 Revista Mexicana de Física **38**, *No. 2 (1992) 332–342*

Physics of rotating objects that slip

R.A. RUELAS-MAYORGA
Instituto de Astronomía, Universidad Nacional Autónoma de México
Apartado postal 70–264

Recibido el 10 de abril de 1991; aceptado el 23 de octubre de 1991

ABSTRACT. In solving the problem of rotation with slippage using traditional techniques, it is shown that during the slipping phase there is at least one quantity which is conserved. This quantity is denoted as $P(v, \omega)$ and is named generalised rotational momentum. Using this invariant in dynamical problems where rotation and slippage occur, simplifies their solutions to a degree comparable to the solutions effected for other problems using the technique of conservation of energy. It is also shown that during the slipping phase, the equation that relates linear acceleration and angular acceleration obeys a linear relationship which is similar to the relation obtained for the case of pure rotation.

RESUMEN. Se muestra, usando técnicas convencionales en la resolución del problema de rotación y resbalados, que durante la fase de resbalado existe por lo menos una cantidad que se conserva. Esta cantidad se denota como $P(v, \omega)$ y se denomina momento rotacional generalizado. El uso de este invariante en problemas dinámicos, en donde hay rotación y patinado, simplifica su solución en un grado similar al que se obtiene, para otra clase de problemas, mediante el método de la conservación de la energía. Se muestra que durante la fase de patinado, la ecuación que relaciona la aceleración lineal con la aceleración angular toma una forma lineal, la cual es similar a la relación que se obtiene en el caso de rotación pura.

PACS: 61.70.Le

1. INTRODUCTION

Many problems dealing with rotation undergo during their initial stage a somewhat quick transient state during which the rotating object slips. The angular speed of the object is high enough so that the friction force, even though at its maximum value, turns out to be insufficient to ensure pure rotation. Let me state here that the assumption of pure rotation is invaluable in the solution of many problem, because it relates the angular and translational variables in a simple way as expressed by Eqs. (1), (2) and (3):

$$x = R\theta \qquad \xrightarrow{+} \qquad \overset{+}{\circlearrowright} \tag{1}$$

$$v = R\omega \qquad \xrightarrow{+} \qquad \overset{+}{\circlearrowright}, \tag{2}$$

$$a = R\alpha \qquad \xrightarrow{+} \qquad \overset{+}{\circlearrowright}, \tag{3}$$

where x, v and a stand for the displacement, the velocity and the acceleration of the centre of mass of the object measured with respect to an arbitrary inertial frame of reference;

and θ, ω and α represent the angular displacement, the angular velocity and the angular acceleration of the object around its axis of rotation. R stands for the radius of the object. The sign convention adopted in this paper is indicated next to Eqs. (1), (2) and (3). Linear variables are positive when they point right; angular variables are positive when they point clockwise.

If, on the other hand, the object under study slips, Eqs. (1), (2) and (3) may not be used and in general it may be said that the angular and translational variables are not related by means of straight forward mathematical expressions. In this case, however, it is known that the force of friction is maximum and given as

$$f_k = \mu_k N. \tag{4}$$

The friction force has two components: (i) traction friction ($f_{\rm t}$) and (ii) rolling friction ($f_{\rm r}$). Therefore Eq. (4) may also be written as

$$f_k = f_{\rm t} + f_{\rm r} = (\mu_{\rm t} + \mu_{\rm r})N, \tag{4a}$$

where $f_{\rm t}$ is responsible for accelerating the centre of mass of the object and $f_{\rm r}$ is responsible for slowing the rotation of the object under study.

As the slipping phase progresses both $f_{\rm t}$ and $f_{\rm r}$ vary. It can easily be proved that pure rotation sets in when $f_{\rm t} = f_{\rm r}$.

2. Solution of the problem

We shall now solve one such problem using the conventional technique. This solution leads in a very direct way to the establishment of an invariant. It may be proved that for such phenomena there is at least one physical quantity which is conserved during the whole time-interval for which slippage occurs.

Let us assume that a symmetric object of mass M, and radius R, is rotating and slipping on a surface with a certain coefficient of kinetic friction μ_k. The initial velocity of the centre of mass is v_0, ω_0 represents its initial angular velocity, v_1 and ω_1 stand for the values of the centre of mass velocity and angular velocity at the exact time when pure rotation is established, therefore, Eq. (2) is valid for v_1 and ω_1. In going from state (v_0, ω_0) to state (v_1, ω_1) certain amounts $\Delta t, \Delta x, \Delta\theta$ of time, distance and angle respectively have elapsed or being travelled (Fig. 1 depicts the situation).

