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Dirac spinors and curvature in the null tetrad formulation of general relativity

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ABSTRACT. Using the fully covariant form of the Dirac spinors and matrices in the null tetrad formalism, we obtain compact expressions for the structure equations and Bianchi identities in Riemannian spaces. Dirac spin coefficients, connections and curvature terms are presented in detail. The formalism yields a direct separation of the connection coefficients and Riemann tensor between selfdual and antiselfdual parts. The relation with Yang-Mills fields becomes explicit.

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1. INTRODUCTION

Spinor calculus has proved to be an extremely useful mathematical tool in general relativity. The basic ingredients are the spin coefficients which transform according to the irreducible representations of the SL$(2,\mathbb{C})$ group. These coefficients can be expressed most naturally in terms of the components of tensorial objects related to a null tetrad basis. Several versions of this formalism have appeared in the literature since the original work of Newman and Penrose [1–4].

The most basic spinors in the null tetrad formulation of general relativity are two-component spinors which transform according to the $D(\frac{1}{2},0)$ or $D(0,\frac{1}{2})$ representations of SL$(2,\mathbb{C})$. On the other hand, the Dirac spinors which describe spin 1/2 particles are four-component spinors; strictly speaking, they are pairs of $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$ two-component spinors. It is possible to express the Dirac equation in a generally covariant form [5], valid in curved space [6–7]. Furthermore, several authors have studied the formulation of Dirac's equation and γ matrices directly in the null tetrad formalism, mainly in connection with supergravity [8–10].

A particularly compact version of the null tetrad formalism, based on the formulation of Ref. [3], was developed by Plebanski [4]. The purpose of the present paper is to formulate the basic equations of general relativity in terms of Dirac spinors and matrices in Plebanski's formalism, and to obtain a new representation of the curvature tensor and its related equations. Previous authors [8–9] have obtained explicit forms for the connection coefficients in terms of the Newman-Penrose coefficients, but, as far as we are aware of, the curvature tensor has not been given in terms of Dirac matrices. Here, we also obtain some useful forms of the Cartan structure equations and Bianchi identities, which have the advantage of being quite compact and easy to handle. We show that a decomposition of the Riemann tensor into selfdual and antiselfdual parts follows in a natural way, and this in turn leads to a manifest connection with Yang-Mills fields.

The null tetrad formulation of general relativity is briefly reviewed in Sect. 2 for the sake of completeness. The Dirac equation is discussed in Sect. 3, where the appropriate representation of the γ matrices is presented. Section 4 deals with the Riemann curvature tensor, its representations in terms of Dirac matrices and its relation with Yang-Mills fields.

2. Preliminaries

In this section we include a brief summary of the null-tetrad formalism. The reader is referred to Ref. [4] for more details.

Let V_4 be a Riemannian manifold with metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = 2e^1e^2 + 2e^3e^4, \tag{1}$$

where the one-forms $e^a = e^a_\mu dx^\mu$ $(a = 1, \ldots, 4)$ define a null tetrad such that

$$e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}, \tag{2}$$

and

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

The inverse tetrad e^μ_a is defined through the condition $e^\mu_a e^b_\mu = \delta^b_a$. In the following, latin and greek indices will always refer to tetradial and tensorial components respectively (thus, for instance, for a vector V^α: $V^a = V^\alpha e^a_\alpha$ and $V^\alpha = V^a e^\alpha_a$).

If V_4 is real and has signature $(-+++)$, then under complex conjugation

$$(e^1, e^2, e^3, e^4)^* = (e^2, e^1, e^3, e^4), \tag{4}$$

while if V_4 is Euclidean then [11]

$$(e^1, e^2, e^3, e^4)^* = (e^2, e^1, e^4, e^3). \tag{5}$$

The Ricci coefficients are defined as

$$\Gamma^a{}_{bc} = -e^a_{\mu;\nu} e^\mu_b e^\nu_c, \tag{6}$$

and have the property $\Gamma_{abc} = \Gamma_{[ab]c}$. Defining the one-forms

$$\Gamma_{ab} = \Gamma_{abc} e^c, \tag{7}$$

one obtains the basic Cartan structure equations:

$$de^a = e^b \wedge \Gamma^a{}_b, \tag{8}$$

$$d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c{}_b = R_{abcd} e^c \wedge e^d, \tag{9}$$

where R_{abcd} are the tetradial components of the Riemann tensor.

