A tale of three theorems

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ABSTRACT. Noether's theorem relates symmetries and conservation laws of Hamiltonian systems. Arnol'd's theorem uses those integrals of motion for the construction of sufficient stability conditions of hydrodynamical problems, which are Hamiltonian with a singular Poisson bracket. Finally, Andrews' theorem imposes restrictions on the existence of Arnol'd stable solutions of symmetric systems. It is shown that denial of Andrews' theorem implies the divergence of the velocity component normal to the symmetric coordinate. This proof by *reductio ad absurdum* may be used to determine the strength of the symmetry breaking elements, necessary to overcome the limitations imposed by this theorem.

RESUMEN. El teorema de Noether relaciona simetrías y leyes de conservación en sistemas hamiltonianos. El teorema de Arnol'd usa esas integrales de movimiento para la construcción de condiciones suficientes de estabilidad para problemas hidrodinámicos, que son hamiltonianos con un paréntesis de Poisson singular. Por último, el teorema de Andrews impone restricciones al conjunto de soluciones estables de acuerdo a Arnol'd, para el caso de sistemas con simetrías. Se muestra aquí que la negación del teorema de Andrews implica la divergencia de la componente de la velocidad normal a la coordenada simétrica. Esta prueba por reducción al absurdo puede ser utilizada para determinar la magnitud de los elementos que rompen la simetría, necesarios para evitar las consecuencias de este toerema.

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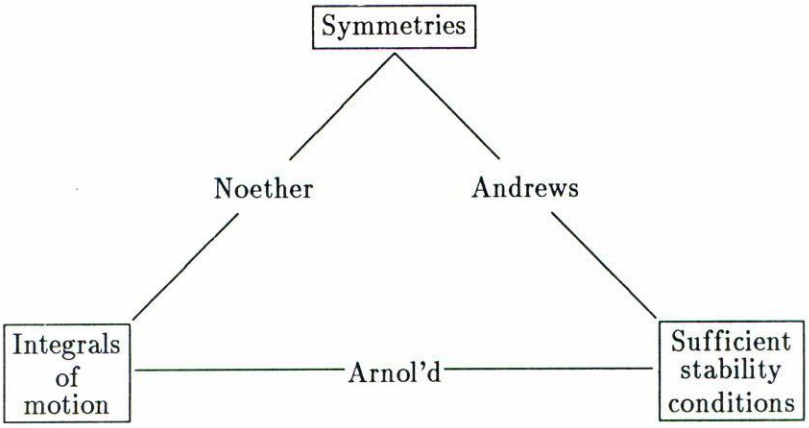
It was the best of times, it was the worst of times
Dickens, 1859.

1. INTRODUCTION

The three theorems of the title are those of Emmy Noether [1], which establishes a relationship between the symmetries of Hamiltonian systems and their conservation laws, of Vladimir Arnol'd [2,3], which uses these integrals of motion to derive sufficient stability conditions applicable to certain nonlinear solutions of hydrodynamical systems, and of David Andrews [4], which shows the limitations on the possible solutions that satisfy these stability criteria, on symmetric systems (see figure). The first two theorems are pretty well known and appreciated; that is not the case of the third one, which I believe has been somewhat ignored and misunderstood.

The purpose of this paper is to emphasize the importance of Andrews' theorem (AT): its denial is shown to result in the divergence of the velocity component normal to a symmetric coordinate. This proof by contradiction of AT is clearly less elegant than the direct one (which only uses the symmetry properties of integrals of motion), but it has the advantage of emphasizing the source of the restrictions imposed by AT, thereby suggesting ways to avoid its consequences.



The proof presented here is based on the manipulation of the equations of motion of a particular system. I have chosen *Laplace tidal equations* (LTE) because it has the three main restoring agents of atmosphere and ocean dynamics: earth's attraction, rotation and curvature (*i.e.*, gravity, Coriolis and the so called β effects). This model is also known as the *shallow water equations* (particularly when rotation effects are ignored); I prefer the name that honors Pierre Simon de Laplace [5], who was the first to correctly pose the problem of the tides on a rotating earth, introducing the concept of Coriolis force sixty years before Gaspard Gustave de Coriolis [6] made it popular, in a different context. LTE are, nowadays, a paradigm of ocean or atmosphere models, with applications which reach much beyond the study of tides: that is the sense in which they are used here, *i.e.*, excluding the tidal potential —or any other forcing— and dissipation.

The stability conditions are related to the sign definiteness of certain Lyapunov functionals, constructed from the integrals of motion of the system. Although not compulsory, it is more illuminating to work within the Hamiltonian formalism, where the choice of those integrals of motion does not appear capricious but, rather, dictated by the symmetries of the problem. Accordingly, that is the formalism adopted here, with an aim for completeness (*e.g.*, the expression of the generators of spatial transformations is derived, even in the case that they are not conserved).

The rest of this paper is organized as follows: In Sect. 2, the three theorems are quickly reviewed; the Hamiltonian structure of LTE (with the novelty of topographic and β effects), the corresponding Arnol'd stability conditions, and the new proof of AT are derived in Sect. 3; general conclusions are presented in Sect. 4. Some mathematical details are left for an Appendix.

