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Solution of nonscalar equations in cylindrical coordinates

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ABSTRACT. A class of functions defined on the plane is introduced and their usefulness is illustrated by solving the vector Helmholtz equation and Dirac's equation in cylindrical coordinates by separation of variables. It is shown that the separable solutions thus obtained are eigenfunctions of the square of the linear momentum perpendicular to the z-axis and of the z-component of the total angular momentum. It is also shown that the functions introduced here form bases for representations of the euclidean group of the plane.

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1. INTRODUCTION

In solving some of the partial differential equations of mathematical physics it is often convenient to use noncartesian coordinate systems. The method of separation of variables is very useful and frequently employed to solve the differential equations governing scalar fields; however, in the case of the differential equations satisfied by vector, tensor or spinor fields, written in noncartesian coordinates, the method of separation of variables cannot be applied in a straightforward manner due to the coupling of the components of the field.

Some nonscalar partial differential equations written in spherical coordinates can be reduced to sets of ordinary differential equations by expressing their solutions in terms of certain fields called vector-, tensor-, and spinor spherical harmonics. A similar reduction can be achieved by making use of the spin-weighted spherical harmonics [1–5], which provide a unified framework applicable to fields of any spin. Some of the advantages of the spin-weighted spherical harmonics come from their relationship with certain differential operators (denoted by $\eth$ and $\bar{\eth}$), which appear in a natural way when the components of the fields are combined to form quantities with a well-defined spin-weight.

The aim of this paper is to introduce a class of functions, analogous to the spin-weighted spherical harmonics, which are adapted to the circular cylindrical coordinates and to show their usefulness in the solution of nonscalar partial differential equations. In Sec. 2, the concept of spin-weight and two differential operators (also denoted by $\eth$ and $\bar{\eth}$) are introduced, in terms of which the spin-weighted cylindrical functions are defined. In Secs. 3 and 4 the vector Helmholtz equation and the Dirac equation are solved by separation of variables and in Sec. 5 it is shown that these separable solutions are eigenfunctions of the z-component of the total angular momentum and of the square of the linear momentum in the xy-plane. In Sec. 6 it is shown that the spin-weighted cylindrical functions span representation spaces for the group of rigid motions of the plane.

2. SPIN-WEIGHTED CYLINDRICAL FUNCTIONS

Let $\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\}$ be the orthonormal basis induced by the cylindrical coordinates ρ, ϕ, z. A quantity η has spin-weight s if under the rotation around $\hat{e}_z$ given by

$$\hat{e}'_\rho + i\hat{e}'_\phi = e^{i\theta}(\hat{e}_\rho + i\hat{e}_\phi) \tag{1}$$

transforms according to

$$\eta' = e^{is\theta}\eta. \tag{2}$$

If η has spin-weight s then its complex conjugate $\bar{\eta}$ has spin-weight $-s$ and if λ has spin-weight s' then $\eta\lambda$ has spin-weight $s + s'$. The vector fields $\hat{e}_z$ and $\hat{e}_\rho \pm i\hat{e}_\phi$ have spin-weight 0 and ± 1, respectively; therefore, for an arbitrary vector field $\mathbf{F}$, the scalar fields $F_z \equiv \mathbf{F} \cdot \hat{e}_z$ and $F_\pm \equiv \mathbf{F} \cdot (\hat{e}_\rho \pm i\hat{e}_\phi)$ have spin-weight 0 and ± 1. In terms of the components F_z and $F_\pm$ one has

$$\mathbf{F} = \tfrac{1}{2}F_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\rho - i\hat{e}_\phi) + F_z\hat{e}_z. \tag{3}$$

Similarly, the components of a traceless totally symmetric tensor of rank n can be combined into $2n+1$ components of spin-weight $-n, -n+1, \ldots, n$.

The operators $\eth$ and $\bar{\eth}$ acting on a quantity η with spin-weight s are defined by

$$\begin{aligned} \eth\eta &\equiv -\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi} - \frac{s}{\rho}\right)\eta = -\rho^s\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right)(\rho^{-s}\eta) \\ \bar{\eth}\eta &\equiv -\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi} + \frac{s}{\rho}\right)\eta = -\rho^{-s}\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}\right)(\rho^s\eta). \end{aligned} \tag{4}$$

The quantities $\eth\eta$ and $\bar{\eth}\eta$ have spin-weight $s+1$ and $s-1$, respectively. A straightforward computation shows that if η has spin-weight s, then

$$\bar{\eth}\,\eth\eta = \eth\,\bar{\eth}\eta = \frac{\partial^2\eta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\eta}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\eta}{\partial\phi^2} + \frac{2is}{\rho^2}\frac{\partial\eta}{\partial\phi} - \frac{s^2}{\rho^2}\eta. \tag{5}$$

In terms of the operators $\eth$ and $\bar{\eth}$ the gradient of a function f with spin-weight 0 is given by

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z \\ &= -\frac{1}{2}\bar{\eth} f(\hat{e}_\rho + i\hat{e}_\phi) - \frac{1}{2}\eth f(\hat{e}_\rho - i\hat{e}_\phi) + \frac{\partial f}{\partial z}\hat{e}_z. \end{aligned} \tag{6}$$

Similarly, using the relations $F_\rho = \frac{1}{2}(F_+ + F_-)$, $F_\phi = \frac{1}{2i}(F_+ - F_-)$ and the fact that F_z and $F_\pm$ have spin-weight 0 and ± 1, one finds that the divergence and the curl of a vector field $\mathbf{F}$ are given by

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \\ &= -\frac{1}{2}\eth F_- - \frac{1}{2}\bar{\eth} F_+ + \frac{\partial F_z}{\partial z} \\ \nabla \times \mathbf{F} &= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{e}_\phi + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\rho F_\phi - \frac{\partial F_\rho}{\partial \phi}\right)\hat{e}_z \\ &= \frac{1}{2i}\left(\bar{\eth} F_z + \frac{\partial F_-}{\partial z}\right)(\hat{e}_\rho + i\hat{e}_\phi) - \frac{1}{2i}\left(\eth F_z + \frac{\partial F_+}{\partial z}\right)(\hat{e}_\rho - i\hat{e}_\phi) \\ &\quad + \frac{1}{2i}(\eth F_- - \bar{\eth} F_+)\hat{e}_z. \end{aligned} \tag{7}$$

Therefore, from the identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ and Eqs. (6–7) it follows that

$$\begin{aligned}\nabla^2 \mathbf{F} &= \frac{1}{2}\left(\bar{\eth}\eth F_- + \frac{\partial^2 F_-}{\partial z^2}\right)(\hat{e}_\rho + i\hat{e}_\phi) + \frac{1}{2}\left(\eth\bar{\eth} F_+ + \frac{\partial^2 F_+}{\partial z^2}\right)(\hat{e}_\rho - i\hat{e}_\phi) \\ &\quad + \left(\bar{\eth}\eth F_z + \frac{\partial^2 F_z}{\partial z^2}\right)\hat{e}_z. \end{aligned} \tag{8}$$