3. Dirac equation and matrices

A Dirac spinor ψ is a pair of two spinors Ψ_A and $\Phi^{A'}$ which transform according to the $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$ representations of the SL(2, $\mathbb{C}$) group; that is,

$$\psi = \begin{pmatrix} \Psi_A \\ \Phi^{A'} \end{pmatrix}. \tag{10}$$

Then, the Dirac equation can be written in the traditional form

$$\gamma^\mu \nabla_\mu \psi + im\psi = 0. \tag{11}$$

Similarly, for the adjoint Dirac spinor,

$$\bar{\Psi} = (\Phi^A, \Psi_{A'}), \tag{12}$$

one has

$$\bar{\psi} \overleftarrow{\nabla}_\mu \gamma^\mu - im\bar{\psi} = 0. \tag{13}$$

Here, γ^μ are the Dirac matrices which satisfy the relation

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2g^{\alpha\beta}, \tag{14}$$

and the covariant derivative is defined by

$$\nabla_\mu \Psi = \partial_\mu \Psi + \Gamma_\mu \Psi, \tag{15}$$

where Γ_μ are connection matrices to be defined in the following. Similarly,

$$\bar{\Psi} \overleftarrow{\nabla}_\mu = \bar{\Psi} \overleftarrow{\partial}_\mu - \bar{\Psi} \Gamma_\mu. \tag{16}$$

The tetradial version of the Dirac matrices is $\gamma^n = e^n_\mu \gamma^\mu$, and its inverse is $\gamma^\mu = e^\mu_n \gamma^n$. The most natural representation of the γ^a matrices in the null tetrad basis is

$$\begin{aligned}
\gamma^1 &= \sqrt{2}\begin{pmatrix} \mathbf{0} & \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \mathbf{0} \end{pmatrix}, \qquad
\gamma^2 = \sqrt{2}\begin{pmatrix} \mathbf{0} & \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & \mathbf{0} \end{pmatrix}, \\
\gamma^3 &= \sqrt{2}\begin{pmatrix} \mathbf{0} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & \mathbf{0} \end{pmatrix}, \qquad
\gamma^4 = \sqrt{2}\begin{pmatrix} \mathbf{0} & \begin{matrix} -1 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & -1 \end{matrix} & \mathbf{0} \end{pmatrix},
\end{aligned} \tag{17}$$

and they satisfy the anticommutation rules

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}. \tag{18}$$

Notice that in this representation all γ^a are real and their transpose is $(\gamma^a)^T = -\gamma_a$.

Now, it is convenient to define the matrices

$$\sigma^{ab} = \tfrac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a), \tag{19}$$

which have the explicit form:

$$\begin{aligned}
&\sigma^{12} + \sigma^{34} = 2\begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \sigma^{31} = 2\begin{pmatrix} \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \sigma^{42} = 2\begin{pmatrix} \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\
&-\sigma^{12} + \sigma^{34} = 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}, \sigma^{32} = 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} \end{pmatrix}, \sigma^{41} = 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{pmatrix}
\end{aligned} \tag{20}$$

It is also convenient to define

$$\begin{aligned}
\gamma^5 &\equiv \tfrac{1}{24}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d, \\
&= \tfrac{i}{24}\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta,
\end{aligned} \tag{21}$$

which turns out to be

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{22}$$

and anticommutes with all matrices γ:

$$\gamma^5\gamma^a + \gamma^a\gamma^5 = 0. \tag{23}$$

(A word of caution on the use of the antisymmetric symbol: ϵ_{abcd} is defined without sign ambiguity in the null tetrad representation since $\epsilon_{1234} = \epsilon^{1234} = 1$. However, its coordinate image is $e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta \epsilon_{abcd} = i\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}$, where it is understood that $\epsilon_{0123} = -1$ and $\epsilon^{0123} = 1$ if the signature is $(-+++)$ and if $-e^3_\mu + e^4_\mu$ is a future pointing four-vector.)

The dual of σ^{ab} is

$$\sigma^*_{ab} \equiv \tfrac{1}{2}\epsilon_{abcd}\sigma^{cd}, \tag{24}$$

and it follows that

$$\sigma^*_{ab} \equiv -\gamma^5\sigma_{ab}. \tag{25}$$

Now, the connection matrix Γ_n defined by Eq. (15) is related to the Ricci coefficients Γ_{abc} by

$$\Gamma_n = -\tfrac{1}{4}\Gamma_{abn}\sigma^{ab}, \tag{26}$$

and obviously $\Gamma_\mu = e^n_\mu\Gamma_n$. The explicit form of the connection matrix in the null tetrad basis turns out to be

$$\Gamma_n = \begin{pmatrix} -\frac{1}{2}(\Gamma_{12n}+\Gamma_{34n}) & -\Gamma_{42n} & 0 & 0 \\ -\Gamma_{31n} & \frac{1}{2}(\Gamma_{12n}+\Gamma_{34n}) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\Gamma_{12n}+\Gamma_{34n}) & -\Gamma_{32n} \\ 0 & 0 & -\Gamma_{41n} & -\frac{1}{2}(-\Gamma_{12n}+\Gamma_{34n}) \end{pmatrix}. \tag{27}$$

