Phase space approach to the orbits in central force fields

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Abstract. The motion in a central field is analized with the use of a phase space for the energy equation. Applications are made to some particular examples.

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1. Introduction

Some years ago, Hauser [1] presented a simple approach to the study of planetary motion without explicit reference to the orbit differential equation, but solving the energy equation by a very simple method. Both equations are clearly equivalent through a trivial first integration of the former one. On the other hand, there exist many methods to obtain the solution to Kepler's problem, including the direct integration of the orbit equation [2] and of the Universal Kepler equation [3], and the analysis of the Laplace-Runge-Lenz vector [4–5]; geometrical methods such as that of the hodographs [7] have also been used to study such a problem.

In this paper we start from the energy equation for a general central field and develop a geometrical method, based upon the introduction of a phase space, where the simple phase portraits of the orbits reveal much of the relevant information about the motion of the particle. In particular, a relationship between the area enclosed by the phase curves and the angle elapsed between two consecutive apsidal positions of the particle is established. Afterwards, we apply the method to solve Kepler's problem in order to be able to set a perturbative technique to compute the precession of the planetary orbits when the correction due to the General Theory of Relativity is taken into account.

Also, a simple method to obtain explicit expressions for the orbit in both the unperturbed an the perturbed cases is presented.

2. The phase space for the orbits in central fields

A first integration of the orbit differential equation for a particle under the action of a central force field yields the so called energy equation [8], which can be written as

$$\frac{u'^2}{2} + \frac{u^2}{2} + \alpha V\left(\frac{1}{u}\right) = \alpha E \tag{1}$$

where $u = 1/r$ and ϕ being the plane polar coordinates, $a = m/l^2$; m, l and E are the mass, the angular momentum and the total energy of the particle in the field $V(r)$, and the prime indicates derivative with respect to ϕ.

If we make $E' = \alpha E$ and $W(u) = u^2/2 + \alpha V(1/u)$, Eq. (1) then reads

$$\frac{u'^2}{2} + W(u) = E' \tag{2}$$

which, besides being the expression of the energy conservation for the system under study, is formally identical to the energy equation for a particle of unit mass and energy E' in one-dimensional motion acted on by the "effective potential" $W(u)$.

Furthermore, because of the term u^2 in $W(u)$, there will exist two or more roots of Eq. (2) with $u' = 0$ and, among them, only those for which $u > 0$ will have a physical meaning as turning points of this one-dimensional motion. If we introduce now a phase space of coordinates u, u', the portrait of an orbit in this space must consist of a continuous curve, symmetric with respect to the u-axis and crossing it at the apsidal positions. We emphasize that only the part of this curve lying in the half-plane $u > 0$ will be physically meaningful.

From Eq. (2) we have

$$u' = \sqrt{2[E' - W(u)]}$$

hence

$$\frac{\partial u'}{\partial E'} = \frac{1}{u'} \tag{3}$$

and, since for a closed phase path completely contained in the region $u > 0$ the area enclosed by it is

$$A = 2\int_{u_{\min}}^{u_{\max}} u'\, du$$

where $u_{\min}$ and $u_{\max}$ correspond to the apocenter and the pericenter of the orbit, respectively, then

$$\frac{\partial A}{\partial E'} = 2\int_{U_{\min}}^{u_{\max}} \frac{\partial u'}{\partial E'}\, du = 2\int_{u_{\min}}^{u_{\max}} \frac{du}{u'}. \tag{4}$$

In this equation, the integration limits obviously depend on the energy but, since u' vanishes at these points, the intermediate results follows. Eq. (3) has been used to obtain the last equality.

On the other hand, from the definition of u' one has $d\phi = du/u'$, and the angle between two consecutive apocenters (or pericenters) is therefore

$$\Delta\Theta = 2\int_{u_{\min}}^{u_{\max}} d\phi = 2\int_{u_{\min}}^{u_{\max}} \frac{du}{u'} \tag{5}$$

or finally, from Eqs. (4) and (5) it follows that

$$\Delta\Theta = \frac{\partial A}{\partial E'}, \tag{6}$$

allowing us to compute $\Delta\Theta$ once A is known as a function of E'. The advantage of this equation, compared with the expression obtained for $\Delta\Theta$ directly from the energy equation [9] is that, in some cases, the area can be computed very simply as illustrated in the next section.

Eq. (6) has the same structure as the equation [10]

$$T = \frac{\partial S}{\partial E}$$

giving the period of a one-dimensional periodic motion in terms of the derivative of the area in the usual phase space with respect to the energy.