2. THE THREE THEOREMS

Noether's theorem

The usual way in which this theorem is presented is by proving that the invariance of the least action principle under a certain transformation (of the state variables and/or spacetime coordinates) implies the existence of an explicit conservation law. However, for the problems of interest here (hydrodynamical systems in the *Eulerian* description), this is not the most useful formalism, because the Lagrangian must include extra ("unobservable") fields.

Instead, consider the (non-canonical) Hamiltonian formalism [7] in which conservation laws imply symmetries, as shown next: Let the momentum $\mathcal{M}$ be the *generator* of (infinitesimal) x-translations, in the same sense that $-\mathcal{H}$ is the generator of time translations, *i.e.*, the first variations are given by $\delta_x\mathcal{F} := \{\mathcal{M},\mathcal{F}\}\delta x$ and $\delta_t\mathcal{F} := \{\mathcal{F},\mathcal{H}\}\delta t$, respectively. (Other momenta, generators of rotations and other translations are similarly defined.) A generator need not be conserved. If, say, the x-momentum is conserved, (or, equivalently, $\mathcal{H}$ is invariant under x-translations) then using Jacobi identity it follows that

$$\{\mathcal{M},\mathcal{H}\} = 0 \Rightarrow \delta_t\delta_x\mathcal{F} = \delta_x\delta_t\mathcal{F} \ \forall\ \mathcal{F}[\varphi].$$

Consequently, the dynamics is invariant under x-translations: it is the same to make an infinitesimal translation in x and then let the system evolve than *viceversa*; the opposite is not necessarily true (unlike in the Lagrangian formalism): $\delta_t\delta_x\mathcal{F} = \delta_x\delta_t\mathcal{F}$ at most implies that the bracket $\{\mathcal{M},\mathcal{H}\}$ is equal to a Casimir.

Arnol'd's theorem

If Φ denotes some basic state [8] (given *a priori*) and the perturbation from it to the actual state φ is defined by $\delta\varphi := \varphi - \Phi$, we search for conditions (on Φ, not on $\delta\varphi$) which guarantee that some measure of $\delta\varphi$ is bounded; these are *sufficient conditions for the stability* of the basic state. Lyapunov method is based in the construction of an *integral of motion* $\mathcal{L}[\varphi]$ [9] that has an extremum at $\varphi = \Phi$, *i.e.*,

$$\delta\mathcal{L} \equiv 0 \quad \delta^2\mathcal{L} > 0 \quad \forall\ \delta\varphi,$$

where $\Delta\mathcal{L} := \mathcal{L}[\Phi + \delta\varphi] - \mathcal{L}[\Phi] = \delta\mathcal{L} + \frac{1}{2}\delta^2\mathcal{L} + \cdots$, with $\delta^n(\cdots) = O(\delta\varphi^n)$. Since $\delta^2\mathcal{L}$ is conserved by the linearized dynamics [10], and is a norm of $\delta\varphi$, then these are conditions for *formal stability* of the basic state, which is a concept stronger than *normal modes* stability but weaker than *nonlinear* stability [11,12].

Given any basic state Φ, one would like to be able to find an appropriate integral of motion $\mathcal{L}$ which satisfies $\delta\mathcal{L} \equiv 0$; this seems to be too ambitious. Instead, given some integral of motion $\mathcal{L}$, one may wonder which is the set of basic states with which it can be used, in order to obtain stability conditions: The Poisson bracket $\{\mathcal{A},\mathcal{B}\}$ is a bilinear form of the functional derivatives $\delta\mathcal{A}/\delta\varphi$ and $\delta\mathcal{B}/\delta\varphi$. Consequently, $\delta\mathcal{L} \equiv 0$ implies that

$$\{\mathcal{L},\mathcal{F}\} = 0 \quad \text{at} \quad \varphi = \Phi, \quad \forall\ \mathcal{F}[\varphi],$$

i.e., Φ is invariant under the transformation generated by $\mathcal{L}$. This is the clue to determine the class of basic states Φ whose stability may be proved using some integral of motion.

Recalling the classical conservation laws, one can use, for steady basic states, the

$$\textit{pseudoenergy:}\quad \mathcal{L} = \mathcal{H} + \mathcal{C}_0 \quad (\partial_t \Phi = 0),$$

(which is a generator of time increments, in spite of the presence of $\mathcal{C}_0$) or the

$$\textit{pseudomomentum:}\quad \mathcal{L} = \mathcal{M} + \mathcal{C}_1 \quad (\partial_x \Phi = 0),$$

for symmetric basic states; in the case of both steady and parallel basic flows, the strongest stability conditions are obtained using a linear combination of both, *viz* $\mathcal{L} = \mathcal{H} - \alpha\mathcal{M} + \mathcal{C}$, where $\mathcal{C} := \mathcal{C}_0 - \alpha\mathcal{C}_1$ and α is arbitrary. Arnol'd [2,3] used the pseudoenergy for the problem of two-dimensional (non-divergent) flow; subsequently, his results were generalized to more complicated systems (*e.g.* [11,12,13,14,15]).

