

# Quantum phase space for an ideal relativistic gas in $d$ spatial dimensions

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**ABSTRACT.** We present the closed formula for the  $d$ -dimensional invariant phase-space integral for an ideal relativistic gas in an exact integral form. In the particular cases of the nonrelativistic and the extreme relativistic limits the phase-space integrals are calculated analytically. Then we consider the  $d$ -dimensional invariant phase space with quantum statistics and derive the cluster decomposition for the grand canonical and canonical partition functions as well as for the microcanonical and grand microcanonical densities of states. As a showcase, we consider the black-body radiation in  $d$  dimensions.

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## 1. INTRODUCTION

The general theory of ideal (relativistic) quantum gases has been widely discussed in the literature [1] and applied to various physical situations, such as multiple production of particles, neutron stars [2], and so forth. From the approaches based on the canonical and the grand canonical ensembles, which are the thermodynamical ones, one can develop for the ideal quantum gases, obeying Bose-Einstein (BE) and/or Fermi-Dirac (FD) statistics, a formalism very similar to that for real classical interacting gases, such as a cluster decomposition and a virial expansion [3]. These analogies exist since ideal Bose and Fermi gases have the properties of real gases (*i.e.* with interactions) if considered from the standpoint of classical Boltzmann statistics. Much less attention has been paid to the formalism based on the microcanonical distribution (see, however Refs. [4] and [5]), where the conservation of energy and momentum is fully taken into account. A general statistical approach for the ideal relativistic gases based on the microcanonical distribution was developed in Ref. [5]. While this formalism can be used for the calculation of the phase-space integrals encountered in particle physics, the other previously considered thermodynamical

ones are not applicable to this problem, since they lack of the constraints of energy-momentum conservation. In Ref. [5] an exact statistical cluster decomposition for the ideal relativistic quantum gases was derived in the formalism of the microcanonical distribution and the quantum statistical effects on the phase space were found to be rather important for elementary particle applications. The formalism developed for the calculation of the phase-space integrals with the cluster decomposition can be directly applied in the Frautschi's [6] formulation of the statistical bootstrap model [7] with the correct quantum statistics [8].

The effects of dimensionality have also been studied in the past for nonrelativistic [9] as well as relativistic [10] thermodynamical systems. However such study for the phase space with or without statistics has not been performed yet. In this paper, which is the straightforward extension of Ref. [5] to  $d$ -spatial dimensional space, we present the exact (grand) microcanonical formalism for the invariant phase space with correct quantum statistics for the ideal quantum relativistic gases, together with the (grand) canonical one.

We organize the paper as follows: In Sect. 2 we derive the closed formula for the invariant phase-space integral for an ideal relativistic gas in  $d$ -spatial dimensions in the form of a one-dimensional integral over the modified Bessel function. For the two limiting cases, namely, in the nonrelativistic limit and in the extreme relativistic limit we evaluate the phase-space integrals analytically. In Sect. 3 we develop a formalism for the invariant phase space with correct quantum statistics in  $d$ -spatial dimensions and derive the cluster decomposition for the grand canonical and canonical partition functions as well as for the microcanonical and grand microcanonical densities of states. These densities of states are expressed in terms of the ordinary relativistic (with BO statistics) phase-space integral in which appears "cluster particle" mass(es). We consider the black-body radiation in  $d$  dimensions and derive the generalized Stefan-Boltzmann law where the dimensional dependence is visible clearly.

## 2. PHASE-SPACE INTEGRAL IN $d$ SPATIAL DIMENSIONS

Let us assume that the system of an ideal relativistic quantum gas is enclosed in a box with quantum  $(d+1)$ -volume  $\omega_d$ , which is defined in the generalized Minkowski space as (hereafter we use the units  $\hbar = c = k = 1$ )

$$\omega_{d\mu} = \frac{V_d u_\mu}{h^d} = \frac{V_d u_\mu}{(2\pi)^d}, \quad (1)$$

where  $u_\mu$  is a  $(d+1)$ -velocity of the frame of reference, in which the box of volume  $V_d$  is at rest, satisfying