We know that the rate of change of the angular momentum (L) of the object must be equal to the external torque (τ) applied to the object. It is easy to see that

$$\tau = -fR \quad \overset{+}{\circlearrowright}, \tag{5}$$

therefore

$$\frac{dL}{dt} = \tau = -fR \tag{6}$$

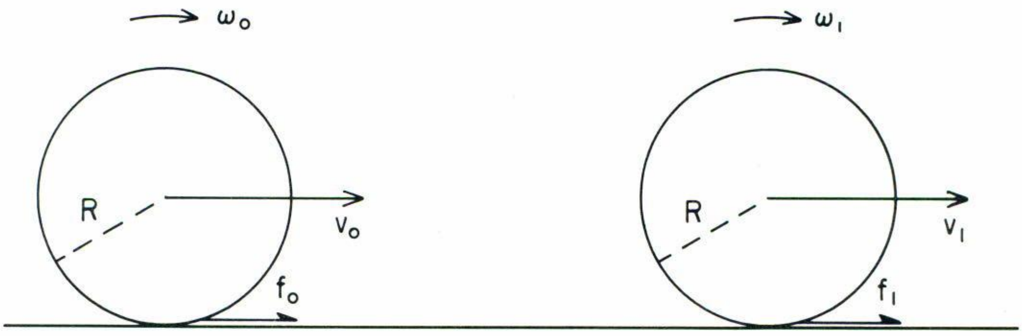

FIGURE 1. This figure depicts the situation which is analysed in the text. To the left we see the initial dynamical state of the object. This state is characterised by the angular velocity (ω_0), the velocity of the centre of mass (v_0) and the friction force (f_0) which attains it maximum value ($\mu_k N$). In going from the position to the left to that at the right, the object rotates and slips. The position at the right, shows the moment at which pure rotation sets in. Again ω_1 represents the object's angular velocity, v_1 represents the velocity of the centre of mass and the friction force is given by f_1. From this point onwards, $v_1 = \omega_1 R$ and $f_1 \leq \mu_k N$.

or

$$dL = -fR\,dt. \tag{6a}$$

Integrating from $t = 0$ to an arbitrary time value of t we obtain

$$\int_{L_0}^{L_f} dL = -\int_0^t fR\,dt, \tag{6b}$$

from which

$$L_f - L_0 = -R\int_0^t f\,dt. \tag{6c}$$

But we know that for a symmetric object $L = I\omega$, where I is the moment of inertia of the object around the axis of rotation. Equation (6c) may be transformed to

$$I\omega - I\omega_0 = -R\int_0^t f\,dt \quad \overset{+}{\circlearrowright} \tag{6d}$$

From Newton's second law it is obvious that

$$\int_0^t f\,dt = \int_{p_0}^{p_f} dp = p_f - p_0 = Mv - Mv_0 \quad \xrightarrow{+}. \tag{7}$$

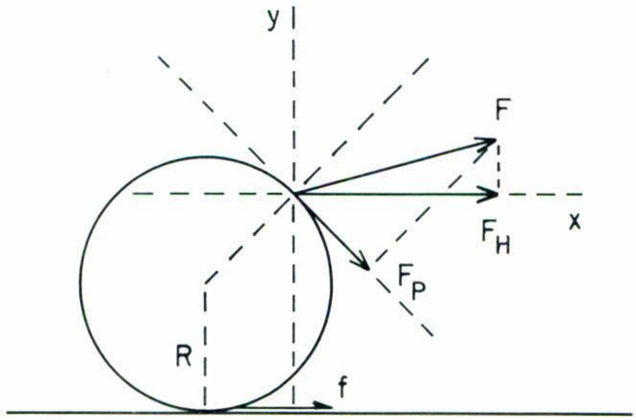

FIGURE 2. This figure shows the rotating and slipping object when an external force is applied to its rim. F represents the external force, F_H is the horizontal component of this force, F_p is the component of F which is tangent to the rim of the object at the point of application, f represents the friction force and R the radius of the object.

