{D\dot{B}}, \tag{23}$$

$^{\circ}ds^2$ is the conformally flat metric given in Eq. (20) and the $g^{A\dot{B}}$ are one-forms such that $g^{A\dot{B}}(\partial_{C\dot{D}}) = -2\delta^A_C\delta^{\dot{B}}_{\dot{D}}$. (The one-forms $g^{A\dot{B}}$ form a cotangent null tetrad in terms of which $ds^2 = -\frac{1}{2}g_{A\dot{B}}g^{A\dot{B}}$, where, as in the preceding expressions, juxtaposition of one-forms means symmetrized tensor product.) Due to Eq. (9), the Lanczos potential given by Eq. (22) satisfies the condition

$$\ell^R H_{\dot{A}\dot{B}\dot{C}R} = 0, \tag{24}$$

cf. Eq. (17).

Given an algebraically special metric that admits a shear-free congruence of null geodesics defined by a multiple principal spinor ℓ_A of the conformal curvature, using Eq. (9) and following the procedure given in Ref. [5], one can find the functions ϕ, $q^{\dot{A}}$, $p^{\dot{A}}$, which determine the conformally flat metric $^{\circ}ds^2$, and then from Eq. (23) identify the combination $\phi^{-2}\Omega_{\dot{A}\dot{B}}\ell_C\ell_D$ that appears in Eq. (22). For example, the Reissner-Nordström metric, which can be specified by the null tetrad

$$\begin{aligned} g^{1\dot{1}} &= -\frac{1}{\sqrt{2}}\left(\frac{\Delta}{r^2}dt + dr\right), \qquad & g^{1\dot{2}} &= -r(d\theta - i\,\mathrm{sen}\,\theta\, d\varphi), \\ g^{2\dot{2}} &= -\sqrt{2}\left(dt - \frac{r^2}{\Delta}dr\right), \qquad & g^{2\dot{1}} &= -r(d\theta + i\,\mathrm{sen}\,\theta\, d\varphi), \end{aligned} \tag{25}$$

with $\Delta = r^2 - 2Mr + Q^2$, admits a shear-free congruence of null geodesics defined by $\ell_A = \delta^2_A$, relative to the tetrad (25), and the trace-free part of the Ricci tensor satisfies Eq. (10). Following the steps indicated in Ref. [5] one finds that $\phi = \frac{1}{r}$, $dq^{\dot{1}} = i d\varphi + \csc\theta\, d\theta$, $dq^{\dot{2}} = dt - \frac{r^2}{\Delta} dr$, $p^{\dot{1}} = -\frac{1}{r}$, $p^{\dot{2}} = -\cos\theta$; therefore, from Eq. (23) it follows that, with respect to the tetrad (25), $\phi^{-2}\Omega_{\dot{1}\dot{1}} = -1$, $\phi^{-2}\Omega_{\dot{2}\dot{2}} = -\frac{\Delta}{2r^2}$, $\phi^{-2}\Omega_{\dot{1}\dot{2}} = 0$, and using Eq. (22) one obtains that the non-vanishing components of the Lanczos potential are

$$H_{\dot{1}\dot{1}\dot{1}2} = -\frac{1}{\sqrt{2}r}, \quad H_{\dot{1}\dot{1}\dot{2}2} = -\frac{\cot\theta}{3r}, \quad H_{\dot{1}\dot{2}\dot{2}2} = \frac{1}{3\sqrt{2}r}\left(\frac{1}{2} - \frac{2M}{r} + \frac{3Q^2}{2r^2}\right). \qquad (26)$$

4. Lanczos potential for conformally flat space-times

In the case of a conformally flat space-time the Weyl spinor vanishes and the metric can be expressed locally as

$$ds^2 = 2\phi^{-2}\, dq^{\dot{A}}\, dp_{\dot{A}} \qquad (27)$$

where ϕ is some (non-vanishing) function and $q^{\dot{A}}$, $p^{\dot{A}}$ are complex coordinates. Clearly, the metric (27) can be obtained form Eq. (8) by setting $Q_{\dot{A}\dot{B}} = 0$ and, therefore,

$$\partial_{1\dot{A}} = \sqrt{2}\frac{\partial}{\partial p^{\dot{A}}}, \qquad \partial_{2\dot{A}} = \sqrt{2}\phi^2\frac{\partial}{\partial q^{\dot{A}}} \qquad (28)$$

is a null tetrad for the metric (27) and the equations governing the Lanczos potential corresponding to a vanishing Weyl spinor can be obtained from Eq. (12), by setting $Q_{\dot{A}\dot{B}} = 0$. This gives