$$u_\mu u^\mu = u_0^2 - \sum_{i=1}^d u_i^2 = 1. \quad (2)$$

Then the  $(d+1)$ -dimensional invariant  $N$ -particle phase-space integral

$$R_N^{(d)}(P, m_1, m_2, \dots, m_N)$$

for the system with total  $(d+1)$ -momenta  $P$  is defined by

$$R_N^{(d)}(P, m_1, m_2, \dots, m_N) = \int \delta^{(d+1)}\left(P - \sum_{i=1}^N p_i\right) \times \prod_{i=1}^N d\sigma^{(d)}(p_i, m_i), \quad (3)$$

where  $d\sigma^{(d)}$  denotes the Touschek's invariant phase-space measure [11] in  $d$  dimensions

$$\begin{aligned} d\sigma^{(d)}(p_i, m_i) &= 2g(\omega_d p_i) \theta(p_{i0}) \delta(p_i^2 - m_i^2) d^{(d+1)} p_i \\ &= \frac{g(\omega_d p_i)}{p_{i0} d^{(d)} p_i}, \end{aligned} \quad (4)$$

with  $g$  being the statistical weight. Note that Eq. (3) is the canonical partition for an ideal BO gas in  $d$ -dimensions. Define the Laplace transform of  $R_N^{(d)}$  as

$$\Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N) = \int \exp(-\beta p) R_N^{(d)}(P, m_1, m_2, \dots, m_N) d^{(d+1)} P, \quad (5)$$

where  $\beta$  is the inverse temperature:  $\beta\mu = u\mu/T$  (in the common rest frame of  $\beta$  and  $\omega_d$ ). Then we have

$$\Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N) = \prod_{i=1}^N \phi^{(d)}(\beta, m_i), \quad (6)$$

where [12]

$$\phi^{(d)}(\beta, m_i) = \int \exp(-\beta p) d\sigma^{(d)}(p, m_i). \quad (7)$$

In the common rest frame of  $\beta$  and  $\omega_d$ ,  $\phi^{(d)}(\beta, m_i)$  is expressed as

$$\phi^{(d)}(\beta, m_i) = C_d(m_i\beta)^{-d''} K_{d'}(m_i\beta), \quad (8)$$

where we use the shorthand notations

$$d' = \frac{(d+1)}{2}, \quad d'' = \frac{(d-1)}{2} \quad (9)$$

and

$$C_d = \frac{gV_d m^d}{2^{d''} \pi^{d'}}. \quad (10)$$

$K_\nu(z)$  stands for the modified Bessel function of the second kind defined by [13]

$$K_\nu(z) = \frac{\Gamma(1/2)}{\Gamma(\nu+1/2)} \left(\frac{z}{2}\right)^\nu \int_1^\infty \exp(-zt)(t^2-1)^{\nu-1/2} dt. \quad (11)$$

In the  $\beta$ -rest frame, one can rewrite Eq. (5) as

$$\begin{aligned} \Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N) &= \int \exp(-\beta P_0) R_N^{(d)}(P, m_1, m_2, \dots, m_N) d^{(d+1)}P \\ &= S_d \int_0^\infty dP P^{(d-2)} R_N^{(d)}(P, m_1, m_2, \dots, m_N) \\ &\quad \times \int_P^\infty dP_0 (P_0^2 - P^2)^{1/2} \exp(-\beta P_0) \\ &= \frac{S_d}{\beta} \int_0^\infty K_1(\beta P) R_N^{(d)}(P, m_1, m_2, \dots, m_N) P^{(d-1)} dP, \end{aligned} \quad (12)$$

where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (13)$$

Equation (12) can be rewritten as a  $K$ -transform [14]

$$\begin{aligned} \Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N) &= \left(\frac{S_d}{4\pi}\right) (4\pi\beta^{-3/2}) \int_0^\infty K_1(\beta P) \\ &\quad \times (\beta P)^{1/2} \left[ P^{d-3/2} R_N^{(d)}(P, m_1, m_2, \dots, m_N) \right] dP, \end{aligned} \quad (14)$$

which can be inverted. We get a rigorous formula for  $R_N^{(d)}$ , which for  $c > 0$ , reads as

$$R_N^{(d)}(P, m_1, m_2, \dots, m_N) = \frac{1}{\pi S_d P^{d-2}} \frac{1}{i} \int_{c-i\infty}^{c+i\infty} d\beta \beta^2 I_1(\beta P) \Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N). \quad (15)$$