Finally, we note that the covariant derivatives of the Dirac matrices must vanish [5–7], from where follow the important relations:

$$\nabla_\alpha\gamma^\beta \equiv \partial_\alpha\gamma^\beta + \Gamma^\beta_{\alpha\mu}\gamma^\mu + \left[\Gamma_\alpha, \gamma^\beta\right] = 0, \tag{28}$$

$$\nabla_\alpha\sigma^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,\sigma^{\alpha\beta}) + \left[\Gamma_\alpha, \sigma^{\alpha\beta}\right] = 0, \tag{29}$$

where $\Gamma^\beta_{\alpha\mu}$ is the standard Christoffel symbol and ∇_α is the covariant derivative which also takes spinorial indices into account [its explicit form is given by equation (28)]. Eq. (28) is just another way of writing the Cartan equation (8), and Eq. (20) simply follows from (28). Eq. (29) is actually a set of 24 equations which permits to calculate the 24 linearly independent components of Γ_α (which are linear combinations of the Ricci coefficients) in terms of $\sigma_{\alpha\beta}$.

4. THE RIEMANN TENSOR

The commutator of covariant derivatives involves the Riemann tensor through the equation

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\psi = -\tfrac{1}{4} R_{abcd}\sigma^{cd}\psi, \tag{30}$$

or its conjugate form

$$\bar{\psi}\big(\overleftarrow{\nabla}_a \overleftarrow{\nabla}_b - \overleftarrow{\nabla}_b \overleftarrow{\nabla}_a\big) = -\tfrac{1}{4}\bar{\psi} R_{abcd}\sigma^{cd} \tag{31}$$

(Our conventions for the Riemann and Ricci tensors are $V_{\alpha;\beta;\gamma} - V_{\alpha;\gamma;\beta} = V_\mu R^\mu{}_{\alpha\beta\gamma}$ and $R_{\alpha\beta} = R^\mu{}_{\alpha\beta\mu}$.)

From the Cartan equation (9), it follows that

$$\partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha + \big[\Gamma_\alpha, \Gamma_\beta\big] = -\tfrac{1}{4}\boldsymbol{R}_{\alpha\beta}, \tag{32}$$

where we have defined

$$\boldsymbol{R}_{\alpha\beta} \equiv R_{\alpha\beta\gamma\delta}\sigma^{\gamma\delta} = e^a_\alpha e^b_\beta R_{abcd}\gamma^c\gamma^d. \tag{33}$$

We can evaluate the explicit forms of $\boldsymbol{R}_{ab} = e^\alpha_a e^\beta_b \boldsymbol{R}_{\alpha\beta}$ in terms of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and the Ricci tensor $R_{\alpha\beta}$. Thus,

$$R^{\alpha\beta}{}_{\gamma\delta} \equiv C^{\alpha\beta}{}_{\gamma\delta} + \frac{1}{2}\delta^{\alpha\beta\rho}_{\gamma\delta\sigma} C^\sigma_\rho - \frac{R}{12}\delta^{\alpha\beta}_{\gamma\delta}, \tag{34}$$

where $C^\sigma_\rho = R^\sigma_\rho - (R/4)\delta^\nu_\rho$.

From Eq. (34) we obtain the more compact formula

$$\boldsymbol{R}_{\alpha\beta} = \boldsymbol{C}_{\alpha\beta} + 2\sigma_{\mu[\alpha} C_{\beta]} - \frac{R}{6}\sigma_{\alpha\beta}, \tag{35}$$

where $\boldsymbol{C}_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\sigma^{\gamma\delta}$. Notice in particular that

$$\sigma^{\alpha\beta}\boldsymbol{R}_{\alpha\beta} = 2R \tag{36}$$

and

$$\sigma^{\alpha\beta}\boldsymbol{C}_{\alpha\beta} = 0. \tag{37}$$

Now, if we define the dual

$$^{*}\boldsymbol{R}_{ab} = \tfrac{1}{2}\epsilon_{abcd}\boldsymbol{R}^{cd}, \tag{38}$$

it follows that in the null tetrad basis:

$$\begin{aligned} &^{*}\boldsymbol{R}_{42} = -\boldsymbol{R}_{42}, && ^{*}\boldsymbol{R}_{41} = \boldsymbol{R}_{41}, \\ &^{*}\boldsymbol{R}_{31} = -\boldsymbol{R}_{31}, && ^{*}\boldsymbol{R}_{32} = \boldsymbol{R}_{32}, \\ &^{*}\boldsymbol{R}_{12} + {}^{*}\boldsymbol{R}_{34} = -(\boldsymbol{R}_{12} + \boldsymbol{R}_{34}), && -{}^{*}\boldsymbol{R}_{12} + {}^{*}\boldsymbol{R}_{34} = -\boldsymbol{R}_{12} + \boldsymbol{R}_{34}. \end{aligned} \tag{39}$$

The explicit form of these matrices is given in the Appendix.