For large values of the energy, the phase curve defined by Eq. (2) and the condition that $u > 0$, isn't closed. In this case we have only one apsidal point, due to the fact that $u_{\min} = 1/r_{\max} = 0$ and the quantity

$$A = 2\int_0^{u_{\max}} u'\, du$$

correspond to the area enclosed by the phase curve and the line $u = 0$, then Eq. (5) takes the form

$$\Delta\Theta = 2\int_0^{u_{\max}} d\phi = 2\theta_0$$

where θ_0 is the angle between the asymptotic line to the orbit and the line from the center to the nearest point in the orbit, therefore the χ angle through which the particle is deflected as it passes the center is given by [11]

$$\chi = |\pi - 2\theta_0|.$$

Therefore Eq. (6) is valid in the unbounded cases and also can be used in the scattering problem.

3. Applications

Among the problems where the above approach is useful we consider the following examples:

i) Kepler's problem

In this case $V(1/u) = -ku$ so that

$$W(u) = \frac{u^2}{2} - \alpha k u$$

or

$$W(u) = (u - \alpha k)^2 - \tfrac{1}{2}\alpha^2 k^2$$

and Eq. (2) then reads

$$u'^2 + (u - \alpha k)^2 = R^2 \tag{7}$$

where $R^2 = 2E' + \alpha^2 k^2$.

In the phase space defined above, Eq. (7) represents a circle of radius R, centered at $(\alpha k, 0)$. The solution of Eq. (7) give to us the orbit, and we will show at the end of this section that is a conic section, with an eccentricity given by

$$e = \sqrt{1 + \left(\frac{2E'}{\alpha^2 k^2}\right)} = \frac{R}{\alpha k}, \tag{8}$$

the second equality follows from the definition of R, and shows that the radius of the circle properly normalized gives the kind of conic section, thus the following possibilities arise.

$R = 0$. The phase portrait of this orbit consists of the single point $(\alpha k, 0)$, see Fig. 1a, and we have $u' = 0$ and, hence, $u = \alpha k = \text{cons.}$, and the orbit in real space must consist of a circle of radius $1/\alpha k$. This can be checked from Eq. (8), since $R = 0$ implies that $e = 0$.

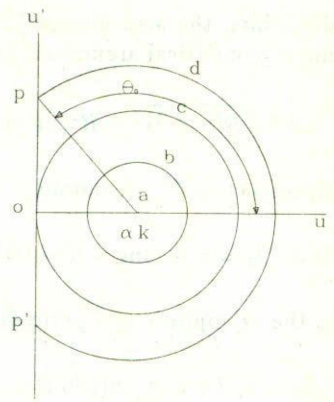

FIGURE 1. Phase portrait for a particle in an inverse square force field.

$0 < R < \alpha k$. The phase path consists of a circle lying in the half-space $u > 0$, Fig. 1b, and it is a closed curve in which u varies between the two apsidal distances $\alpha k - R$ and $\alpha k + R$ so that it represents an elliptic orbit due to the fact that $0 < e = R/\alpha k < 1$, and the major semi-axes is given by

$$a = \frac{1}{2}\left[\frac{1}{u_{\max}} + \frac{1}{u_{\min}}\right].$$

$R = \alpha k$. The corresponding circle passes through the origin of the phase space but this point must be excluded, giving place to an open curve which can be described as the path of a particle coming from $u \approx 0$ ($r = \infty$) with $u' \approx 0^+$ up to the apsidal position where the radial speed changes its sign, and going back to $u \approx 0$ ($r = \infty$) with $u' \approx 0^-$. The orbit is then a parabola as can be also seen from the fact that $E' = 0$ or $e = 1$. See Fig. 1c.

$R > \alpha k$. The phase portrait reduces to that arc of the circle lying in the half-space $u > 0$, and it is described as the path of a particle coming from $u \approx 0$ with a non-vanishing initial speed up to the apsidal distance and going back to $u \approx 0$. Hence, the orbit is a hyperbola due to $e = R/\alpha k > 1$. See Fig. 1d.