A particular case of  $d = 3$  reproduces the result known in the literature [15]. All  $R_N^{(d)}(P, m_1, m_2, \dots, m_N)$  have dimension  $[P^{-(d+1)}]$ , since  $R_N^{(d)}(P, m_1, m_2, \dots, m_N) \times d^{(d+1)} P$  is dimensionless. The phase-space integral  $R_N^{(d)}$  in Eq. (3) can be calculated explicitly when all particles are nonrelativistic (NR) or extreme relativistic (ER).

### NR limit

One can determine this limit if one replaces both hand sides by their corresponding asymptotic forms for  $\beta \rightarrow \infty$ . As the range of integration in the r.h.s. of Eq. (14) extends in fact from  $M = \sum_{i=1}^N m_i > 0$  to  $\infty$  one can replace the function  $K_1(\beta P)$  by its asymptotic form for  $\beta P \rightarrow \infty$ , by using the relation

$$\lim_{z \rightarrow \infty} K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z). \quad (16)$$

One thus gets

$$\prod_{i=1}^N \phi^{(d),\text{NR}}(\beta, m_i) = \frac{2^{1/2} \pi^{d'} \beta^{-3/2}}{\Gamma(d/2)} \int_0^\infty \exp(-\beta P) [R_N^{(d),\text{NR}} P^{d-3/2}] dP, \quad (17)$$

where

$$\phi^{(d),\text{NR}}(\beta, m_i) = gV_d \exp(-m_i \beta) \left(\frac{m_i}{2\pi\beta}\right)^{d/2}. \quad (18)$$

The inverse Laplace transform of this equation yields

$$R_N^{(d),\text{NR}}(P, m_1, m_2, \dots, m_N) = \frac{(gV_d)^N \Gamma(d/2) 2^{-(Nd+1)/2} \pi^{-(Nd/2+d')}}{\Gamma((Nd-3)/2)} \times \frac{\prod_{i=1}^N m_i^{d/2}}{\left(\sum_{i=1}^N m_i\right)^{d-3/2}} \left(P - \sum_{i=1}^N m_i\right)^{(Nd-5)/2}. \quad (19)$$

*ER limit*

By replacing the l.h.s. of Eq. (14) by its ER limit formula, we have

$$\prod_{i=1}^N \phi^{(d),\text{ER}}(\beta, m_i) = \left(\frac{S_d}{4\pi}\right) (4\pi\beta^{-3/2}) \int_0^\infty K_1(\beta P) \times (\beta P)^{1/2} \left[ P^{d-3/2} R_N^{(d),\text{ER}}(P, m_1, m_2, \dots, m_N) \right] dP, \quad (20)$$

where

$$\phi^{(d),\text{ER}}(\beta, m_i) = gV_d\beta^{-d} \frac{\Gamma(d')}{\pi^{d'}}. \quad (21)$$

Eq. (21) is obtained from Eq. (8) by using the relation

$$\lim_{z \rightarrow 0} z^\nu K_\nu(z) = \Gamma(\nu) 2^{\nu-1}. \quad (22)$$

The inverse  $K$ -transform [14] of this equation leads us to

$$R_N^{(d),\text{ER}}(P, m_1, m_2, \dots, m_N) = \frac{\left[ gV_d\Gamma(d') \right]^N \Gamma(d/2) 2^{-(Nd-2)} \pi^{-(Nd'+d/2)}}{\Gamma(Nd/2 - 1) \Gamma(Nd/2)} P^{(N-1)d-1}. \quad (23)$$

As can easily be checked, Eqs. (19) and (23) with  $d = 3$  reproduce the known results in the literature [16].