The Casimirs are added here in order to enforce $\delta\mathcal{L} \equiv 0$: the Hamiltonian $\mathcal{H}$ (momentum $\mathcal{M}$) is not necessarily extremum, $\delta\mathcal{H}/\delta\varphi \not\equiv 0 (\delta\mathcal{M}/\delta\varphi \not\equiv 0)$, at a steady (symmetric) basic state, because the Poisson bracket is singular. Notice that a steady solution in Eulerian variables will likely correspond to time-dependence in both particle position fields or the additional fields needed to construct a Lagrangian: this shows the advantage of the Hamiltonian formalism: The Casimirs, which correspond to relabelling symmetries, "lost" in the reduction from Lagrangian to Eulerian variables, allow for the construction of a pseudoenergy which has an extremum at a given basic state, even though the energy is not at an extremum there.

Andrews' theorem

Assume that the system under study is invariant under x-translations: This implies two things: First, a translated solution is also a solution; in particular, the translated basic state $\Phi(x+\delta x, \ldots)$ is a possible state of the system, which correspond to the perturbation $\delta\varphi = \Phi_x \delta x + O(\delta x)^2$. Second, the corresponding momentum is conserved, and therefore

$$\delta_x(\mathcal{H} + \mathcal{C}_0) = \delta x\{\mathcal{H} + \mathcal{C}_0, \mathcal{M}\} \equiv 0;$$

indeed $\Delta_x(\mathcal{H} + \mathcal{C}_0) \equiv 0 \ \forall \delta x$: if the formal stability condtion ($\delta\mathcal{L} \equiv 0$ and $\delta^2\mathcal{L} > 0$, $\forall\ \delta\varphi$) is strictly satisfied, then this perturbation must vanish, *i.e.*,

$$\delta(\mathcal{H} + \mathcal{C}_0) = 0 \quad \delta^2(\mathcal{H} + \mathcal{C}_0 > 0 \quad \forall\ \delta\varphi \Rightarrow \Phi_x \equiv 0;$$

this is Andrew's theorem.

In sum, if the system (evolution equation and boundary conditions) is x-symmetric, then solutions of the stability condition for $\mathcal{L} = \mathcal{H} + \mathcal{C}_0$, must also have that symmetry. More generally, an extremum of pseudoenergy must have the spatial symmetries of the system.

This theorem is based on the assumption that the Lyapunov functional $\mathcal{L}$ is not only conserved $\{\mathcal{L}, \mathcal{H}\} = 0$, but also invariant under x-translations $\{\mathcal{M}, \mathcal{L}\} = 0$. It is, therefore, not necessarily restricted to steady basic states Φ. Assume, for instance, that Φ is neither steady nor x-independent, namely $\{\varphi, \mathcal{H}\} \neq 0$ and $\{\varphi, \mathcal{M}\} \neq 0$ at $\varphi = \Phi$, but uniformly translating with a speed c, *i.e.*, Φ is a (non-trivial) function of $x - ct$, as well as of other variables different from x and t: Since $(\partial_t + c\partial_x)\Phi = 0$ implies $\{\varphi, \mathcal{H} - c\mathcal{M}\} \equiv 0$ at $\varphi = \Phi$, one might be tempted to choose, for the stability conditions, the functional $\mathcal{L} = \mathcal{H} - c\mathcal{M} + \mathcal{C}$ (*e.g.*, see [16]). However, an obvious generalization of AT, shows that a solution of $\delta^2(\mathcal{H} - c\mathcal{M} + \mathcal{C}) > 0 \;\forall\; \delta\varphi$ must also be x-independent.

It may be thought that the argument in previous paragraph is no more than AT in a different frame (*i.e.*, one moving with speed c along the x direction). However, covariance under Galilean transformation is not the rule, but a —curious— exception in models of atmosphere and ocean dynamics. I shall come back to this point in the following section.

Notice —this is an important corollary of AT— that if the system is invariant under translations along both the x and y directions (which implies an infinite domain), then there are no solutions of the pseudoenergy extremizing condition. Carnevale and Shepherd [17] argue that in an infinite domain one might be able to prove Arnol'd-stability through the specification of radiation conditions in an "appropriate frame of reference"; in the (symmetric) examples they give, this procedure is equivalent to the use of pseudomomentum, as done in Refs. [13,14,15] and further discussed next.

3. Laplace tidal equations

Evolution equations

For simplicity, a Cartesian geometry and minimal vertical resolution will be used. Let $h(x,y,t)$ be the depth of the fluid and $u(x,y,t)$ and $v(x,y,t)$ be the velocity components along the eastward (x) and northward (y) directions. The equations of motion are

$$h_t + uh_x + vh_y + h(u_x + v_y) = 0,$$

$$u_t + uu_x + vu_y - fv + p_x = 0,$$

$$v_t + uv_x + vv_y + fu + p_y = 0,$$

where

$$p = g(h + \tau)$$

is the kinematic pressure field (g is the effective gravity whereas $\tau(x,y)$, which may vanish, represents the bottom topography) and

$$f = f_0 + \beta y$$

is the Coriolis "parameter" (twice the vertical component of earth's angular velocity). If the horizontal domain is denoted by D, the only boundary condition is that of vanishing normal mass flux $h\mathbf{u} \cdot \mathbf{n} = 0$ at ∂D. Notice that no external forcing is included, *i.e.*, in spite of its name, this system is not used here to study the tides.