In the case of elliptic orbits, the phase curves are closed and the area enclosed by them is

$$A = \pi R^2 = \pi(2E' + \alpha^2 k^2),$$

so that the angle between two apocenters (or pericenters) is by Eq. (6)

$$\Delta\Theta = 2\pi.$$

In the case of hyperbolic orbits, the area enclosed by the circle and the line $u = 0$, can be evaluated using a geometrical arguments and we have

$$A = \tfrac{1}{2}\pi R^2 + \alpha k\sqrt{R^2 - \alpha^2 k^2} + R^2 \text{ang } \sin(\alpha k/R),$$

deriving this expression with respect to E', we obtain

$$\Delta\Theta = 2\theta_0 = \pi + 2\text{ang } \sin(\alpha k/R),$$

therefore, the angle between the asymptotic and pericentrical lines is given by

$$\theta_0 = \pi/2 + \text{ang } \sin(\alpha k/R)$$

this angle has a simple interpretation in the phase portrait, because θ_0 is the angle between the positive direction of u axis and the line from the center to the P point, where the circle cuts the u' axis. See Fig. 1d. So far, we have obtained qualitative information about the orbits solely from geometrical consideration. Nevertheless, we already know that Kepler's problem is completely integrable. Therefore, defined a complex quantity z such that

$$z = u - \alpha k - iu',$$

and, from Eq. (7), we get

$$zz^* = R^2,$$

which, in turn, implies that

$$z = Re^{if(\phi)}$$

or

$$\begin{aligned} \text{Re}\, z &= u - \alpha k = R\cos f(\phi), \\ \text{Im}\, z &= -u' = R\sin f(\phi), \end{aligned} \tag{9}$$

where the function $f(\phi)$ is easily computed by differentiating the first of Eqs. (9) and comparing the result with the second. From this we obtain $f(\phi) = \phi - \phi_0$ and the explicit expression for the orbit is then

$$u = \frac{1}{p}\left[1 + \sqrt{1 + \frac{2E'}{\alpha^2 k^2}}\cos(\phi - \phi_0)\right],$$

the familiar equation for a conic section [11], with $p = 1/\alpha k$ as its *latus rectum*, and the eccentricity is given by Eq. (8).

ii) Relativistic correction to Kepler's problem

The potential is, in this case [12,13]

$$V\left(\frac{1}{u}\right) = -ku - \left(\frac{p\epsilon}{3\alpha}\right)u^3,$$

where $\epsilon = 3GM/pc^2$ is a very small quantity, with G the gravitational constant, M is the mass of the sun and c is the speed of light. Therefore,

$$W(u) = \frac{u^2}{2} - \frac{u}{p} - \left(\frac{p\epsilon}{3}\right)u^3,$$

and, from Eq. (2) we get

$$\frac{u'^2}{2} + \frac{u^2}{2} - \frac{u}{p} - \left(\frac{p\epsilon}{3}\right)u^3 = E'. \tag{10}$$

Integration of this equation is possible only in terms of elliptic functions, but a perturbative approach to the solution can be used to describe the phase portrait of the orbits. In order to do so we first note that, for $\epsilon = 0$, the turning points of the one-dimensional motion are located at $\alpha k \pm R$, *i.e.* the non-perturbed path cuts the u-axis at

$$u_0 = (1/p)(1 \pm e).$$

For $\epsilon \neq 0$ we assume the expansion [12,13]

$$u = u_0 + \epsilon u_1 + \cdots$$

and it follows that the equation for the new apsidal positions up to first order in ϵ is

$$\frac{(u_0^2 + 2\epsilon u_0 u_1)}{2} - \frac{(u_0 + \epsilon u_1)}{p} - \frac{p\epsilon u_0^3}{3} = E'$$

but, since u_0 satisfies

$$\frac{1}{2}u_0^2 - \frac{u_0}{p} = E',$$

the equation for u_1 becomes

$$u_0 u_1 - \frac{u_1}{p} - \frac{p u_0^3}{3} = 0,$$

whose solution is

$$u_1 = \pm \frac{(1 \pm e)^3}{3pe}.$$

The turning points are then obtained from

$$u = \frac{1}{p}(1 \pm e) \pm \frac{\epsilon}{3pe}(1 \pm e)^3$$

or

$$u - \frac{1}{p}(1 + \epsilon') = \pm \frac{1}{p}\left(e + \epsilon e + \frac{\epsilon}{3e}\right)$$

where $\epsilon' = \epsilon(1 + e^2/3)$, so that these points are symmetric with respect to the center $((1 + \epsilon')/p, 0)$. The resulting phase curve has a very complex shape, but it greatly resembles and ellipse for small e (in actual cases this parameter is smaller than 10^{-7}). Hence, for the sake of simplicity one can consider a true ellipse centered at the point $((1 + \epsilon')/p, 0)$, since for that abcissa u' has an extremum given by

$$u' = \pm \frac{e(1 + \epsilon/3e^2)}{p}.$$

In Fig. 2 we show how the curve obtained from Eq. (10) and the corresponding simplified elliptical curve coincide already for $e = 0.001$. Also, notice that the smaller the eccentricity e of the unperturbed orbit is the better such an ellipse fits so that, if the unperturbed orbit is an ellipse our simplifying assumption is very well fulfilled.