### 3. INVARIANT PHASE SPACE WITH STATISTICS IN $d$ SPATIAL DIMENSIONS

Consider a system of a relativistic ideal gas of one kind, which consists of  $N$  identical particles of mass  $m$ , obeying the BE/FD or BO statistics. The  $(d+1)$ -dimensional quantum phase space (density of states) for the system with total  $(d+1)$ -momenta  $Q$  is defined by

$$\sigma_N^{(d)}(Q, \omega_d) = \sum_{\{n\}} \delta^{(d+1)} \left( Q - \sum_{\alpha} q_{\alpha} n_{\alpha} \right) \delta_K \left( N - \sum_{\alpha} n_{\alpha} \right). \quad (24)$$

Here the occupation number  $n(q_{\alpha}) = n_{\alpha}$  indicates how many particles have momenta  $q_{\alpha}$  ( $\alpha = 0, 1, \dots, d$ ). The BE/FD statistics is accounted for by

$$n_{\alpha} = \begin{cases} 0, 1 & \text{for FD,} \\ 0, 1, \dots, \infty & \text{for BE.} \end{cases} \quad (25)$$

In Eq. (24) the Kronecker  $\delta_K$  fixes the particle numbers and the  $\delta^{(d+1)}$  selects the allowed  $(d+1)$ -momenta.

In parallel with the invariant phase space, we define the canonical partition function  $Z_N^{(d)}$  as the Laplace transform of  $\sigma_N^{(d)}(Q, \omega_d)$

$$Z_N^{(d)} = \int \exp(-\beta Q) \sigma_N^{(d)}(Q, \omega_d) d^{(d+1)}Q. \quad (26)$$

Inserting Eq. (24) into Eq. (26) and using a Fourier representation for the  $\delta_K$ , we obtain

$$Z_N^{(d)} = \sum_{\{n\}} \int_0^{2\pi} \exp(i\lambda N) \prod_{\alpha} \left\{ x_{\alpha} \exp(-i\lambda) \right\}^{n_{\alpha}} \frac{d\lambda}{2\pi}, \quad (27)$$

where we introduced the notation

$$x_{\alpha} = \exp(-\beta q_{\alpha}). \quad (28)$$

Introducing

$$z = \exp(-i\lambda), \quad (29)$$

Eq. (27) can be evaluated as

$$\begin{aligned} Z_N^{(d)} &= \frac{1}{2\pi i} \int \frac{dz}{z^{N+1}} Z^{(d)}(z, \gamma) \\ &= \frac{1}{N!} \left[ \frac{d^N Z^{(d)}(z, \gamma)}{dz^N} \right]_{z=0}, \end{aligned} \quad (30)$$

where

$$Z^{(d)}(z, \gamma) = \prod_{\alpha} (1 + \gamma z x_{\alpha})^{\gamma} \quad (31)$$

where  $\gamma = 1$  for FD and  $\gamma = -1$  for BE,  $Z^{(d)}(z, \gamma)$  is the grand canonical partition function,  $z$  is the fugacity and  $T \ln z = \mu$  is the chemical potential per particle. Next, replacing the discrete sum over states by an integral in the evaluation of  $\ln Z^{(d)}(z, \gamma)$  from Eq. (31), we obtain

$$\ln Z^{(d)}(z, \gamma) = \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1} z^k}{k} \int \exp(-k\beta q) d\sigma^{(d)}(q, m). \quad (32)$$

Note that the Boltzmann (BO) statistics case follows by putting  $\gamma = 0$  ( $0^0 = 1$ ) in Eq. (32). The integrand in this equation can be evaluated in the common rest frame of  $\beta$  and  $\omega_d$  by integrating over  $d$ -dimensional solid angle in the spherically symmetric momentum space, which yields

$$\int d^d q = S_d \int q^{d-1} dq. \tag{33}$$

Hence

$$Z^{(d)}(z, \gamma) = \exp \left[ C_d(m\beta)^{-d''} \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1} z^k}{k^{d'}} K_{d'}(km\beta) \right], \tag{34}$$

Proceeding as in Ref. [5], one can derive the formulas for the canonical function, cluster decomposition for the invariant microcanonical as well as grand microcanonical densities of states in  $d$ -spatial dimensions. Since the details of the evaluation are similar to Ref. [5] we quote in the following only the final results.