It is also possible to separate the Riemann tensor into its selfdual and antiselfdual parts by defining

$$\boldsymbol{R}^{\pm}_{ab} = \tfrac{1}{2}(\boldsymbol{R}_{ab} \pm {}^{*}\boldsymbol{R}_{ab}) \tag{40}$$

and similarly for the Weyl tensor

$$\boldsymbol{C}^{\pm}_{ab} = \tfrac{1}{2}(\boldsymbol{C}_{ab} \pm {}^{*}\boldsymbol{C}_{ab}). \tag{41}$$

It then follows that

$$\begin{aligned} \boldsymbol{R}_{ab} &= \boldsymbol{C}_{ab} + 2\sigma_{n[a}C^{n}_{b]} - \frac{R}{6}\sigma_{ab}, \\ ^{*}\boldsymbol{R}_{ab} &= -\gamma^{5}(\boldsymbol{C}_{ab} - 2\sigma_{n[a}C^{n}_{b]} - \frac{R}{6}\sigma_{ab}) \end{aligned} \tag{42}$$

and

$$^{*}\boldsymbol{C}_{ab} = -\gamma^{5}\boldsymbol{C}_{ab}. \tag{43}$$

Thus, for instance,

$$\begin{aligned} \boldsymbol{C}_{ab} &= \frac{1}{2}\boldsymbol{R}_{ab} - \frac{1}{2}\gamma^{5*}\boldsymbol{R}_{ab} + \frac{R}{6}\sigma_{ab} \\ &= \frac{1}{2}(1+\gamma^{5})\boldsymbol{R}^{+}_{ab} + \frac{1}{2}(1-\gamma^{5})\boldsymbol{R}^{-}_{ab} + \frac{R}{6}\sigma_{ab} \end{aligned} \tag{44}$$

Notice that, as a consequence of Eq. (42), the vacuum Einstein equations (with cosmological constant) are entirely equivalent to the condition

$$\boldsymbol{R}_{ab} = -\gamma^{5*}\boldsymbol{R}_{ab}. \tag{45}$$

Let us now consider the Bianchi identities $R_{\alpha\beta[\gamma\delta;\epsilon]} = 0$. According to our analysis, they are equivalent to the equations

$$\nabla_b {}^*\boldsymbol{R}^{ab} = 0 \qquad \text{or} \qquad \nabla_{[a}\boldsymbol{R}_{bc]} = 0, \tag{46}$$

which can also be written in coordinate form as

$$\frac{1}{\sqrt{-g}}\partial_\beta\left(\sqrt{g}\ {}^*\boldsymbol{R}^{\alpha\beta}\right) + \left[\Gamma_\beta, {}^*\boldsymbol{R}^{\alpha\beta}\right] = 0. \tag{47}$$

On the other hand, the contracted form of the Bianchi identities,

$$\nabla^n R_{anbc} = \nabla_b R_{ca} - \nabla_c R_{ba}, \tag{48}$$

leads to the relation

$$\nabla^n \boldsymbol{R}_{an} = 2\sigma_{ns}\nabla^n R^s{}_a. \tag{49}$$

It is interesting to note that Eqs. (49) and (32) are similar to Yang-Mills field equations with the right hand side of (49) acting as a source.

5. CONCLUSIONS

The algebra of Dirac spinors and matrices which is currently used in quantum field theory can be extended in a natural way to Riemannian spaces through a combination with the null tetrad formalism. The resulting formulae have the advantage of being compact and of providing a direct separation between selfdual and antiselfdual objects.

Thus, one can chose $\sigma_{\alpha\beta}$ as the basic objects and evaluate Γ_α and $\boldsymbol{R}_{\alpha\beta}$ through the structure equations (29) and (32). As it was first noticed by Plebanski [12], this approach permits a direct separation between selfdual and antiselfdual parts of the Einstein equations in vacuum. The Einstein equations can also be introduced through the Eq. (49), which has precisely the form of a Yang-Mills equation with a source (which vanishes in vacuum), and where the gauge group is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Furthermore, the decomposition of $\boldsymbol{R}_{\alpha\beta}$ between its conformal and Ricci part is obvious in Eq. (42).