The geometry used is Cartesian, for simplicity; however, the parameter β models the effect of earth's curvature, through the change of f with latitude. Even though there is no vertical structure in the fields (u, v, h), this system can be generalized to a problem with N homogeneous layers, such that u, v, h and p are N-vectors and where g is a $N \times N$ matrix, which parameterizes the vertical stratification [15].

The presence of the Coriolis term implies that covariance under $x \to x - \varepsilon t$ and $(h, u, v) \to (h, u + \varepsilon, v)$ is accomplished only if it is also imposed that $p \to g(h + \tau) - \varepsilon(f_0 y + \frac{1}{2}\beta y^2)$. But the (thermodynamic) pressure cannot, obviously, be frame dependent: as nicely shown by White [18], the invariant transformation represents a Galilean boost and a rotation of the apparent vertical, $z \to z + \varepsilon(f_0 y + \frac{1}{2}\beta y^2)/g$, which is reflected as a change of the topography. Similarly, in the f-plane ($\beta \equiv 0$) a change in the (vertical) rotation rate is equivalent to the topography of a revolution paraboloid [14]. Certain atmosphere-ocean models happen to be Galilean invariant [18], but those are an exception, which might be misleading.

More globally, in a frame rotating faster or slower than the earth, there is a centrifugal force in addition to the (modified) Coriolis one; the (hydrostatic) equations of atmosphere-ocean dynamics are written in a particular frame. This is indeed part of the great intuition of Laplace [5], in contradiction with the idea of Newton, Leibniz and "other geometers", who thought (incorrectly) that it was possible to study the tides in a still earth, and afterwards introduce rotation as merely a change of variables.

Hamiltonian structure

In order to discuss the Hamiltonian structure of this system, it is better to rewrite the evolution equations in the form

$$\begin{aligned} h_t &= -(uh)_x - (vh)_y, \\ u_t &= qhv - b_x, \\ v_t &= -qhu - b_y, \end{aligned}$$

where $b = p + \frac{1}{2}u^2 + \frac{1}{2}v^2$ is the Bernoulli function, and $q = (f + v_x - u_y)/h$, the *potential vorticity*, is a conserved quantity $q_t + uq_x + vq_y = 0$. Furthermore, in order to be able to obtain the momenta, even in the presence of β effects, it is necessary to make a change in the state variables, from the u field to

$$\hat{u} := u - \tfrac{1}{2}\beta y^2.$$

The evolution equations are easily obtained from the Hamiltonian

$$\mathcal{H}[h, \hat{u}, v] := \iint_D d^2x \left(\tfrac{1}{2}h(u^2 + v^2) + \pi(h)\right),$$

and the Poisson bracket

$$\{\mathcal{A}, \mathcal{B}\} := \iint_D d^2x \left(q\frac{\delta \mathcal{A}}{\delta \hat{u}}\frac{\delta \mathcal{B}}{\delta v} - q\frac{\delta \mathcal{B}}{\delta \hat{u}}\frac{\delta \mathcal{A}}{\delta v} - \frac{\delta \mathcal{A}}{\delta \mathbf{w}} \cdot \nabla \frac{\delta \mathcal{B}}{\delta h} + \frac{\delta \mathcal{B}}{\delta \mathbf{w}} \cdot \nabla \frac{\delta \mathcal{A}}{\delta h} \right),$$

where $\mathbf{w} := (\hat{u}, v)$ and $\pi(h) := \int p\, dh \equiv \frac{1}{2}hg(h + 2\tau)$ (see the Appendix).

The Casimirs of the Poisson bracket are of the form

$$\mathcal{C}[h, \hat{u}, v] := \int\!\!\int_D d^2x\, hF(q) - \sum_i a_i \oint_{\partial D_i} \mathbf{u} \cdot d\mathbf{x},$$

where the function F and the constants a_i are arbitrary. The last term is but a linear combination of the Kelvin circulations in each connected part ∂D_i of the boundary; $\partial D = \bigcup_i \partial D_i$. Recall that the Casimirs are conserved, because $\{\mathcal{C}, \mathcal{H}\} = 0$, by definition.