Using Eqs. (6–7) and the commutativity of $\eth$ and $\bar{\eth}$ one finds that the laplacian of a function f of spin-weight 0 is given by

$$\nabla^2 f = \bar{\eth}\eth f + \frac{\partial^2 f}{\partial z^2}. \tag{9}$$

We shall denote by ${}_sF_{\alpha m}$ a function of ρ and ϕ with spin-weight s such that

$$\bar{\partial}\,\partial({}_sF_{\alpha m}) = -\alpha^2\,{}_sF_{\alpha m}, \tag{10}$$

$$-i\frac{\partial}{\partial\phi}({}_sF_{\alpha m}) = m\,{}_sF_{\alpha m}, \tag{11}$$

where α is a (real or complex) constant and m is an integer or a half-integer according to whether s is an integer or a half-integer. Condition (11) implies that ${}_sF_{\alpha m}(\rho,\phi) = f(\rho)e^{im\phi}$ and from Eqs. (5) and (10) it follows that $f(\rho)$ must satisfy the equation

$$\left[\rho^2\frac{d^2}{d\rho^2} + \rho\frac{d}{d\rho} + \alpha^2\rho^2 - (m+s)^2\right] f(\rho) = 0. \tag{12}$$

Therefore, if $\alpha \neq 0$, $f(\rho)$ is a linear combination of $J_{m+s}(\alpha\rho)$ and $N_{m+s}(\alpha\rho)$ or of $H^{(1)}_{m+s}(\alpha\rho)$ and $H^{(2)}_{m+s}(\alpha\rho)$. We shall adopt the following notation:

$${}_sZ_{\alpha m}(\rho,\phi) \equiv Z_{m+s}(\alpha\rho)e^{im\phi} \quad (\alpha \neq 0), \tag{13}$$

where Z_ν is a Bessel function: J_ν, N_ν, $H^{(1)}_\nu$ or $H^{(2)}_\nu$ (*e.g.*, ${}_sJ_{\alpha m}(\rho,\phi) \equiv J_{m+s}(\alpha\rho) \times e^{im\phi}$). Thus,

$$\begin{aligned} {}_sF_{\alpha m} &= A\,{}_sJ_{\alpha m} + B\,{}_sN_{\alpha m} \\ &= C\,{}_sH^{(1)}_{\alpha m} + D\,{}_sH^{(2)}_{\alpha m}, \end{aligned} \tag{14}$$

where A, B, C, D, are arbitrary constants.

In the case where $\alpha = 0$ and $m+s \neq 0$, $f(\rho)$ is a linear combination of ρ^{m+s} and ρ^{-m-s}. Therefore

$${}_sF_{0m} = A\rho^{m+s}e^{im\phi} + B\rho^{-m-s}e^{im\phi}, \quad (m+s \neq 0). \tag{15}$$

Finally, in the case where $\alpha = 0$ and $m+s = 0$,

$${}_sF_{0,-s} = Ae^{-is\phi} + B\ln\rho\, e^{-is\phi}. \tag{16}$$

In some applications, the boundary conditions exclude the solutions corresponding to $\alpha = 0$.

By using the recurrence relations for the Bessel functions and Eqs. (4) and (13) one finds that, for $\alpha \neq 0$,

$$\begin{aligned} \partial({}_sZ_{\alpha m}) &= \alpha\,{}_{s+1}Z_{\alpha m}, \\ \bar{\partial}({}_sZ_{\alpha m}) &= -\alpha\,{}_{s-1}Z_{\alpha m}. \end{aligned} \tag{17}$$

In the case where $\alpha = 0$ one gets

$$\begin{aligned} \eth(\rho^{m+s}e^{im\phi}) &= 0, \qquad \bar{\eth}(\rho^{m+s}e^{im\phi}) = -2(m+s)\rho^{m+s-1}e^{im\phi}, \\ \bar{\eth}(\rho^{-m-s}e^{im\phi}) &= 0, \qquad \eth(\rho^{-m-s}e^{im\phi}) = 2(m+s)\rho^{-m-s-1}e^{im\phi}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \eth(e^{-is\phi}) &= 0 \qquad \bar{\eth}(e^{-is\phi}) = 0, \\ \eth(\ln\rho e^{-is\phi}) &= -\rho^{-1}e^{-is\phi}, \qquad \bar{\eth}(\ln\rho e^{-is\phi}) = -\rho^{-1}e^{-is\phi}. \end{aligned} \tag{19}$$

3. SOLUTION OF THE VECTOR HELMHOLTZ EQUATION

According to Eq. (8), the vector Helmholtz equation,

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = 0, \tag{20}$$

amounts to the set of uncoupled equations

$$\begin{aligned} \bar{\eth}\eth F_- + \frac{\partial^2}{\partial z^2}F_- + k^2F_- &= 0, \\ \bar{\eth}\eth F_+ + \frac{\partial^2}{\partial z^2}F_+ + k^2F_+ &= 0, \\ \bar{\eth}\eth F_z + \frac{\partial^2}{\partial z^2}F_z + k^2F_z &= 0. \end{aligned} \tag{21}$$

Taking into account the fact that F_-, F_+ and F_z have spin-weight -1, 1 and 0, respectively, we seek for solutions of Eqs. (21) of the form

$$\begin{aligned} F_- &= g_{-1}(z)\,{}_{-1}F_{\alpha m}(\rho,\phi), \\ F_+ &= g_1(z)\,{}_1F_{\alpha m}(\rho,\phi), \\ F_z &= g_0(z)\,{}_0F_{\alpha m}(\rho,\phi), \end{aligned} \tag{22}$$

where m is an integer. Substituting Eqs. (22) into Eqs. (21) and using Eq. (10) one obtains

$$\left(\frac{d^2}{dz^2} + k^2 - \alpha^2\right) g_i(z) = 0, \quad (i = -1, 1, 0);$$

hence, if $\alpha^2 \neq k^2$, $g_i(z) = A_i e^{\gamma z} + B_i e^{-\gamma z}$, with

$$\gamma^2 \equiv \alpha^2 - k^2, \tag{23}$$

and if $\alpha^2 = k^2$ (*i.e.*, $\gamma = 0$), $g_i(z) = A_i + B_i z$, where the A_i and B_i are arbitrary constants. Thus, assuming that α is different from zero, from Eqs. (14) and (22) it follows that the vector Helmholtz equation admits separable solutions of the form