$$D_{(\dot{A}}\phi H_{\dot{B}\dot{C}\dot{D})1} = \partial_{(\dot{A}}\phi^{-1} H_{\dot{B}\dot{C}\dot{D})2} \qquad (29)$$

where, in the present case,

$$\partial_{\dot{A}} \equiv \frac{\partial}{\partial p^{\dot{A}}}, \qquad D_{\dot{A}} \equiv \frac{\partial}{\partial q^{\dot{A}}}, \qquad (30)$$

are commuting differential operators. Taking $\dot{A} = \dot{B} = \dot{C} = \dot{D} = \dot{1}$ in Eq. (29) one gets $D_{\dot{1}}\phi H_{\dot{1}\dot{1}\dot{1}1} = \partial_{\dot{1}}\phi^{-1} H_{\dot{1}\dot{1}\dot{1}2}$, which implies that, locally,

$$\phi H_{\dot{1}\dot{1}\dot{1}1} = \partial_{\dot{1}} M_{\dot{1}\dot{1}}, \qquad \phi^{-1} H_{\dot{1}\dot{1}\dot{1}2} = D_{\dot{1}} M_{\dot{1}\dot{1}}, \qquad (31)$$

where $M_{\dot{1}\dot{1}}$ is some function. Substituting Eqs. (31) into Eq. (29) with $\dot{A} = \dot{B} = \dot{C} = \dot{1}$, $\dot{D} = \dot{2}$, it follows that $D_{\dot{1}}(3\phi H_{\dot{1}\dot{1}\dot{2}1} - \partial_{\dot{2}} M_{\dot{1}\dot{1}}) = \partial_{\dot{1}}(3\phi^{-1} H_{\dot{1}\dot{1}\dot{2}2} - D_{\dot{2}} M_{\dot{1}\dot{1}})$; hence, there exists locally a function $M_{\dot{1}\dot{2}} \equiv M_{\dot{2}\dot{1}}$ such that $3\phi H_{\dot{1}\dot{1}\dot{2}1} - \partial_{\dot{2}} M_{\dot{1}\dot{1}} = 2\partial_{\dot{1}} M_{\dot{1}\dot{2}}$ and

$3\phi^{-1}H_{\dot{1}\dot{1}\dot{2}2} - D_{\dot{2}}M_{\dot{1}\dot{1}} = 2D_{\dot{1}}M_{\dot{1}\dot{2}}$, where the factor 2 is introduced for convenience. Thus,

$$\phi H_{\dot{1}\dot{1}\dot{2}1} = \frac{1}{3}(2\partial_{\dot{1}}M_{\dot{1}\dot{2}} + \partial_{\dot{2}}M_{\dot{1}\dot{1}}), \qquad \phi^{-1}H_{\dot{1}\dot{1}\dot{2}2} = \frac{1}{3}(2D_{\dot{1}}M_{\dot{1}\dot{2}} + D_{\dot{2}}M_{\dot{1}\dot{1}}). \tag{32}$$

Substituting Eqs. (32) into Eq. (29) with $\dot{A} = \dot{B} = \dot{1}$, $\dot{C} = \dot{D} = \dot{2}$, one obtains that $D_{\dot{1}}(\phi H_{\dot{1}\dot{2}\dot{2}1} - \frac{2}{3}\partial_{\dot{2}}M_{\dot{1}\dot{2}}) = \partial_{\dot{1}}(\phi^{-1}H_{\dot{1}\dot{2}\dot{2}2} - \frac{2}{3}D_{\dot{2}}M_{\dot{1}\dot{2}})$ and, therefore, there exists locally a function $M_{\dot{2}\dot{2}}$ such that $\phi H_{\dot{1}\dot{2}\dot{2}1} - \frac{2}{3}\partial_{\dot{2}}M_{\dot{1}\dot{2}} = \frac{1}{3}\partial_{\dot{1}}M_{\dot{2}\dot{2}}$ and $\phi^{-1}H_{\dot{1}\dot{2}\dot{2}2} - \frac{2}{3}D_{\dot{2}}M_{\dot{1}\dot{2}} = \frac{1}{3}D_{\dot{1}}M_{\dot{2}\dot{2}}$, *i.e.*,

$$\phi H_{\dot{1}\dot{2}\dot{2}1} = \frac{1}{3}(2\partial_{\dot{2}}M_{\dot{1}\dot{2}} + \partial_{\dot{1}}M_{\dot{2}\dot{2}}), \qquad \phi^{-1}H_{\dot{1}\dot{2}\dot{2}2} = \frac{1}{3}(2D_{\dot{2}}M_{\dot{1}\dot{2}} + D_{\dot{1}}M_{\dot{2}\dot{2}}). \tag{33}$$