Finally, the linear momenta are given by

$$\mathcal{M}^x[h, \hat{u}, v] := \int\!\!\int_D d^2x\, h(\hat{u} - f_0 y),$$

$$\mathcal{M}^y[h, \hat{u}, v] := \int\!\!\int_D d^2x\, h(v + f_0 x),$$

whereas the angular momentum (*i.e.*, its vertical component) is

$$\mathcal{M}^a[h, \hat{u}, v] := \int\!\!\int_D d^2x\, h(xv - y\hat{u} - \tfrac{1}{2}f_0(x^2 + y^2)).$$

Each one of the functionals $\mathcal{M}^x, \mathcal{M}^y$ and $\mathcal{M}^a$, regardless of being conserved or not, is the generator of a spatial transformation, if the boundary has the corresponding symmetry (see Appendix). With respect to the conservation of these functionals, it is found that

$$\{\mathcal{M}^x, \mathcal{H}\} := -\int\!\!\int_D d^2x\, hp_x,$$

$$\{\mathcal{M}^y, \mathcal{H}\} := -\int\!\!\int_D d^2x\, h(p_y + \beta y u),$$

$$\{\mathcal{M}^a, \mathcal{H}\} := -\int\!\!\int_D d^2x\, h(p_\vartheta + \beta xyu + \tfrac{1}{2}\beta y^2 v),$$

where $\partial_\vartheta = x\partial_y - y\partial_x$. Recall that $\partial p \equiv g(\partial h + \partial \tau)$: if $\mathcal{M}^s$ is a generator (the boundary has the corresponding symmetry), then the term $hg\partial_s h$ integates out. Consequently, if $\mathcal{M}^x$ is a generator and $\tau_x \equiv 0$, then it is conserved, but in the case of $\mathcal{M}^y$ ($\mathcal{M}^a$), in order to be conserved it is further needed that $\beta \equiv 0$ and $\tau_y \equiv 0$ ($\tau_\vartheta \equiv 0$), because the β term breaks y-homogeneity and isotropy, and so does a non-symmetric topography.

Sufficient stability conditions

Let (H, U, V) be a certain steady solution of the model equations; eliminating time derivatives it is found that

$$HU = -\Psi_y,$$

$$HV = \Psi_x,$$

$$Q\nabla\Psi = \nabla B,$$

where $\Psi \equiv \Psi(Q)$ and $B \equiv B(Q)$ are transport and Bernoulli functions, and Q is the potential vorticity field, in the basic state. Notice that since $\Psi \equiv \Psi(Q)$ and Ψ is a constant at ∂D_i then Q is also constant, say Q_i, at each connected part of the boundary ∂D_i.

The *pseudoenergy* is $\mathcal{L} = \mathcal{H} + \mathcal{C}_0$, with $\mathcal{C}_0$ chosen so that $\delta\mathcal{L} \equiv 0$, which is satisfied by, and only by, $F_0(q) \equiv q\Psi(q) - B(q)$, and $a_i \equiv F_0'(Q_i)$. Notice, this is important, that the requirement $\delta\mathcal{L} \equiv 0$ determines $F_0(q)$ uniquely. the second variations of $\mathcal{H}$ and $\mathcal{C}_0$ are

$$\delta^2\mathcal{H} \equiv \int\!\!\int_D d^2x\, H(\delta u^2 + \delta v^2) + 2\delta h(U\delta u + V\delta v) + g\delta h^2,$$

$$\delta^2\mathcal{C}_0 = \int\!\!\int_D d^2x\, H\Psi'(Q)\delta q^2,$$

where $\Psi'(Q) := d\Psi(Q)/dQ$. Stability conditions are then easily found by requiring that both $\delta^2\mathcal{C}_0$ and $\delta^2\mathcal{H}$ be positive definite; these are

$$\frac{d\Psi}{dQ} > 0 \quad \text{and} \quad U^2 + V^2 < gH,$$

everywhere. Since $\delta^2\mathcal{H}(t) = \delta^2\mathcal{C}_0(t) \equiv \delta^2\mathcal{H}(0) + \delta^2\mathcal{C}_0(0)$, under the linearized dynamics [10] both conditions imply that the "wave energy" $\delta^2\mathcal{H}(t)$ is bounded from above and below, *viz.*

$$\delta^2\mathcal{C}_0(0) > \delta^2\mathcal{H}(t) > 0,$$

and therefore they guarantee formal stability in the metric

$$\|(\delta u, \delta v, \delta h)\| := \sqrt{\delta^2\mathcal{H}(t)}.$$

These conditions were derived in [14] for the case without topography; Holm *et al.* [11] found similar criteria for two-dimensional compressible flow, which is mathematically equivalent to the shallow water model in the absence of Coriolis and topographic effects.

In the more symmetric case in which the basic flow is not only steady but also parallel, *i.e.*, $(H(y), U(y), 0)$, which is possible only if $\tau_x \equiv 0$, solving for $\delta(\mathcal{H} - \alpha\mathcal{M} + \mathcal{C}) = 0$ and

$\delta^2(\mathcal{H} - \alpha\mathcal{M} + \mathcal{C}) > 0 \ \forall \ \delta\varphi$, results in $\mathcal{C} = \mathcal{C}_0 - \alpha\mathcal{C}_1$ (with $F_1'(Q) \equiv y$) and the sufficient stability conditions

$$(U - \alpha)\left(\hat{\beta} - U_{yy} + (f - U_y)\frac{fU}{gH}\right) < 0,$$

$$(U - \alpha)^2 < gH,$$

for some α, where

$$\hat{\beta} := \beta + \frac{(f - U_y)\tau_y}{H};$$

these conditions are the generalization of those in Ref. [13], for the case with topography. It might be thought that the parameter α merely represents a uniform change of U into $U - \alpha$. However, the presence of the term fU/gH shows clearly that the problem is not invariant under a Galilean transformation along the x direction, in the presence of Coriolis effects, as discussed above.