$$\begin{aligned} F_- &= (A_{-1}e^{\gamma z} + B_{-1}e^{-\gamma z})\big[C_{-1}({}_{-1}J_{\alpha m}) + D_{-1}({}_{-1}N_{\alpha m})\big], \\ F_+ &= (A_1 e^{\gamma z} + B_1 e^{-\gamma z})\big[C_1({}_1J_{\alpha m}) + D_1({}_1N_{\alpha m})\big], \\ F_z &= (A_0 e^{\gamma z} + B_0 e^{-\gamma z})\big[C_0({}_0J_{\alpha m}) + D_0({}_0N_{\alpha m})\big], \end{aligned} \tag{24a}$$

and, if $\alpha = \pm k$ ($\gamma = 0$),

$$\begin{aligned} F_- &= (A_{-1} + B_{-1}z)\big[C_{-1}({}_{-1}J_{\alpha m}) + D_{-1}({}_{-1}N_{\alpha m})\big], \\ F_+ &= (A_1 + B_1 z)\big[C_1({}_1J_{\alpha m}) + D_1({}_1N_{\alpha m})\big], \\ F_z &= (A_0 + B_0 z)\big[C_0({}_0J_{\alpha m}) + D_0({}_0N_{\alpha m})\big]. \end{aligned} \tag{24b}$$

From Eqs. (7) and (17) we see that the vector field (24a) has a vanishing divergence if and only if

$$\begin{aligned} &\frac{\alpha}{2}(A_1C_1 - A_{-1}C_{-1}) + \gamma A_0C_0 = 0, \qquad &&\frac{\alpha}{2}(B_1C_1 - B_{-1}C_{-1}) - \gamma B_0C_0 = 0, \\ &\frac{\alpha}{2}(A_1D_1 - A_{-1}D_{-1}) + \gamma A_0D_0 = 0, \qquad &&\frac{\alpha}{2}(B_1D_1 - B_{-1}D_{-1}) - \gamma B_0D_0 = 0, \end{aligned} \tag{25}$$

Introducing the constants

$$\begin{aligned} a_1 &\equiv \frac{i}{2\alpha}(A_1C_1 + A_{-1}C_{-1}), \qquad & a_2 &\equiv \frac{i}{2\alpha}(A_1D_1 + A_{-1}D_{-1}), \\ b_1 &\equiv \frac{i}{2\alpha}(B_1C_1 + B_{-1}C_{-1}), \qquad & b_2 &\equiv \frac{i}{2\alpha}(B_1D_1 + B_{-1}D_{-1}), \\ c_1 &\equiv \frac{1}{2\alpha\gamma}(A_1C_1 - A_{-1}C_{-1}), \qquad & c_2 &\equiv \frac{1}{2\alpha\gamma}(A_1D_1 - A_{-1}D_{-1}), \\ d_1 &\equiv \frac{1}{2\alpha\gamma}(B_{-1}C_{-1} - B_1C_1), \qquad & d_2 &\equiv \frac{1}{2\alpha\gamma}(B_{-1}D_{-1} - B_1D_1), \end{aligned}$$

and assuming that Eqs. (25) hold, the components (24*a*) can be written as

$$\begin{aligned} F_- &= \alpha\big[(-ia_1 - \gamma c_1)e^{\gamma z} + (-ib_1 + \gamma d_1)e^{-\gamma z}\big]\,{}_{-1}J_{\alpha m} \\ &\quad + \alpha\big[(-ia_2 - \gamma c_2)e^{\gamma z} + (-ib_2 + \gamma d_2)e^{-\gamma z}\big]\,{}_{-1}N_{\alpha m} \\ &= i\bar{\eth}\big[(a_1 e^{\gamma z} + b_1 e^{-\gamma z})\,{}_0J_{\alpha m} + (a_2 e^{\gamma z} + b_2 e^{-\gamma z})\,{}_0N_{\alpha m}\big] \\ &\quad + \frac{\partial}{\partial z}\bar{\eth}\big[(c_1 e^{\gamma z} + d_1 e^{-\gamma z})\,{}_0J_{\alpha m} + (c_2 e^{\gamma z} + d_2 e^{-\gamma z})\,{}_0N_{\alpha m}\big] \\ &= i\bar{\eth}\,\psi_1 + \frac{\partial}{\partial z}\bar{\eth}\,\psi_2, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \psi_1 &\equiv (a_1 e^{\gamma z} + b_1 e^{-\gamma z})\,{}_0J_{\alpha m} + (a_2 e^{\gamma z} + b_2 e^{-\gamma z})\,{}_0N_{\alpha m}, \\ \psi_2 &\equiv (c_1 e^{\gamma z} + d_1 e^{-\gamma z})\,{}_0J_{\alpha m} + (c_2 e^{\gamma z} + d_2 e^{-\gamma z})\,{}_0N_{\alpha m}, \end{aligned} \tag{27}$$

which are solutions of the scalar Helmholtz equation.

Similarly one finds that

$$\begin{aligned} F_+ &= \alpha\big[(-ia_1 + \gamma c_1)e^{\gamma z} + (-ib_1 - \gamma d_1)e^{-\gamma z}\big]\,{}_1J_{\alpha m} \\ &\quad + \alpha\big[(-ia_2 + \gamma c_2)e^{\gamma z} + (-ib_2 - \gamma d_2)e^{-\gamma z}\big]\,{}_1N_{\alpha m} \\ &= -i\eth\,\psi_1 + \frac{\partial}{\partial z}\eth\,\psi_2 \end{aligned} \tag{28}$$

and

$$\begin{aligned} F_z &= -\alpha^2\big[(c_1 e^{\gamma z} + d_1 e^{-\gamma z})\,{}_0J_{\alpha m} + (c_2 e^{\gamma z} + d_2 e^{-\gamma z})\,{}_0N_{\alpha m}\big] \\ &= \bar{\eth}\eth\,\psi_2. \end{aligned} \tag{29}$$

Using Eqs. (6–7) it can be shown that Eqs. (26) and (28–29) amount to the simple expression

$$\begin{aligned} \mathbf{F} &= \hat{e}_z \times \nabla\psi_1 + \nabla \times (\hat{e}_z \times \nabla\psi_2) \\ &= -\nabla \times (\psi_1 \hat{e}_z) - \nabla \times \nabla \times (\psi_2 \hat{e}_z). \end{aligned} \tag{30}$$

In an entirely similar manner, one finds that if the vector field given by Eqs. (24*b*) has a vanishing divergence then Eq. (30) also applies, where ψ_1 and ψ_2 are solutions of the scalar Helmholtz equation of the form $(a_1 + b_1 z)\,{}_0J_{\alpha m} + (a_2 + b_2 z)\,{}_0N_{\alpha m}$.

Thus, in those cases where the boundary conditions do not allow for solutions of the form (22) with $\alpha = 0$, owing to the completeness of the solutions (24) and to the linearity of the Helmholtz equation, any solution of the vector Helmholtz equation whose divergence is equal to zero can be written in the form (30), with ψ_1 and ψ_2 being solutions of the scalar Helmholtz equation. It is easy to see that, conversely, given two solutions ψ_1 and ψ_2 of the scalar Helmholtz equation, the vector field (30) satisfies the vector Helmholtz equation and its divergence is equal to zero.