Substituting now Eqs. (33) into Eq. (29) with $\dot{A} = \dot{1}$, $\dot{B} = \dot{C} = \dot{D} = \dot{2}$, one obtains $D_{\dot{1}}(\phi H_{\dot{2}\dot{2}\dot{2}1} - \partial_{\dot{2}}M_{\dot{2}\dot{2}}) = \partial_{\dot{1}}(\phi^{-1}H_{\dot{2}\dot{2}\dot{2}2} - D_{\dot{2}}M_{\dot{2}\dot{2}})$, which implies the existence of a function H such that $\phi H_{\dot{2}\dot{2}\dot{2}1} - \partial_{\dot{2}}M_{\dot{2}\dot{2}} = \partial_{\dot{1}}(\sqrt{2}\phi H)$ and $\phi^{-1}H_{\dot{2}\dot{2}\dot{2}2} - D_{\dot{2}}M_{\dot{2}\dot{2}} = D_{\dot{1}}(\sqrt{2}\phi H)$, where the factor $\sqrt{2}\phi$ is introduced for later convenience; therefore,

$$\phi H_{\dot{2}\dot{2}\dot{2}1} = \partial_{\dot{2}}M_{\dot{2}\dot{2}} + \partial_{\dot{1}}(\sqrt{2}\phi H), \qquad \phi^{-1}H_{\dot{2}\dot{2}\dot{2}2} = D_{\dot{2}}M_{\dot{2}\dot{2}} + D_{\dot{1}}(\sqrt{2}\phi H). \tag{34}$$

Finally, substituting Eqs. (34) into Eq. (29) with $\dot{A} = \dot{B} = \dot{C} = \dot{D} = \dot{2}$, one finds that $D_{\dot{2}}\partial_{\dot{1}}(\phi H) = \partial_{\dot{2}}D_{\dot{1}}(\phi H)$ or, equivalently,

$$D^{\dot{A}}\partial_{\dot{A}}(\phi H) = 0. \tag{35}$$

Thus, from Eqs. (31–34) it follows that the most general solution of Eq. (29) is given locally by

$$H_{\dot{A}\dot{B}\dot{C}R} = \frac{1}{\sqrt{2}}\phi^{-1}\partial_{R(\dot{A}}M_{\dot{B}\dot{C})} + \phi^{-1}\delta^{\dot{2}}_{\dot{A}}\delta^{\dot{2}}_{\dot{B}}\delta^{\dot{2}}_{\dot{C}}\delta^{\dot{S}}_{\dot{1}}\partial_{R\dot{S}}(\phi H) \tag{36}$$

where the $M_{\dot{A}\dot{B}}$ are arbitrary functions and H satisfies Eq. (35). In order to find the Lanczos potential for a conformally flat space-time in an arbitrary tetrad one has to replace the derivatives appearing in Eq. (36) by covariant derivatives, eliminating all explicit reference to the local expression of the metric given by Eq. (27).

Using the connection coefficients for the tetrad (28) it is easy to see that if $X_{\dot{A}\dot{B}}$ is a symmetric spinor field then, with respect to the null tetrad (28), $\nabla_{R(\dot{A}}X_{\dot{B}\dot{C})} = \phi^{-1}\partial_{R(\dot{A}}\phi X_{\dot{B}\dot{C})}$; therefore, the first term in the right-hand side of Eq. (36) amounts to $\nabla_{R(\dot{A}}X_{\dot{B}\dot{C})}$, where $X_{\dot{A}\dot{B}} \equiv \frac{1}{\sqrt{2}}\phi^{-1}M_{\dot{A}\dot{B}}$. Similarly, by introducing a spinor field $\ell_{\dot{A}} = \delta^{\dot{2}}_{\dot{A}}$ one finds that the last term in Eq. (36) is equal to $\phi^{-4}\nabla_R{}^{\dot{S}}(\phi^4\ell_{\dot{A}}\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{S}}H)$.