Rederivation of Andrews' theorem

Assume that the topography τ, if present, is x-independent, so that AT can be applied. In the case of the non-parallel basic flow, the potential vorticity and Bernoulli functions are written in terms of the transport function $\Psi(x, y)$ and depth $H(x, y)$ fields, in the form

$$Q = H^{-1}(f + \nabla \cdot (H^{-1}\nabla\Psi)),$$

$$B = gH + g\tau + \tfrac{1}{2}H^{-2}(\nabla\Psi)^2.$$

If $\tau_x \equiv 0$, using $Q\nabla\Psi = \nabla B$ it follows that

$$Q_x = -QH^{-1}H_x + H^{-1}\nabla \cdot (H^{-1}\nabla\Psi)_x,$$

$$Q\Psi_x = B_x = (gH - H^{-2}(\nabla\Psi)^2)H^{-1}H_x + H^{-2}\nabla\Psi \cdot \nabla\Psi_x.$$

Multiplying the first equation by $H\Psi_x$ and then using the second one, it is found that

$$\begin{aligned} H\Psi_x Q_x &= -(g - H^{-3}(\nabla\Psi)^2)(H_x)^2 - H^{-2}H_x\nabla\Psi \cdot \nabla\Psi_x + \Psi_x\nabla \cdot (H^{-1}\nabla\Psi)_x \\ &= -(gH - U^2 - V^2)H^{-1}(H_x)^2 - H^{-1}(\nabla\Psi_x)^2 + \nabla \cdot (\Psi_x(H^{-1}\nabla\Psi)_x). \end{aligned}$$

Assume then that the domain is the channel $(-\infty < x < \infty, y_1 < y < y_2)$, so that the problem is x-homogeneous. If the first stability condition is satisfied, $d\Psi/dQ > 0$, and the basic flow were not x-independent, as demanded by AT, then $\Psi_x Q_x$ should be positive everywhere, *i.e.*,

$$(HVV_x)_x - (HVU_x)_y > (gH - U^2 - V^2)H^{-1}(H_x)^2 + H^{-1}(\nabla(HV))^2.$$

Finally, if the second stability condition, $gH > U^2 + V^2$, is also satisfied, integrating in y it follows that

$$\Xi'(x) > \int_{y_1}^{y_2} dy\, H^{-1}\Big((gH - U^2 - V^2)(H_x)^2 + (\nabla(HV))^2\Big) > 0,$$

where

$$\Xi(x) := \frac{1}{2}\int_{y_1}^{y_2} dy\, H(V^2)_x.$$

This inequality implies that as $x \to \pm\infty$, $\Xi(x) \to \pm\infty$, *i.e.*,V^2 (or H) diverges; this absurd result is the consequence of denying AT. ■

Notice that if the range of x were finite, thereby breaking the symmetry, then there would be no problem with the inequality $\Xi'(x) > 0$. Yet another way to break the symmetry —indeed, the way originally suggested by Andrews [4]— is to use a x-dependent topography $\tau_x \not\equiv 0$.

Now assume that the boundary ∂D is, instead, invariant under a translation in y: A similar derivation, but using $d\Psi = \Psi_y dy$ and $dQ = Q_y dy$, gives from the first stability condition $\Psi_y Q_y > 0$,

$$(HUU_y)_y - (HUV_y)_x > (gH - U^2 - V^2)H^{-1}(H_y)^2 + H^{-1}(\nabla(HU))^2 + \hat{\beta}HU;$$

the key difference here is the term $\hat{\beta}HU$: even if ∂D is independent under y-translations (*i.e.*, $\mathcal{M}^y$ exists), the dynamics is not, for $\hat{\beta} \neq 0$. Therefore it is not true that U^2 must diverge as $|y| \to \infty$, for solutions of the stability conditions, $d\Psi/dQ > 0$ and $gH > U^2+V^2$ (*i.e.*, AT cannot be applied); however, it is needed $\int dx\hat{\beta}HU < 0$ somewhere, in order to compensate the postive definite terms in the right hand side of this inequality.

As a corollary, if $\beta = 0 = \tau$ and the domain is invariant under translations in both x and y (*i.e.*, the infinite f-plane) then there are no solutions of the stability conditions. This does not mean that there are no stable solutions (those conditions are only *sufficient*, not *necessary* ones), it just means that their stability cannot be proved by maximizing pseudoenergy.