On the other hand, there exist divergenceless solutions of the vector Helmholtz equation of the form (22) with $\alpha = 0$ that can also be written in the form (30), for which one of the functions ψ_1 and ψ_2 determines the other. An example of such a solution is given by

$$\begin{aligned} F_- &= ik(a_1 e^{ikz} - b_1 e^{-ikz})\rho^{-m+1} e^{im\phi}, \\ F_+ &= ik(-a_2 e^{ikz} + b_2 e^{-ikz})\rho^{m+1} e^{im\phi}, \\ F_z &= (m-1)(a_1 e^{ikz} + b_1 e^{-ikz})\rho^{-m} e^{im\phi}, \\ &\quad + (m+1)(a_2 e^{ikz} + b_2 e^{-ikz})\rho^{m} e^{im\phi}. \end{aligned} \tag{31}$$

The electric and magnetic fields of a monochromatic electromagnetic wave in vacuum obey the vector Helmholtz equation and have vanishing divergence. If, for example, the fields are confined inside a circular wave guide or a cylindrical cavity resonator with perfectly conducting walls, the boundary conditions cannot be satisfied by the separable solutions (22) with $\alpha = 0$ and therefore there exist two independent scalar potentials ψ_1, ψ_2, which fulfill the Helmholtz equation, such that

$$\mathbf{E} = -\nabla \times (\psi_1 \hat{e}_z) - \nabla \times \nabla \times (\psi_2 \hat{e}_z). \tag{32}$$

Then, the relation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ gives

$$\mathbf{B} = \frac{ik}{c} \nabla \times (\psi_2 \hat{e}_z) + \frac{i}{kc} \nabla \times \nabla \times (\psi_1 \hat{e}_z). \tag{33}$$

Equation (29) shows that the potentials ψ_1 and ψ_2 generate transverse electric and transverse magnetic fields, respectively. It may be noticed that if Eq. (31) is the electric field of an electromagnetic wave in vacuum, then the z-component of the corresponding magnetic field is proportional to

$$\begin{aligned} (\nabla \times \mathbf{F})_z &= k(m-1)(a_1 e^{ikz} - b_1 e^{-ikz})\rho^{-m} e^{im\phi} \\ &\quad + k(m+1)(-a_2 e^{ikz} + b_2 e^{-ikz})\rho^{m} e^{im\phi}, \end{aligned}$$

and therefore this electromagnetic field *cannot* be split into a transverse electric and a transverse magnetic field with respect to $\hat{e}_z$ (compare, *e.g.*, Ref. [6] and the references cited therein).

As a simple example of the application of the solutions (24) we shall solve the Maxwell-London equations for the case of an infinite superconducting cylinder of radius a placed in an originally uniform magnetic field perpendicular to the axis of the cylinder. We shall employ a system of cylindrical coordinates such that the z-axis coincides with the axis of the cylinder and the angle ϕ is measured from the direction of the original magnetic field. Since outside the cylinder the magnetic induction and the magnetic field intensity satisfy the equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = 0$, with $\mathbf{B} = \mu_0 \mathbf{H}$, there exists a potential function ϕ^* such that $\mathbf{H} = -\nabla\phi^*$ and $\nabla^2\phi^* = 0$. Solving the Laplace equation, taking into account that ϕ^* does not depend on z and that, due to the symmetry under the reflection on the xz-plane, ϕ^* must be an even function of ϕ as well as that $\mathbf{B} \to B_0\hat{\imath}$ as $\rho \to \infty$, one finds the expression

$$\phi^* = -\frac{B_0}{\mu_0}\rho\cos\phi + \sum_{m=1}^{\infty} b_m \rho^{-m}\cos m\phi.$$

Since $\mathbf{B} = -\mu_0\nabla\phi^*$, from Eqs. (4) and (6) one obtains, for $\rho \geq a$,

$$\begin{aligned} B_- &= \mu_0\bar{\partial}\phi^* = B_0 e^{i\phi} + \mu_0 \sum_{m=1}^{\infty} m b_m \rho^{-m-1} e^{-im\phi} \\ B_+ &= \mu_0\partial\phi^* = B_0 e^{-i\phi} + \mu_0 \sum_{m=1}^{\infty} m b_m \rho^{-m-1} e^{im\phi}. \end{aligned} \tag{34}$$

On the other hand, inside the superconductor, the magnetic induction is assumed to satisfy the equation $\nabla^2\mathbf{B} = \lambda^{-2}\mathbf{B}$, where λ is a constant, which is the vector Helmholtz equation (20) with $k = 1/i\lambda$. From the symmetry of the problem it follows that B_z must be equal to zero and that the remaining components must be independent of z. Thus, since $\nabla \cdot \mathbf{B} = 0$ and $B_\pm$ must be bounded at $\rho = 0$, from Eqs. (24b), (7) and (17) we obtain, for $\rho \leq a$,

$$\begin{aligned} B_- &= \sum_{m=-\infty}^{\infty} a_m \, {}_{-1}J_{\alpha m} = \sum_{m=-\infty}^{\infty} a_m J_{m-1}(\alpha\rho) e^{im\phi} \\ B_+ &= \sum_{m=-\infty}^{\infty} a_m \, {}_{1}J_{\alpha m} = \sum_{m=-\infty}^{\infty} a_m J_{m+1}(\alpha\rho) e^{im\phi}, \end{aligned} \tag{35}$$

where $\alpha = k = 1/i\lambda$. (Note that ${}_sJ_{-\alpha,m} = (-1)^{m+s}\,{}_sJ_{\alpha m}$; therefore it is not necessary to include terms with $\alpha = -1/i\lambda$ in Eq. (35).) By equating Eqs. (34) and (35) at $\rho = a$ one finds that the only nonvanishing coefficients are b_1, a_1 and a_{-1}, which are given by

$$a_1 = a_{-1} = \frac{B_0}{J_0(\alpha a)} = \frac{B_0}{J_0(\frac{a}{i\lambda})} = \frac{B_0}{I_0(\frac{a}{\lambda})},$$

$$b_1 = \frac{B_0 a^2}{\mu_0}\frac{J_2(\alpha a)}{J_0(\alpha a)} = -\frac{B_0 a^2}{\mu_0}\frac{I_2(\frac{a}{\lambda})}{I_0(\frac{a}{\lambda})},$$

where the I_ν are modified Bessel functions.