Using Eqs. (6d) and (7) we obtain

$$I\omega - I\omega_0 = -R(Mv - Mv_0), \tag{6e}$$

which may easily be transformed into

$$\frac{I\omega_0}{R} + Mv_0 = \frac{I\omega}{R} + Mv. \tag{6f}$$

But v and ω are values of the velocity and angular velocity at an arbitrary time t. We therefore conclude that the quantity

$$P(v,\omega) = Mv + \frac{I\omega}{R} \quad \xrightarrow{+} \quad \overset{+}{\circlearrowleft}$$

is conserved.

Knowing that $P(v,\omega)$ is a conserved quantity simplifies the solution of problems in a degree comparable to that achieved by the method of conservation of energy for other types of dynamical problems. Because of the fact that $P(v,\omega)$ has units of linear momentum, I have named it as generalised rotational momentum. This name has nothing to do with the generalised coordinates of the Lagrangean or Hamiltonian treatment of mechanical problems.

If other forces are applied to the rim of the rotating object, as depicted in Fig. 2, the solution of the problem follows the same general lines,

$$\tau = -Rf + RF_P(\pm) = \frac{\Delta L}{\Delta t} \quad \xrightarrow{+} \tag{9a}$$

(where + is used when F_P propitiates a clockwise rotation) which implies

$$\Delta L = -R_f \Delta t + RF_P(\pm)\Delta t. \tag{9b}$$

In the x-direction we have the following equation:

$$\Delta p_x = F_x \Delta t = [f + F_H(\pm)]\Delta t \tag{9c}$$

(where + is used when F_H points right, – when it points left) from which we obtain

$$Mv - Mv_0 = f\Delta t + F_H(\pm)\Delta t \tag{9d}$$

or

$$f\Delta t = (Mv - Mv_0) - F_H(\pm)\Delta t. \tag{9d'}$$

Substitution of (9d') in (9b) leads to

$$\Delta L = -R[Mv - Mv_0] + RF_H(\pm)\Delta t + RF_P(\pm)\Delta t, \tag{9e}$$

which may be transformed to

$$\frac{I\omega}{R} - \frac{I\omega_0}{R} = (Mv_0 - Mv) + [F_H(\pm) + F_P(\pm)]\Delta t$$

or

$$\frac{I\omega}{R} + Mv - \left(\frac{I\omega_0}{R} + Mv_0\right) = [F_H(\pm) + F_P(\pm)]\Delta t,$$

which may be reduced to

$$\frac{\Delta}{\Delta t}\left[Mv + \frac{I\omega}{R}\right] = F_H(\pm) + F_P(\pm). \tag{9f}$$

If $\Delta t \to 0$ then

$$\frac{d}{dt}\left[Mv + \frac{I\omega}{R}\right] = F_H(\pm) + F_P(\pm) \tag{9g}$$

or

$$Ma + \frac{I\alpha}{R} = F_H(\pm) + F_P(\pm). \tag{9h}$$

From Eq. (9g), it is clear that if the magnitude of the perpendicular force (F_P) is equal to that of the horizontal component of the net force (F) but of opposite direction, the generalised rotational momentum ($P(v,\omega)$) is also conserved.

If there exists interaction between more than one rotating and slipping object, the motion is described by the values of the variables v and ω evolving in such a way as to preserve the constancy of the following quantity:

$$P_T(v_i, \omega_i) = \sum_i \left(M_i v_i + \frac{I_i \omega_i}{R_i} \right), \tag{10}$$

where $P_T(v_i, \omega_i)$ represents the generalised rotational momentum of the system.

Applications of this novel method may be found in the Appendix, where a typical problem is solved in detail using the canonical method as well as the method developed here so that comparisons between both approaches may be established.

3. Further results derived from the invariance of $P(v, \omega)$

The fact that $P(v, \omega)$ is conserved when objects rotate and slip, may provide some information as to how the interaction between the rotating object and the surface occurs and evolves in time.

It is obvious that $P(v, \omega)$ is an implicit function of time, therefore

$$\frac{d}{dt}[P(v, \omega)] = \frac{I}{R}\frac{d\omega}{dt} + M\frac{dv}{dt} = 0 \tag{11a}$$

or

$$\frac{I}{R}\alpha + Ma = 0, \tag{11b}$$

which implies that

$$a = -\frac{I}{MR}\alpha; \tag{11c}$$

but

$$I = MK^2, \tag{12}$$

where K is the radius of gyration of the object under study. Therefore Eq. (11c) may be written as

$$a = -K^2\left(\frac{\alpha}{R}\right). \tag{11d}$$

In general $K = kR$ where k is a constant, therefore

$$a = -k^2(R\alpha). \tag{11e}$$

Comparing Eqs. (11e) and (3) we conclude that the relationship between linear acceleration and angular acceleration is of the same type for both pure rotation and rotation plus slippage except for the presence of the constant k.