Let me finish by discussing the restrictions on *parallel* solutions of the stability conditions, and the ways to avoid the limitations imposed by AT. Consider the case $\hat{\beta} = 0$, and the stability conditions for the choice $\alpha = 0$: From the first one, $UQ_y < 0$, it follows that

$$\frac{1}{2}(U^2)_{yy} > (U_y)^2 + (f - U_y)\frac{fU^2}{gH};$$

with the second one, $U^2 < gH$, it is then found that

$$(U^2)_{yy} > [(U_y)^2 + (f - U_y)^2 + f^2]\frac{U^2}{gH}$$

which implies that U^2 diverges as $|y| \to \infty$. Now, there are three ways to avoid this result; the three are forms to break the symmetry of y-homogeneity, thereby rendering AT not applicable:

1. The range of y is finite; the inequality implies no divergence of U^2.

2. It is $\hat{\beta} \neq 0$ (on account of β and/or topographic effects); the inequality is not valid. Recall that in this case the Hamiltonian is not invariant under translations in $y, \{\mathcal{M}^y, \mathcal{H}\} \neq 0$.

3. A non-vanishing value of α is used; once again the inequality is valid. With $\alpha \neq 0$, AT cannot be applied, because even though the pseudoenergy is invariant under translations in y, $\{\mathcal{M}^y, \mathcal{H}+\mathcal{C}_0\} = 0$, the zonal pseudomomentum is not, $\{\mathcal{M}^y, \mathcal{M}^x + \mathcal{C}_1\} \neq 0$, on account of the $-f_0 y$ term in the definition of $\mathcal{M}^x$.

An example of the third possibility above (*i.e.*, that one must use the pseudomomentum, for a symmetric basic state, if the system has two spatial symmetries), is presented in the proof of formal stability of a solid-body rotating vortex [14].

4. Conclusions

Three main original results are presented in this paper: First, the Hamiltonian structure of Laplace tidal equations (also known as shallow water equations) is presented, including the possibility of β (*i.e.*, earth's curvature) and topographic effects; even though they are well defined, the generators of spatial transformations may not be conserved, precisely because of those effects. Second, new stability conditions are derived, using Arnol'd's method, which include the possibility of topography. Finally, some inequalities are obtained and used to show that denial of Andrews' theorem ("an Arnol'd-stable steady state must have the symmetries of the system") results in the, unphysical, divergence of the velocity component normal to the symmetric coordinate. (The Hamiltonian structure is not needed in order to derive these inequalities, but rather it is used to link symmetries and conservation laws, which is the main theme of this paper.)

By an *Arnol'd-stable* state it is exclusively meant one which is an extremum of the sum of the Hamiltonian plus a suitable chosen Casimir (*i.e.*, the *pseudoenergy*). Therefore, Andrews' theorem is not a statement on the lack of stable "non-symmetric state in symmetric system", but, rather, a statement on the failure of Arnol'd's method to search for them. Indeed, it is possible to prove the stability of such state by *ad hoc* methods, as done by Benjamin [19] with the soliton of the K de V equation (see also Ref. [17]) or by Tang [20] with elliptical vortices in two-dimensional flow.

In order to find stable states which are extremes of the pseudoenergy in systems with symmetric boundaries, it is necessary to introduce symmetric breaking elements in the interior dynamics. Consider, for instance, the case of Laplace tidal equations, discussed in Sect. 3.

If there is no topography, the boundaries are (only) invariant under y-translations (an unusual example) and this symmetry is broken with the β-effect (gradient of Coriolis "parameter"), then the inequality $\Xi'(x) > 0$, with x and y interchanged, is replaced by

$$\beta \iint_D d^2x\, HU < -\iint_D d^2x\, H^{-1}((gH - U^2 - V^2)(H_y)^2 + (\nabla(HU))^2) < 0,$$

i.e., existence of an Arnol'd-stable state requires the net zonal transport to be westward.

Consider now the more common example of a zonal channel with a zonally asymmetric topography (indeed, the case proposed in [4]); instead of $\Xi'(x) > 0$ it is found

$$g \iint_D d^2x\, H_x \tau_x < -\iint_D d^2x\, H^{-1}((gH - U^2 - V^2)(H_x)^2 + (\nabla(HV))^2) < 0,$$

i.e., an anticorrelation is needed between the zonal slopes of the topography and the fluid depth, in an Arnol'd-stable state.

Finally, assume that D is the infinite plane and the topography, if present, is zonally symmetric; it is found that

$$\int_{-\infty}^{\infty} dy\, \hat\beta U < -\frac{1}{2}\int_{-\infty}^{\infty} dy((U_y)^2 + (f - U_y)^2 + f^2)\frac{U^2}{gH},$$

i.e., existence of an Arnol'd-stable state requires the net zonal transport to be "westward" relative to the effective β. Of course, in this parallel case, stability may be proved even if $\hat\beta = 0$, using a value of $\alpha \neq 0$, *i.e.*, using the zonal pseudomomentum in order to break the symmetry.

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APPENDIX

I will show here that $\mathcal{H}, \mathcal{C}$ and $\mathcal{M}^s$ have the required properties. The procedure consists in calculating the first variation, extracting the functional derivatives, and substituting in the Poisson with a generic functional. Proof of the Jacobi identity is left for a future publication.