*Canonical partition function:*

$$Z_N^{(d)}(\gamma) = \sum_{\{n, N\}} \prod_{k=1}^N \frac{1}{n_k!} \frac{C_d(m\beta)^{-d''} (-\gamma)^{k-1}}{k^{d'} K_{d'}(km\beta)} n_k, \tag{35}$$

where  $\{n, N\}$  is the partition number of  $N$  satisfying

$$\sum_{k=1}^N kn_k = N.$$

In the BO case ( $\gamma = 0$ ), only one partition survives, namely,  $\{n, N\} = \{N, 0, \dots, 0\}$ .

*Invariant microcanonical density of states:*

$$\begin{aligned} \sigma_N^{(d)}(Q, \omega_d) &= \sum_{\{n, N\}} G^{(d)}\{(n, N), \gamma\} \int \delta^{(d+1)} \left( Q - \sum_{k=1}^N P_k \right) \\ &\times \prod_{k=1}^N R_N^{(d)}(P_k, km) d^{(d+1)} P_k, \end{aligned} \tag{36}$$



where

$$G^{(d)}\{(n, N), \gamma\} = \prod_{k=1}^N \frac{1}{n_k!} \frac{(-\gamma)^{k-1}}{k^{d+1}} n_k. \tag{37}$$

*Invariant grand microcanonical density of states with a fixed total four momentum and with chemical potential  $\mu = 0$ :*

$$\sigma^{(d)}(Q^2, \omega_d^2, Q\omega_d, \gamma) = \sum_{N=0}^{\infty} \frac{1}{N!} R_N^{(d),\text{eff}}(Q, m), \tag{38}$$

where

$$R_N^{(d),\text{eff}}(Q, m) = \sum_{k_1, k_2, \dots}^{\infty} \prod_{i=1}^N \frac{(-\gamma)^{k_i-1}}{k_i^{d+1}} R_N^{(d)}(Q, k_1 m, k_2 m, \dots, k_N m) \quad (N \text{ masses}). \tag{39}$$

Note that both the microcanonical and the grand microcanonical densities of states, Eqs. (36) and (38), are expressed in terms of the ordinary (with BO statistics) invariant phase-space integral  $R_N^{(d)}$ , in which appear “cluster particles” of mass  $km$  and of masses  $k_1 m, k_2 m, \dots, k_N m$ , respectively.

*Black-body radiation*

As an example, we treat the ideal gases with the  $m = 0$  case by using the grand microcanonical densities of states Eq. (38). We obtain

$$R_N^{(d),\text{eff}}(Q, 0) = R_N^{(d)}(Q, 0) \{f^{(d)}(\gamma)\}^N, \tag{40}$$

where

$$f^{(d)}(\gamma) = \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^{d+1}} = \begin{cases} \left(1 - \frac{1}{2^d}\right) \zeta(d+1) & \text{for FD,} \\ 1 & \text{for BO,} \\ \zeta(d+1) & \text{for BE,} \end{cases} \tag{41}$$

with  $\zeta(x)$  being the Riemann  $\zeta$ -function. Hence

$$\sigma^{(d)}(Q^2, \omega_d^2, Q\omega_d, \gamma) = \sum_{N=0}^{\infty} \frac{1}{N!} \{f^{(d)}(\gamma)\}^N R_N^{(d)}(Q, 0). \tag{42}$$

The Laplace transform of this equation yields

$$\begin{aligned}
 Z^{(d)}(\gamma) &= \sum_{N=0}^{\infty} \frac{1}{N!} \left[ f^{(d)}(\gamma) \right]^N \int d^{(d+1)}Q \exp(-\beta Q) \\
 &\quad \times \delta^{(d+1)} \left( Q - \sum_{i=1}^N q_i \right) \prod_{i=1}^N d\sigma^{(d)}(q_i, 0) \\
 &= \sum_{N=0}^{\infty} \frac{1}{N!} \left[ f^{(d)}(\gamma) \int \exp(-\beta q) d\sigma^{(d)}(q, 0) \right]^N \\
 &= \exp \left[ \lim_{m \rightarrow 0} f^{(d)}(\gamma) C_d(m\beta)^{-d''} K_{d'}(m\beta) \right] \\
 &= \exp \left[ \frac{f^{(d)}(\gamma) g V_d \beta^{-d} \Gamma(d')}{\pi^{d'}} \right]. \tag{43}
 \end{aligned}$$

where Eq. (22) is used in evaluating the last formula. The mean energy  $\langle E \rangle$  of a system is given by

$$\begin{aligned}
 \langle E \rangle &= - \frac{\partial \ln Z^{(d)}(\gamma)}{\partial \beta} \\
 &= f^{(d)}(\gamma) g V_d \beta^{-(d+1)} \frac{d\Gamma(d')}{\pi^{d'}}. \tag{44}
 \end{aligned}$$