Consequently, within the approximation made, the semi-axes of the ellipse are $(e + \epsilon e + \epsilon/3e)/p$ and $(e + \epsilon/3e)/p$ so that its area is then

$$A = \pi e^2 \frac{(1 + \epsilon)}{p^2} + 2\pi \frac{\epsilon}{3p^2} + \mathcal{O}(\epsilon^2).$$

Now, from Eq. (8) we have that $e = Rp$, thus we get $e^2/p^2 = 2E' + 1/p^2$ so that the area is

$$A = \pi \left(2E' + \frac{1}{p^2}\right)(1 + \epsilon) + 2\pi \frac{\epsilon}{3p^2} + \mathcal{O}(\epsilon^2)$$

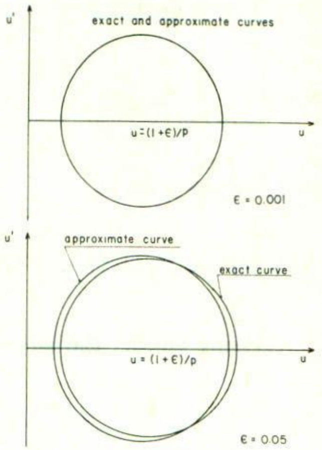

FIGURE 2. The actual and the simplified phase paths for two different values of the perturbation parameter. These are indistinguishable for $\epsilon = 0.001$.

and the angle defined in Eq. (6) is, therefore [14],

$$\Delta\Theta = 2\pi(1+\epsilon)$$

accounting for the existence of a precession of the perihelium of the perturbed orbit [13,15,16].

Furthermore, if this representation of the phase orbit is assumed, then the equation for this curve is:

$$\frac{u'^2}{[e(1+\epsilon/3e^2)/p]^2} + \frac{[u-(1+\epsilon')/p]^2}{[e(1+\epsilon+\epsilon/3e^2)/p]^2} = 1, \tag{11}$$

and, in a similar way as in Kepler's problem, we define

$$z = \frac{u-(1+\epsilon')/p}{e(1+\epsilon+\epsilon/3e^2)/p} - i\frac{u'}{e(1+\epsilon/3e^2)/p},$$

and, since $zz^* = 1$, if follows that $z = e^{if(\phi)}$, hence

$$u = \frac{(1+\epsilon')}{p} + \frac{e}{p}(1+\epsilon+\epsilon/3e^2)\cos f(\phi),$$

$$u' = -\frac{e}{p}(1+\epsilon/3e^2)\sin f(\phi).$$

Differentiation of the first of these equations and comparison with the second leads to the approximate result

$$f(\phi) = \frac{\phi - \phi_0}{1 + \epsilon}$$

so that the approximate equation of the orbit is

$$u = \frac{1}{p}\left[1 + \epsilon' + e(1 + \epsilon + \epsilon/3e^2)\cos\left(\frac{\phi - \phi_0}{1 + \epsilon}\right)\right],$$

which, in fact, represents a precessing ellipse, the advance angle being $\Theta = 2\pi\epsilon$ [13,14].

4. Conclusion

In the approach used in this work, a lot of information from geometrical considerations can be found for the central problem in a simple way, so in this scheme the Kepler problem can be studied analyzing a simple circle in the bounded case, and a section of a circle in the unbounded case. Also the phase space used was very convenient to study the corresponding relativistic problem, in which an approximate solution was found studying a simple ellipse. Thus our approach can be handled in a much simpler way than others used in the literature to study the orbits in central force fields.

We have derived a geometrical approach to analyze the orbits of a particle in a central force field. The simple curves in our phase space have a straighforward interpretation in real space, and we have given a simple method to compute the angle between two consecutive apocenters (or pericenters), which in turn allowed us to compute the precession of the perihelion in the perturbed Kepler problem.

The use of an ellipse to simulate the phase portrait of the orbit is justified by calculating the difference in the value of u' through the use of both Eqs. (10) and (11). Such a difference is negligible even for values of ϵ much larger than those of practical interest in planetary motion.

In a different context, the phase space that we used is closely related with the recent regularization infinite point techniques for the parabolic case. The details of these techniques can be found in Ref. [19].

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Resumen. Se analiza el movimiento en campos centrales interpretando la ecuación de la energía en un espacio fase asociado. Se aplican los resultados a casos particulares.