4. SOLUTION OF THE DIRAC EQUATION

The orthonormal basis $\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\}$ can be considered as induced by the two-component spinor field

$$o \equiv \begin{bmatrix} e^{-i\phi/2} \\ 0 \end{bmatrix} \tag{36}$$

by means of the relations

$$\hat{e}_\rho + i\hat{e}_\phi = o^t \varepsilon \sigma o, \quad \hat{e}_z = o^\dagger \sigma o, \tag{37}$$

where

$$\varepsilon \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{38}$$

and the σ_i are the Pauli matrices, so that the rotation given by Eq. (1) is induced by the transformation

$$o' = e^{i\theta/2} o \tag{39}$$

and therefore we shall assign to o the spin-weight $\frac{1}{2}$.

The spinor field o also induces the spinor basis $\{o, -\imath\}$, where

$$\imath \equiv \varepsilon \bar{o} = \begin{bmatrix} 0 \\ -e^{i\phi/2} \end{bmatrix}. \tag{40}$$

From Eqs. (39–40) it follows that $\imath$ has spin-weight $-\frac{1}{2}$. An arbitrary two-component spinor field u can be expressed in the form

$$u = u_- o - u_+ \imath, \tag{41}$$

where $u_\pm$ are complex-valued functions; u_- has spin-weight $-\frac{1}{2}$ and u_+ has spin-weight $\frac{1}{2}$. Using Eqs. (36), (40) and (41) one gets

$$u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = u_- \begin{bmatrix} e^{-i\phi/2} \\ 0 \end{bmatrix} - u_+ \begin{bmatrix} 0 \\ -e^{i\phi/2} \end{bmatrix} = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix} \begin{bmatrix} u_- \\ u_+ \end{bmatrix}, \tag{42}$$

hence

$$\begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \equiv \Lambda u. \tag{43}$$

In order to write the Dirac equation in terms of spin-weighted quantities, by using the equality

$$\boldsymbol{\sigma} \cdot \nabla = \boldsymbol{\sigma} \cdot \hat{e}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{\sigma} \cdot \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \boldsymbol{\sigma} \cdot \hat{e}_z \frac{\partial}{\partial z}$$

and Eq. (36), one finds that

$$\boldsymbol{\sigma} \cdot \nabla o = \frac{1}{2\rho} o. \tag{44}$$

Taking the complex conjugate of Eq. (44) and making use of the relations $\imath = \varepsilon \bar{o}$, $\varepsilon^2 = -I$ and $\varepsilon \bar{\sigma}_i \varepsilon = \sigma_i$ one obtains

$$\boldsymbol{\sigma} \cdot \nabla \imath = -\frac{1}{2\rho} \imath. \tag{45}$$

Using now the Dirac equation written in the standard form

$$\begin{aligned} i\hbar \frac{\partial u}{\partial t} &= -i\hbar c \boldsymbol{\sigma} \cdot \nabla v + mc^2 u \\ i\hbar \frac{\partial v}{\partial t} &= -i\hbar c \boldsymbol{\sigma} \cdot \nabla u - mc^2 v, \end{aligned} \tag{46}$$

where $\psi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is a four-component Dirac spinor, expressing the two-component spinors u and v in the form (41) and using Eqs. (4), (36), (40) and (44–45) one finds that the Dirac equation amounts to

$$\begin{aligned}
\frac{1}{c}\frac{\partial u_-}{\partial t} &= -\frac{\partial v_-}{\partial z} + \bar{\eth} v_+ - \frac{imc}{\hbar} u_-, \\
\frac{1}{c}\frac{\partial u_+}{\partial t} &= \eth v_- + \frac{\partial v_+}{\partial z} - \frac{imc}{\hbar} u_+, \\
\frac{1}{c}\frac{\partial v_-}{\partial t} &= -\frac{\partial u_-}{\partial z} + \bar{\eth} u_+ + \frac{imc}{\hbar} v_-, \\
\frac{1}{c}\frac{\partial v_+}{\partial t} &= \eth u_- + \frac{\partial u_+}{\partial z} + \frac{imc}{\hbar} v_+.
\end{aligned} \tag{47}$$

(An alternate derivation of Eq. (47) is given in Ref. [7].)

Equations (47) can be solved by separation of variables, looking for solutions of the form

$$\begin{aligned}
u_- &= g(z)\,{}_{-1/2}F_{\alpha m}(\rho,\phi)e^{-iEt/\hbar}, \\
u_+ &= G(z)\,{}_{1/2}F_{\alpha m}(\rho,\phi)e^{-iEt/\hbar}, \\
v_- &= f(z)\,{}_{-1/2}F_{\alpha m}(\rho,\phi)e^{-iEt/\hbar}, \\
v_+ &= F(z)\,{}_{1/2}F_{\alpha m}(\rho,\phi)e^{-iEt/\hbar},
\end{aligned} \tag{48}$$

where m is a half-integer and α, E are some constants. Since the components $u_\pm$, $v_\pm$ must be bounded at $\rho = 0$ and at infinity, α must be real and if $\alpha \neq 0$ the functions ${}_sF_{\alpha m}$ appearing in Eqs. (48) are multiples of ${}_sJ_{\alpha m}$. Taking ${}_sF_{\alpha m} = {}_sJ_{\alpha m}$, substituting Eqs. (48) into Eqs. (47) and using the relations (17) one obtains

$$\begin{aligned}
-\frac{df}{dz} - \alpha F - \frac{imc}{\hbar} g &= -\frac{iE}{\hbar c} g, \\
\alpha f + \frac{dF}{dz} - \frac{imc}{\hbar} G &= -\frac{iE}{\hbar c} G, \\
-\frac{dg}{dz} - \alpha G + \frac{imc}{\hbar} f &= -\frac{iE}{\hbar c} f, \\
\alpha g + \frac{dG}{dz} + \frac{imc}{\hbar} F &= -\frac{iE}{\hbar c} F,
\end{aligned} \tag{49}$$

In order to solve these equations it is convenient to express the functions g, G, f, F in the form

$$\begin{aligned} g(z) &= A(z) - B(z), \qquad & f(z) &= i(D(z) - C(z)), \\ G(z) &= A(z) + B(z), \qquad & F(z) &= i(D(z) + C(z)), \end{aligned} \tag{50}$$

(*cf.* Ref. [5]); then, Eqs. (49) amount to

$$\begin{aligned} \frac{dA}{dz} + \alpha A &= \frac{E + mc^2}{\hbar c} C, \\ -\frac{dC}{dz} + \alpha C &= \frac{E - mc^2}{\hbar c} A, \end{aligned} \tag{51}$$

and

$$\begin{aligned} \frac{dB}{dz} - \alpha B &= \frac{E + mc^2}{\hbar c} D, \\ -\frac{dD}{dz} - \alpha D &= \frac{E - mc^2}{\hbar c} B. \end{aligned} \tag{52}$$