The spinor field $\ell_{\dot{A}}$ satisfies the covariant equation

$$\nabla_{A\dot{B}}\ell_{\dot{C}} = \frac{1}{2}\phi^{-1}\left(\ell_{\dot{B}}\partial_{A\dot{C}}\phi + \epsilon_{\dot{B}\dot{C}}\ell^{\dot{S}}\partial_{A\dot{S}}\phi\right) \tag{37}$$

which implies that

$$\ell^{\dot{C}}\nabla_{A\dot{B}}\ell_{\dot{C}} = \ell_{\dot{B}}\ell^{\dot{C}}\partial_{A\dot{C}}\ln\phi \tag{38}$$

cf. Eq. (9) and

$$\ell^{\dot{B}}\ell^{\dot{C}}\nabla_{A\dot{B}}\ell_{\dot{C}} = 0. \tag{39}$$

This last condition means that $\ell_{\dot{A}}$ defines a shear-free congruence of null geodesics. Equations (38) and (39) are invariant under rescalings of $\ell_{\dot{A}}$ but Eq. (37) is not. Given a solution of Eq. (39), there exists a function ϕ such that Eq. (38) is satisfied [5] and, by appropriately rescaling $\ell_{\dot{A}}$, Eq. (37) also holds. Therefore, since any non-vanishing factor can be absorbed into H, the second term in the right-hand side of Eq. (36) can be written covariantly as $\phi^{-4}\nabla_R{}^{\dot{S}}(\phi^4\ell_{\dot{A}}\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{S}}H)$ provided that $\ell_{\dot{A}}$ satisfies Eq. (39), with ϕ now being defined by Eq. (38).

Hence, the most general Lanczos potential for a conformally flat space-time, written in a covariant form, is given locally by

$$H_{\dot{A}\dot{B}\dot{C}R} = \nabla_{R(\dot{A}}X_{\dot{B}\dot{C})} + \phi^{-4}\nabla_R{}^{\dot{S}}\left(\phi^4\ell_{\dot{A}}\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{S}}H\right) \tag{40}$$

where $X_{\dot{A}\dot{B}}$ is an arbitrary symmetric spinor field, $\ell_{\dot{A}}$ satisfies Eq. (39), ϕ is defined by Eq. (38) and H obeys a wavelike equation. In fact, $\nabla_{R(\dot{A}}H_{\dot{B}\dot{C}\dot{D})}{}^R = \nabla_{R(\dot{A}}\nabla^R{}_{\dot{B}}X_{\dot{C}\dot{D})} + \nabla_{R(\dot{A}}\phi^{-4}\nabla^{R\dot{S}}\phi^4\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{D})}\ell_{\dot{S}}H$. The first term vanishes identically since the Weyl spinor is equal to zero and therefore, the function H is restricted by the condition

$$\nabla_{R(\dot{A}}\phi^{-4}\nabla^{R\dot{S}}\phi^4\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{D})}\ell_{\dot{S}}H = 0 \tag{41}$$

cf. Ref. [7], Eq. (14). Due to Eq. (38), the left-hand side of Eq. (41) is proportional to $\ell_{\dot{A}}\ell_{\dot{B}}\ell_{\dot{C}}\ell_{\dot{D}}$; hence, Eq. (41) yields just one condition on H (equivalent to Eq. (35)).

By using the basic identity (5), one finds that the tensor equivalent of the first term in the right-hand side of Eq. (40) is

$$H_{\mu\nu\rho} = \frac{1}{3}(2f_{\mu\nu;\rho} + f_{\mu\rho;\nu} - f_{\nu\rho;\mu} + g_{\mu\rho}f_\nu{}^\sigma{}_{;\sigma} - g_{\nu\rho}f_\mu{}^\sigma{}_{;\sigma}) \tag{42}$$

where $f_{\mu\nu} = -f_{\nu\mu}$ is the tensor equivalent of X_{AB}. If $f_{\mu\nu}$ satisfies the source-free

Maxwell equations, then Eq. (42) reduces to $H_{\mu\nu\rho} = f_{\mu\nu;\rho}$ (cf. Ref. [1]) and $H_{\mu\nu\rho}$ satisfies condition (4) if and only if the scalar curvature vanishes.

5. Concluding remarks

The explicit expressions presented in Sect. 3 give an independent proof of the existence of the Lanczos potential for the wide class of space-times considered here. On the other hand, the results of Sect. 4 show the ambiguity involved in the definition of the Lanczos potential, at least in the case of the conformally flat space-times.

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Resumen. Se obtiene una expresión explícita para el potencial de Lanczos, el cual genera el tensor de curvatura conforme, para todos los espacio-tiempos que admiten una congruencia sin distorsión de geodésicas luxoides, siempre que el tensor de Ricci esté restringido adecuadamente. Se obtiene también la expresión local más general del potencial de Lanczos de un espacio-tiempo conformalmente plano.