The BE case with  $d = 3$  and  $g = 2$  reproduces the correct Stefan-Boltzmann law. Thus we see from Eq. (43) how the dimensionality is reflected in the familiar formulas of the Stefan-Boltzmann law.

#### REFERENCES

1. See, e.g., J.L. Synge, *The Relativistic Gas*, North-Holland, (1957); P.T. Landsberg and P. Havas, in *Statistical Mechanics of Equilibrium and Nonequilibrium*, ed. by J. Meixner, North-Holland (1965); R.M. Ziff, G.E. Uhlenbeck and M. Kac, *Phys. Rep.* **32C** (1977) 171.
2. S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, N.Y. (1957).
3. M.M. Nieto, *Lett. Nuovo Cimento* **1** (1969) 677; *J. Math. Phys.* **11** (1970) 1346; F. Karsch and D.E. Miller, *Phys. Rev.* **A22** (1980) 1210.
4. V.B. Magalinskii and Ya.P. Terletskii, *Sov. Phys. JETP* **5** (1957) 483; M. Hayashi, *Bull. Acad. Polon.* **14** (1966) 261; Honorio Vera Mendoza and M. Hayashi, *Rev. Mex. Fís.* **36** (1990) 50.
5. M. Chaichian, R. Hagedorn, and M. Hayashi, *Nucl. Phys.* **B92** (1975) 445.

6. S. Frautschi, *Phys. Rev.* **D3** (1971) 2821.
7. R. Hagedorn, *Nuovo Cimento Suppl.* **3** (1965) 147.
8. W. Nahm, *Nucl. Phys.* **B45** (1972) 525; M. Chaichian and M. Hayashi, *Phys. Rev.* **D1** (1977) 402.
9. S. de Groot *et al.*, *Proc. R. Soc.* **203A** (1950) 266; A. Münster, *Z. Phys.* **144** (1956) 197; P.T. Landsberg, *Proc. Cambridge Philos. Soc.* **50** (1954) 65.
10. R. Beckmann, F. Karsch and D.E. Miller, *Phys. Rev.* **A25** (1982) 561.
11. B. Touschek, *Nuovo Cimento* **58B** (1968) 295.
12. Note that  $\phi^{(d)}(\beta, m_i)$  is nothing but the BO statistics case ( $\gamma = 0$ ) of  $\ln Z^{(d)}(z, \gamma)$  defined in Eq. (32):  $\phi^{(d)}(\beta, m_i) = \ln Z^{(d)}(z = 1, \gamma = 0, m \rightarrow m_i)$ .
13. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, 9th ed., Dover Pub., N.Y. (1970).
14. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, N.Y. (1953).
15. F. Lurçat and P. Mazur, *Nuovo Cimento* **31** (1964) 140; V.A. Kolkunov, *Soviet Phys. JETP* **43** (1962) 1448.
16. If we had taken the covariant phase space  $d^{(d)}p/2p_0$ , we would have recovered the BO statistics case of F. Lurçat and P. Mazur in Ref. [15].

RESUMEN. Presentamos la fórmula cerrada para la integral de espacio-fase invariante de dimensión  $d$  para un gas ideal relativista en una forma integral exacta. En los casos particulares de los límites no relativista y ultrarelativista las integrales de espacio-fase son calculadas analíticamente. También, consideramos el espacio-fase invariante de dimensión  $d$  con estadística cuántica y derivamos la descomposición en cúmulos para las funciones de partición gran canónica y canónica, así como las densidades de estados microcanónica y gran microcanónica. Como ejemplo, consideramos la radiación de cuerpo negro en  $d$  dimensiones.