From Eqs. (51) it follows that $d^2A/dz^2 = -(k^2 - \alpha^2)A$, where $k \equiv \sqrt{E^2 - m^2c^4}/\hbar c$; hence,

$$A = a_1 e^{i\sqrt{k^2-\alpha^2}\, z} + a_2 e^{-\sqrt{k^2-\alpha^2}\, z}, \tag{53}$$

where a_1 and a_2 are two arbitrary constants. Substitution of Eq. (53) into the first of Eqs. (51) yields

$$C = a_1 \frac{(\alpha + i\sqrt{k^2 - \alpha^2})\hbar c}{E + mc^2} e^{i\sqrt{k^2-\alpha^2}\, z} + a_2 \frac{(\alpha - i\sqrt{k^2 - \alpha^2})\hbar c}{E + mc^2} e^{-i\sqrt{k^2-\alpha^2}\, z}. \tag{54}$$

The solutions (53–54) are bounded only if

$$|\alpha| \leq k. \tag{55}$$

Since Eqs. (51) and (52) differ only by the sign of α, one immediately obtains

$$\begin{aligned} B &= b_1 e^{i\sqrt{k^2-\alpha^2}\, z} + b_2 e^{-i\sqrt{k^2-\alpha^2}\, z} \\ D &= b_1 \frac{(-\alpha + i\sqrt{k^2 - \alpha^2})\hbar c}{E + mc^2} e^{i\sqrt{k^2-\alpha^2}\, z} + b_2 \frac{(-\alpha - i\sqrt{k^2 - \alpha^2})\hbar c}{E + mc^2} e^{-i\sqrt{k^2-\alpha^2}\, z}, \end{aligned} \tag{56}$$

where b_1 and b_2 are arbitrary constants. Thus, from Eqs. (48) and (50) we see that Eqs. (47) admit separable solutions of the form

$$\begin{bmatrix} u_- \\ u_+ \\ v_- \\ v_+ \end{bmatrix} = \begin{bmatrix} A(z)X_{\alpha m} \\ iC(z)X_{-\alpha m} \end{bmatrix} e^{-iEt/\hbar} + \begin{bmatrix} B(z)X_{-\alpha m} \\ iD(z)X_{\alpha m} \end{bmatrix} e^{-iEt/\hbar}, \tag{57}$$

where

$$X_{\alpha m} \equiv \begin{bmatrix} {}_{-1/2}J_{\alpha m} \\ {}_{1/2}J_{\alpha m} \end{bmatrix}, \quad X_{-\alpha m} \equiv \begin{bmatrix} -({}_{-1/2}J_{\alpha m}) \\ {}_{1/2}J_{\alpha m} \end{bmatrix}, \quad (\alpha \neq 0). \tag{58}$$

When $\alpha = 0$, ${}_sF_{0m}$ is bounded only if $m = -s$ (*cf.* Eqs. (15–16)) and, in that case, ${}_sF_{0,-s}$ must be a multiple of $e^{-is\phi}$; hence, we look for solutions of Eqs. (47) of the form

$$m = \frac{1}{2}: \begin{cases} u_- = g(z)e^{i\phi/2}e^{-iEt/\hbar}, \\ u_+ = 0, \\ v_- = f(z)e^{i\phi/2}e^{-iEt/\hbar}, \\ v_+ = 0, \end{cases} \tag{59}$$

and

$$m = -\frac{1}{2}: \begin{cases} u_- = 0, \\ u_+ = G(z)e^{-i\phi/2}e^{-iEt/\hbar}, \\ v_- = 0, \\ v_+ = F(z)e^{-i\phi/2}e^{-iEt/\hbar}. \end{cases} \tag{60}$$

Substituting Eqs. (59) and (60) into Eqs. (47), using Eqs. (19), we get

$$\begin{aligned} g(z) &= a_1 e^{ikz} + a_2 e^{-ikz}, \\ f(z) &= \frac{\hbar ck}{E+mc^2}(a_1 e^{ikz} - a_2 e^{-ikz}), \\ G(z) &= b_1 e^{ikz} + b_2 e^{-ikz}, \\ F(z) &= \frac{\hbar ck}{E+mc^2}(-b_1 e^{ikz} + b_2 e^{-ikz}), \end{aligned} \tag{61}$$

Thus, in the case $\alpha = 0$, Eqs. (47) admit separable solutions of the form

$$\begin{bmatrix} u_- \\ u_+ \\ v_- \\ v_+ \end{bmatrix} = \begin{bmatrix} \zeta_1(z) X_{0m} \\ \zeta_2(z) X_{0m} \end{bmatrix} e^{-iEt/\hbar} \quad (m = \pm\tfrac{1}{2}), \tag{62}$$

(*cf.* Eq. (57)) where

$$X_{0,1/2} \equiv \begin{bmatrix} e^{i\phi/2} \\ 0 \end{bmatrix}, \quad X_{0,-1/2} \equiv \begin{bmatrix} 0 \\ e^{-i\phi/2} \end{bmatrix}. \tag{63}$$

The solutions (59–61), corresponding to $\alpha = 0$, are superpositions of plane waves traveling along the z-axis in the positive and negative directions.

5. Characterization of the separable solutions

For a vector field $\mathbf{F}$, the operator corresponding to the z-component of the total angular momentum is given by

$$\begin{aligned} J_3 \mathbf{F} &= (-i\hbar \hat{e}_z \cdot \mathbf{r} \times \nabla)\mathbf{F} + i\hbar \hat{e}_z \times \mathbf{F} \\ &= -i\hbar \frac{\partial \mathbf{F}}{\partial \phi} + i\hbar \hat{e}_z \times \mathbf{F}. \end{aligned} \tag{64}$$

From the relation $\hat{e}_\rho + i\hat{e}_\phi = e^{-i\phi}(\hat{\imath} + i\hat{\jmath})$ one finds that $\partial(\hat{e}_\rho + i\hat{e}_\phi)/\partial\phi = \hat{e}_\phi - i\hat{e}_\rho = \hat{e}_z \times (\hat{e}_\rho + i\hat{e}_\phi)$; therefore, expressing the vector field $\mathbf{F}$ in the form (3) we obtain

$$\begin{aligned} &J_3\big(\tfrac{1}{2}F_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\rho - i\hat{e}_\phi) + F_z\hat{e}_z\big) \\ &= \frac{1}{2}\left(-i\hbar\frac{\partial}{\partial\phi}F_-\right)(\hat{e}_\rho + i\hat{e}_\phi) + \frac{1}{2}\left(-i\hbar\frac{\partial}{\partial\phi}F_+\right)(\hat{e}_\rho - i\hat{e}_\phi) + \left(-i\hbar\frac{\partial}{\partial\phi}F_z\right)\hat{e}_z. \end{aligned} \tag{65}$$