When no external force is applied to the system, the following equation is valid during the slipping phase:

$$f_k = ma = m(-k^2 R\alpha) \tag{11f}$$

(remember that during the slipping transient, α will be essentially negative), but

$$f_k = \mu_k N = \mu_k mg. \tag{11g}$$

Therefore, from the previous two Eqs. (11f, 11g) we can conclude that

$$\alpha = -\frac{\mu_k g}{k^2 R}. \tag{11h}$$

This expression allows us to obtain equations for the angular velocity and position (number of rotations) of the object during the slipping phase. Integrating Eq. (11h) from $t = 0$ to an arbitrary time t we get

$$\omega_t = \omega_0 - \frac{t(\mu_k g)}{k^2 R}. \tag{11j}$$

Integrating again

$$\theta(t) = \theta_0 + \omega_0 t - \frac{t^2(\mu_k g)}{2k^2 R}. \tag{11k}$$

For practical purposes $\theta_0 = 0$, therefore

$$\theta(t) = t\left[\omega_0 - \frac{t(\mu_k g)}{2k^2 R}\right], \tag{11m}$$

and the number of rotations executed during the slipping phase up to time t is given as

$$n(t) = \frac{t}{2\pi}\left[\omega_0 - \frac{t(\mu_k g)}{2k^2 R}\right]. \tag{11n}$$

Knowing the initial conditions of motion v_0 and ω_0, it could be interesting to establish the duration of the slipping phase. We will do this as follows:

From Eq. (6f)

$$\frac{I\omega_0}{R} + Mv_0 = \frac{I\omega_1}{R} + Mv_1, \tag{13a}$$

where ω_1 and v_1 represent the values of angular velocity and linear velocity at the moment pure rotation is established; therefore from Eq. (2)

$$v_1 = R\omega_1. \tag{2a}$$

Substituting (2a) in (13a) we get

$$\frac{I\omega_0}{R} + Mv_0 = \frac{I + MR^2}{R}\omega_1, \tag{13b}$$

therefore

$$\omega_1 = \left[\frac{I + \frac{MRv_0}{\omega_0}}{I + MR^2}\right]\omega_0. \tag{13c}$$

Setting (11j) equal to (13c) we are able to obtain an expression for the time duration of the slipping phase:

$$\omega_0 - \frac{\mu_k g}{k^2 R}t = \left[\frac{I + \frac{MRv_0}{\omega_0}}{I + MR^2}\right]\omega_0,$$

which yields

$$t = \frac{M(kR)^2}{\mu_k g}\left[\frac{R\omega_0 - v_0}{I + MR^2}\right]. \tag{13d}$$

Using the fact that $I = M(kR)^2$ this expression is reduced to

$$t = \left[\frac{k^2}{k^2 + 1}\right]\left[\frac{R\omega_0 - v_0}{\mu_k g}\right]. \tag{13d'}$$

4. CONCLUSIONS

i) It is demonstrated that for a rotating object that slips there exists a dynamical invariant given as

$$P(v, \omega) = Mv + \frac{I\omega}{R}. \tag{8}$$

ii) For a system of rotating objects that interact, the quantity

$$P_T(v_i, \omega_i) = \sum_i \left(M_i v_i + \frac{I_i\omega_i}{R_i}\right) \tag{10}$$

is defined as the generalized momentum of the system. It has been shown that it is also conserved.

iii) Application of this method leads to the following interesting facts:

a) During the slipping phase the mathematical relationship between the linear acceleration (a) and the angular acceleration (α) obeys a linear expression given as

$$a = -k^2(R\alpha). \tag{11e}$$

This expression becomes that for pure rotation if $k^2 = -1$ or $k = i$. We can think of pure rotation as a particular case of rotation with slippage for which $k = i$.