From the definition of the Hamiltonian, it is easily found that

$$\delta\mathcal{H}[h, \mathbf{w}] = \iint_D d^2x(b\delta h + hu \cdot \delta\mathbf{w}),$$

and therefore

$$\{\mathcal{A},\mathcal{H}\} := \iint_D d^2x\left(\frac{\delta\mathcal{A}}{\delta\hat{u}}(qhv - b_x) + \frac{\delta\mathcal{A}}{\delta v}(-qhu - b_y) + h\mathbf{u}\cdot\nabla\frac{\delta\mathcal{A}}{\delta h}\right);$$

making a partial integration of the last term (recall that $h\mathbf{u}\cdot n = 0$ at ∂D), using $\mathcal{A} = h$, $\hat{u}$ or v, and imposing the relation $\varphi_t = \{\varphi,\mathcal{H}\}$, LTE are obtained. ■

With respect to a Casimir, using $\delta q = (\delta v_x - q\delta h)/h$, it is found that

$$\delta\mathcal{C} = \iint_D d^2x((F(q) - qF'(q))\delta h + F''(q)q_y\delta\hat{u} - F''(q)q_x\delta v)$$
$$+ \sum_i \oint_{\partial D_i}(F'(q) - a_i)\delta\mathbf{u}\cdot d\mathbf{x},$$

from which it follows $\{\mathcal{C},\mathcal{B}\} \equiv 0$ for any $\mathcal{B}[h,\hat{u},v]$, which is the required relation. ■

Furthermore, it is determined that the class of *admissible functionals* (for the Poisson bracket) are the Casimirs and those —like the Hamiltonian and the momenta— which satisfy $\mathbf{n}\cdot(\delta\mathcal{A}/\delta\mathbf{w}) = 0$ at the boundary $\mathbf{x}\in\partial D$.

The Poisson brackets of the momenta with a generic functional of state $\mathcal{B}[h,\hat{u},v]$ are

$$\{\mathcal{M}^s,\mathcal{B}\} := \iint_D d^2x\left(\frac{\delta\mathcal{B}}{\delta\hat{u}}\partial_s\hat{u} + \frac{\partial\mathcal{B}}{\delta v}\partial_s v - h\partial_s\frac{\delta\mathcal{B}}{\delta h}\right), \quad s = x \text{ or } y,$$

for the linear ones, and

$$\{\mathcal{M}^a,\mathcal{B}\} := \iint_D d^2x\left(\frac{\delta\mathcal{B}}{\delta\hat{u}}(\partial_\vartheta\hat{u} + v) + \frac{\delta\mathcal{B}}{\delta v}(\partial_\vartheta v - \hat{u}) - h\partial_\vartheta\frac{\delta\mathcal{B}}{\delta h}\right)$$

for the angular one, where $\partial_\vartheta = x\partial_y - y\partial_x$.

In order for $\mathcal{M}^s$ to be a generator of the corresponding spatial transformation, it must be possible to integrate by parts the last term, going from $-h\partial_s(\delta\mathcal{B}/\delta h)$ to $(\delta\mathcal{B}/\delta h)\partial_s h$ ($s = x, y$ or ϑ), *i.e.*, the boundary ∂D must be invariant under that transformation, so that the integral of $\partial_s(h(\delta\mathcal{B}/\delta h))$ vanishes identically. (In the case of an unbounded domain, in order to avoid infinities it may be necessary to redefine the functionals of state by subtracting their formal values at the state $u = v = 0$, and $h =$ constant.) Notice that $\delta_\vartheta\hat{u} = (\partial_\vartheta\hat{u} + v)\delta\vartheta$ and $\delta_\vartheta v = (\partial_\vartheta v - \hat{u})\delta\vartheta$, *i.e.*, $(\hat{u}, v)$ behaves as a vector under infinitesimal rotations.

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7. The evolution of state variables $\varphi(x,\ldots,t)$ is determined by a *Hamiltonian* functional $\mathcal{H}[\varphi]$ and Poisson bracket $\{\ ,\ \}$ in the form $\varphi_t = \{\varphi, \mathcal{H}\}$. The bracket must satisfy a certain set of properties (including the Jacobi identity) and may be singular, namely, there may exist functionals of state $\mathcal{C}[\varphi]$, called *Casimirs*, such that $\{\mathcal{F}, \mathcal{C}\} = 0 \ \forall\ \mathcal{F}[\varphi]$.
8. Quantities in the basic state are denoted by capital symbols, roman or greek.
9. As a matter of fact, Lyapunov method only requires that $\dot{\mathcal{L}} \le 0$, not necessarily $\dot{\mathcal{L}} = 0$.
10. Notice that $\delta\mathcal{L} \equiv 0$ by construction; therefore, if the evolution equations are linearized in $\delta\varphi$, it follows that the law $(\Delta\mathcal{L})_t = 0$ is reduced to its lowest order contribution, *viz*, $(\frac{1}{2}\delta^2\mathcal{L})_t = 0$.
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