This formula leads to the following definition: if η has spin-weight s, the operator $J_3^{(s)}$ is defined by

$$J_3^{(s)}\eta \equiv -i\hbar\frac{\partial}{\partial\phi}\eta, \tag{66}$$

then Eq. (65) amounts to

$$\begin{aligned} J_3\left(\tfrac{1}{2}F_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\rho - i\hat{e}_\phi) + F_z\hat{e}_z\right) \\ = \frac{1}{2}\big(J_3^{(-1)}F_-\big)(\hat{e}_\rho + i\hat{e}_\phi) + \frac{1}{2}\big(J_3^{(1)}F_+\big)(\hat{e}_\rho - i\hat{e}_\phi) + \big(J_3^{(0)}F_z\big)\hat{e}_z. \end{aligned} \tag{67}$$

According to Eq. (11), the spin-weighted functions ${}_sF_{\alpha m}$ are eigenfunctions of $J_3^{(s)}$ with eigenvalue $m\hbar$

$$J_3^{(s)}\,{}_sF_{\alpha m} = m\hbar\,{}_sF_{\alpha m}, \tag{68}$$

and therefore, from Eqs. (67) and (68) we see that the separable solutions of the vector Helmholtz equation given by Eq. (22) are eigenfunctions of J_3

$$J_3\mathbf{F} = m\hbar\mathbf{F}. \tag{69}$$

Similarly, by using the relations

$$\frac{\partial}{\partial x}(\hat{e}_\rho + i\hat{e}_\phi) = \frac{i\,\mathrm{sen}\,\phi}{\rho}(\hat{e}_\rho + i\hat{e}_\phi), \quad \frac{\partial}{\partial y}(\hat{e}_\rho + i\hat{e}_\phi) = -\frac{i\cos\phi}{\rho}(\hat{e}_\rho + i\hat{e}_\phi) \tag{70}$$

and their complex conjugates, one finds that the operators $P_1 = -i\hbar\partial/\partial x$ and $P_2 = -i\hbar\partial/\partial y$, corresponding to the x- and y-components of the linear momentum, acting on a vector field $\mathbf{F}$ are given by

$$\begin{aligned} P_k\big(\tfrac{1}{2}F_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\rho - i\hat{e}_\phi) + F_z\hat{e}_z\big) \\ = \tfrac{1}{2}(P_k^{(-1)}F_-)(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}(P_k^{(1)}F_+)(\hat{e}_\rho - i\hat{e}_\phi) + (P_k^{(0)}F_z)\hat{e}_z \end{aligned} \tag{71}$$

$(k = 1, 2)$, where

$$\begin{aligned} P_1^{(s)} &\equiv -i\hbar\left(\frac{\partial}{\partial x} - is\frac{\mathrm{sen}\,\phi}{\rho}\right) = -i\hbar\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\mathrm{sen}\,\phi}{\rho}\frac{\partial}{\partial\phi} - is\frac{\mathrm{sen}\,\phi}{\rho}\right) \\ P_2^{(s)} &\equiv -i\hbar\left(\frac{\partial}{\partial y} + is\frac{\cos\phi}{\rho}\right) = -i\hbar\left(\mathrm{sen}\,\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi} + is\frac{\cos\phi}{\rho}\right). \end{aligned} \tag{72}$$

A straightforward computation shows that

$$P_1^{(s)2} + P_2^{(s)2} = -\hbar^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{2is}{\rho^2}\frac{\partial}{\partial \phi} - \frac{s^2}{\rho^2} \right) = -\hbar^2 \bar{\eth}\eth \tag{73}$$

(*cf.* Eq. (5)), and from Eq. (10) we obtain

$$\left(P_1^{(s)2} + P_2^{(s)2}\right) {}_sF_{\alpha m} = \hbar^2\alpha^2 \, {}_sF_{\alpha m}. \tag{74}$$

Thus, in view of Eqs. (71) and (74), the separable solutions of the vector Helmholtz equation (22) are eigenfunctions of $P_1^2 + P_2^2$ with eigenvalue $\hbar^2\alpha^2$

$$\left(P_1^2 + P_2^2\right)\mathbf{F} = \hbar^2\alpha^2\mathbf{F}. \tag{75}$$

For a two-component spinor field u, the operator corresponding to the z-component of the total angular momentum is given by

$$J_3 u = -i\hbar\frac{\partial u}{\partial \phi} + \frac{\hbar}{2}\sigma_3 u.$$

From Eqs. (42–43) it follows that the components of $J_3 u$ with respect to the basis $\{o, -\imath\}$ are

$$\begin{aligned}
\begin{bmatrix} (J_3u)_- \\ (J_3u)_+ \end{bmatrix} &= \Lambda\left(-i\hbar\frac{\partial}{\partial\phi}\right)\Lambda^{-1}\begin{bmatrix} u_- \\ u_+ \end{bmatrix} + \frac{\hbar}{2}\Lambda\sigma_3\Lambda^{-1}\begin{bmatrix} u_- \\ u_+ \end{bmatrix} \\
&= -i\hbar\frac{\partial}{\partial\phi}\begin{bmatrix} u_- \\ u_+ \end{bmatrix},
\end{aligned}$$

thus,

$$J_3(u_- o - u_+ \imath) = (J_3^{(-1/2)}u_-)o - (J_3^{(1/2)}u_+)\imath \tag{76}$$

(*cf.* Eq. (67)), with $J_3^{(s)}$ defined by Eq. (68).

Using the relations

$$\frac{\partial o}{\partial x} = \frac{i \operatorname{sen}\phi}{2\rho}o, \qquad \frac{\partial \imath}{\partial x} = -\frac{i \operatorname{sen}\phi}{2\rho}\imath,$$

$$\frac{\partial o}{\partial y} = -\frac{i \cos\phi}{2\rho}o, \qquad \frac{\partial \imath}{\partial y} = \frac{i \cos\phi}{2\rho}\imath,$$

which follow from Eqs. (36) and (40), it is easy to see that

$$P_k(u_- o - u_+ \imath) \equiv -i\hbar \frac{\partial}{\partial x^k}(u_- o - u_+ \imath)$$
$$= (P_k^{(-1/2)} u_-)o - (P_k^{(1/2)} u_+)\imath \qquad (77)$$

($k = 1,2$) with $P_k^{(s)}$ given by Eqs. (72). From Eqs. (68), (74) and (76–77) we conclude that the separable solutions of the Dirac equation (48) and (59–60) are eigenfunctions of J_3 and of $P_1^2 + P_2^2$ with eigenvalues $m\hbar$ and $\hbar^2\alpha^2$.