Equation (32), which relates $\boldsymbol{R}_{\alpha\beta}$ and Γ_α, has the familiar form of a Yang-Mills equation in which the field $\boldsymbol{R}_{\alpha\beta}$ is defined in terms of the potentials Γ_α. The integrability conditions of this equation are just the Bianchi identities. In order to have a complete Yang-Mills theory, one needs an additional equation which relates the derivatives of the field with the sources. This second equation can be obtained from the Bianchi identities. For instance, it is clear that any field satisfying the condition (45) also satisfies the source free Yang-Mills equations as a consequence of the Bianchi identities written as in Eq. (47). However, we have shown that Eq. (45) is precisely the condition that the space be vacuum.

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APPENDIX

The explicit forms of the matrices $\boldsymbol{R}_{ab}$ in terms of the tetrad components of the Ricci tensor and the spinor components of the Weyl tensor are

$$\boldsymbol{R}_{31} = 2\begin{pmatrix} C^{(2)} & -C^{(3)}+\frac{R}{6} & 0 & 0 \\ C^{(1)} & -C^{(2)} & 0 & 0 \\ 0 & 0 & -R_{31} & R_{33} \\ 0 & 0 & -R_{11} & R_{31} \end{pmatrix}, \tag{A1}$$

$$\boldsymbol{R}_{42} = 2\begin{pmatrix} C^{(4)} & -C^{(5)} & 0 & 0 \\ C^{(3)}-\frac{R}{6} & -C^{(4)} & 0 & 0 \\ 0 & 0 & R_{42} & R_{22} \\ 0 & 0 & -R_{44} & -R_{42} \end{pmatrix}, \tag{A2}$$

$$\frac{1}{2}\left(\boldsymbol{R}_{12}+\boldsymbol{R}_{34}\right) = 2\begin{pmatrix} C^{(3)}+\frac{R}{12} & -C^{(4)} & 0 & 0 \\ C^{(2)} & -C^{(3)}-\frac{R}{12} & 0 & 0 \\ 0 & 0 & \frac{1}{2}(R_{12}-R_{34}) & -R_{32} \\ 0 & 0 & -R_{41} & \frac{1}{2}(-R_{12}+R_{34}) \end{pmatrix}, \tag{A3}$$

$$\boldsymbol{R}_{32} = 2\begin{pmatrix} R_{32} & R_{22} & 0 & 0 \\ -R_{33} & -R_{32} & 0 & 0 \\ 0 & 0 & -\bar{C}^{(2)} & -\bar{C}^{(1)} \\ 0 & 0 & \bar{C}^{(3)}-\frac{R}{6} & \bar{C}^{(2)} \end{pmatrix}, \tag{A4}$$

$$\boldsymbol{R}_{41} = 2\begin{pmatrix} -R_{41} & R_{44} & 0 & 0 \\ -R_{11} & R_{41} & 0 & 0 \\ 0 & 0 & -\bar{C}^{(4)} & -\bar{C}^{(3)}+\frac{R}{6} \\ 0 & 0 & \bar{C}^{(5)} & \bar{C}^{(4)} \end{pmatrix}, \tag{A5}$$

and

$$\frac{1}{2}\big(\boldsymbol{R}_{12}-\boldsymbol{R}_{34}\big)=2\begin{pmatrix} \frac{1}{2}(R_{12}-R_{34}) & -R_{42} & 0 & 0 \\ -R_{31} & \frac{1}{2}(-R_{12}+R_{34}) & 0 & 0 \\ 0 & 0 & \bar{C}^{(3)}_{12}+\frac{R}{12} & -\bar{C}^{(2)} \\ 0 & 0 & -\bar{C}^{(4)} & -\bar{C}^{(3)}-\frac{R}{12} \end{pmatrix}, \tag{A6}$$

where $C^{(a)}$ and $\bar{C}^{(a)}$ are the spinor images of the Weyl tensor [5].

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RESUMEN. En este trabajo se utiliza la forma covariante de los espinores y matrices de Dirac en el formalismo de tétrada nula, para obtener en forma compacta expresiones para las ecuaciones de estructura y las identidades de Bianchi en espacios riemannianos. Se presentan en detalle los coeficientes espinoriales y los términos de curvatura. El formalismo permite una separación directa de los coeficientes de conexión y el tensor de Riemann en partes autoduales y antiautoduales. También, la relación con campos de Yang-Mills se vuelve explícita.