b) The angular velocity of the rotating object during the slipping phase evolves according to the following equation:

$$\omega_t = \omega_0 - t\frac{\mu_k g}{k^2 R}. \tag{11j}$$

c) The number of rotations executed during the slipping phase up to time t is given as

$$n(t) = \frac{t}{2\pi}\left(\omega_0 - t\frac{\mu_k g}{2k^2 R}\right). \tag{11n}$$

d) Beginning with an initial linear velocity v_0 and an initial angular velocity ω_0, the slipping phase lasts a time given by the following equations:

$$t = \frac{M(kR)^2}{\mu_k g}\left[\frac{R\omega_0 - v_0}{I + MR^2}\right], \tag{13d}$$

$$t = \left[\frac{k^2}{k^2+1}\right]\left[\frac{R\omega_0 - v_0}{\mu_k g}\right]. \tag{13d'}$$

APPENDIX

Let us solve problem 12-44 from Resnick and Halliday (1977) [1]. The problem reads as follows:

A uniform solid cylinder of radius R is given an angular velocity ω_0 about its axis and is then dropped vertically onto a flat horizontal table. The table is not frictionless, so the cylinder begins to move as it slips. What is the velocity of the centre of mass of the cylinder when pure rotations sets in?

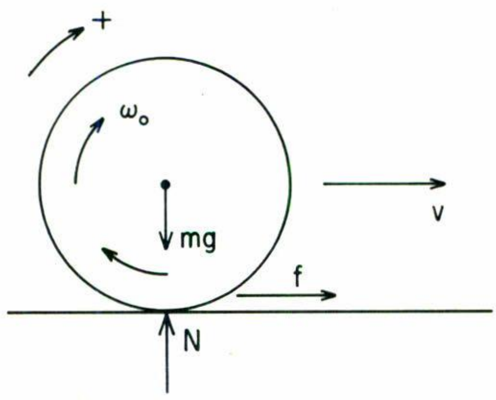

FIGURE 3. This figure represents the situation for the problem solved in the Appendix.

Firstly let us illustrate the canonical approach taken from Derringh (1981) [2]. (Ref. to Fig. 3).

Let ω_0 be clockwise and let clockwise rotations be positive. If the cylinder rolls, a clockwise rotation leads to a linear velocity v to the right, so let translational motion to the right be positive. As the cylinder slips, points on its rim in contact with the table move to the left relative to the table; the friction force f opposes this and hence f points to the right, in the direction of motion; note that $f = \mu N$, but it is not necessary to use this relation. For translational motion,

$$f = ma = m\frac{dv}{dt},$$

$$v = \frac{ft}{m} + v_0 = \frac{ft}{m}$$

since $v_0 = 0$, the cylinder being dropped vertically. For rotation, noting that f will give rise to counterclockwise rotation,

$$\tau = -fR = I\alpha = (\tfrac{1}{2}mR^2)\left(\frac{d\omega}{dt}\right),$$

$$\omega = -\frac{2ft}{mR} + \omega_0.$$

When at time $t = t^*$ rolling sets in, $v^* = R\omega^*$,

$$\frac{ft^*}{m} = -\frac{2ft^*}{m} + R\omega_0,$$

$$t^* = \frac{Rm\omega_0}{3f}.$$

Hence, at this instant,

$$v^* = \frac{ft^*}{m} = \tfrac{1}{3} R\omega_0 \qquad \text{(Ans.)}.$$

Now using the technique developed in this paper.

We know the quantity $Mv + I\omega/R$ is conserved. Therefore, its value at the beginning of the slipping phase ($I\omega_0/R$, because $v_0 = 0$) must equal its value at the end of this phase ($Mv_f + I\omega_f/R$). So

$$\frac{I\omega_0}{R} = \left(Mv_f + \frac{I\omega_f}{R} \right),$$

which implies

$$v_f = \left[\frac{R}{1 + MR^2/I} \right] \omega_0.$$

For a cylinder $I = (MR^2)/2$ therefore

$$v_f = \frac{R\omega_0}{1 + (MR^2)/(MR^2)/2} = \tfrac{1}{3} R\omega_0.$$

Which is the same answer obtained using the canonical approach; however, the conceptual difficulty and the process of setting the problem for solution are very much simpler and more direct.

ACKNOWLEDGMENTS

I thank the members of the Science Department of the McCallie School for a stimulating atmosphere and for their advice, support and friendship. Helpful discussions with Charles Standard, James Puiia and David Waller are also acknowledged. I thank Dr. Clifford E. Swartz for valuable suggestions. J. Orta is thanked for her typing skills.

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2. E. Derringh, *Selected Solutions for Physics*, John Wiley & Sons (1981), p. 107.