The fact that Eqs. (49) can be reduced to two independent pairs of differential equations [Eqs. (51–52)] is related with the existence of an operator K that commutes with the Dirac hamiltonian, J_3 and $P_1^2 + P_2^2$. From Eqs. (17), (19), (58) and (63) one finds that

$$\tilde{Q} X_{\kappa m} = \kappa X_{\kappa m} \quad (\kappa = \pm\alpha), \qquad (78)$$

where

$$\tilde{Q} \equiv \begin{bmatrix} 0 & -\bar{\eth} \\ \eth & 0 \end{bmatrix}. \qquad (79)$$

Then, by defining the operator

$$\tilde{K} \equiv \hbar \begin{bmatrix} -\tilde{Q} & 0 \\ 0 & \tilde{Q} \end{bmatrix}, \qquad (80)$$

one finds that each term in the right-hand side of Eq. (57) is an eigenfunction of $\tilde{K}$ with eigenvalue $-\hbar\alpha$ and $\hbar\alpha$, respectively. With respect to the basis $\left\{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \right\}$, the operator $\tilde{Q}$ is given by

$$Q \equiv \Lambda^{-1} \tilde{Q} \Lambda = \frac{1}{\hbar}(\sigma_1 P_2 - \sigma_2 P_1). \qquad (81)$$

Therefore, in the representation used in Eq. (46), the operator $\tilde{K}$ corresponds to

$$K = \hbar \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} P_1 - \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} P_2$$
$$= \gamma_5(\gamma_2 P_1 - \gamma_1 P_2), \qquad (82)$$

with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. From Eq. (81) it follows that $Q^2 = (P_1^2 + P_2^2)/\hbar^2$ and hence

$$K^2 = P_1^2 + P_2^2. \qquad (83)$$

6. RELATIONSHIP WITH THE EUCLIDEAN GROUP OF THE PLANE

According to Eqs. (68) and (74), the spin-weighted cylindrical functions ${}_sF_{\alpha m}$ are the common eigenfunctions of $J_3^{(s)}$ and $P_1^{(s)2} + P_2^{(s)2}$ with eigenvalues $m\hbar$ and $\hbar^2\alpha^2$, respectively, with the operators $J_3^{(s)}$ and $P_k^{(s)}$ being defined by Eqs. (66) and (72). A straightforward computation shows that, for a fixed value of s, the operators $J_3^{(s)}$ and $P_k^{(s)}$ satisfy the commutation relations of the Lie algebra of the euclidean group of the plane

$$\begin{aligned} \left[P_1^{(s)}, P_2^{(s)}\right] &= 0, \\ \left[J_3^{(s)}, P_1^{(s)}\right] &= i\hbar P_2^{(s)}, \\ \left[J_3^{(s)}, P_2^{(s)}\right] &= -i\hbar P_1^{(s)}. \end{aligned} \tag{84}$$

As a consequence of Eqs. (84), $P_1^{(s)2} + P_2^{(s)2}$ commutes with $P_1^{(s)}, P_2^{(s)}$ and $J_3^{(s)}$, and since

$$\left[J_3^{(s)}, P_1^{(s)} \pm iP_2^{(s)}\right] = \pm\hbar\left(P_1^{(s)} \pm iP_2^{(s)}\right), \tag{85}$$

$\left(P_1^{(s)} \pm iP_2^{(s)}\right) {}_sF_{\alpha m}$ is proportional to a function ${}_sF_{\alpha,m\pm1}$. In fact, using the recurrence relations for the Bessel functions one finds that the spin-weighted functions ${}_sZ_{\alpha m}$, defined by Eq. (13), satisfy

$$\left(P_1^{(s)} \pm iP_2^{(s)}\right) {}_sZ_{\alpha m} = \pm i\hbar\alpha\, {}_sZ_{\alpha,m\pm1}. \tag{86}$$

Furthermore, using the fact that the operators $P_k^{(s)}$ and $J_3^{(s)}$ do not change the spin-weight, it can be seen that $\eth P_k^{(s)} = P_k^{(s+1)}\eth$, $\eth J_3^{(s)} = J_3^{(s+1)}\eth$, $\bar{\eth} P_k^{(s)} = P_k^{(s-1)}\bar{\eth}$, $\bar{\eth} J_3^{(s)} = J_3^{(s-1)}\bar{\eth}$.

Thus, for fixed values of s and α, the functions ${}_sF_{\alpha m}$ form a basis of an infinite-dimensional representation of the euclidean group of the plane, $E(2)$. Owing to the invariance of the area element $dx\,dy = \rho\, d\rho\, d\phi$ under the rigid motions of the plane, the operators that translate and rotate the functions defined on the plane are unitary with respect to the inner product

$$(f, g) \equiv \int_0^{2\pi} \int_0^{\infty} \overline{f(\rho,\phi)} g(\rho,\phi) \rho\, d\rho\, d\phi.$$

In particular, the set of functions $\{{}_sJ_{\alpha m}\}$, with s and α fixed and $\alpha \neq 0$, is a basis of an irreducible infinite-dimensional representation of $E(2)$. As a consequence of the unitarity of the operators corresponding to the rigid motions of the plane, the expression

$$\sum_{m=-\infty}^{\infty} \overline{{}_0J_{\alpha m}(\rho_1,\phi_1)}\,{}_0J_{\alpha m}(\rho_2,\phi_2) = \sum_{m=-\infty}^{\infty} J_m(\alpha\rho_1)J_m(\alpha\rho_2)e^{im(\phi_2-\phi_1)},$$

for real α, is invariant under translations and rotations; therefore, using the fact that $J_m(0) = \delta_{m0}$ one obtains the addition theorem

$$J_0(\alpha R) = \sum_{m=-\infty}^{\infty} J_m(\alpha\rho_1)J_m(\alpha\rho_2)e^{im(\phi_2-\phi_1)}, \tag{87}$$

where $R = \sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\phi_2-\phi_1)}$ is the distance between the points of coordinates (ρ_1,ϕ_1) and (ρ_2,ϕ_2) (*cf.*, for example, Ref. [8]).

7. Concluding remarks

According to the results of Sects. 2, 5 and 6, the spin-weighted cylindrical functions ${}_sF_{\alpha m}$ have many properties analogous to those of the spin-weighted spherical harmonics. Even though any expression given in terms of the ${}_sF_{\alpha m}$ can also be written in terms of Bessel functions and other well-known functions, the examples given in Sects. 3 and 4 show the advantages of using the specific combinations given by the spin-weighted functions ${}_sF_{\alpha m}$.

The results of Sects. 4 and 5 show that the Dirac equation can be solved in cylindrical coordinates by finding, as a first step, the common eigenfunctions of the operators J_3 and K [Eq. (82)]. In fact, it is easy to see that the Dirac hamiltonian can be written as

$$H = c\alpha_3 P_3 + ic\alpha_3\beta K + \beta mc^2. \tag{88}$$

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RESUMEN. Se presenta una clase de funciones definidas en el plano y se ilustra su utilidad resolviendo la ecuación vectorial de Helmholtz y la ecuación de Dirac en coordenadas cilíndricas por separación de variables. Se muestra que las soluciones separables que se obtienen son eigenfunciones del cuadrado del momento lineal perpendicular al eje z y de la componente z del momento angular total. Se muestra también que las funciones introducidas aquí forman bases para representaciones del grupo euclideano del plano.