The method of adjoint operators*

G.F. TORRES DEL CASTILLO
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla, 72000 Puebla, Pue., México

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ABSTRACT. The method of adjoint operators, which yields expressions for the complete solutions of systems of homogeneous linear partial differential equations in terms of potentials, is reviewed and applied to obtain expressions for the solution of the source-free Maxwell equations in anisotropic media.

RESUMEN. Se revisa el método de operadores adjuntos, el cual permite obtener expresiones para las soluciones completas de sistemas de ecuaciones diferenciales parciales lineales homogénas en términos de potenciales, y se aplica para obtener expresiones para la solución de las ecuaciones de Maxwell sin fuentes en medios anisótropos.

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1. INTRODUCTION

Systems of homogeneous linear partial differential equations arise in mathematical physics mainly in connection with the equations governing vector, tensor or spinor fields. Some examples of such systems are the source-free Maxwell equations, the linearized Einstein equations (in vacuum or with sources), the Dirac equation and the equations of equilibrium for an elastic medium. A procedure commonly employed to solve these systems of equations consists in combining appropriately the equations of the system, or their derivatives, to obtain an equation containing only one of the unknowns or a combination of them and their derivatives; the solution of such a decoupled equation can then be used to find the remaining unknowns.

Finding a decoupled equation is not always an easy task, especially if noncartesian coordinates or noncartesian components are being used. However, there exist several cases in which the existence of decoupled equations is known. For instance, it is very well known that the source-free Maxwell equations in vacuum imply that the electric and magnetic fields satisfy the wave equation and, therefore, each cartesian component of these fields obeys a second-order decoupled equation.

Wald [1] found that, under certain conditions, the existence of a decoupled equation derived from a set of coupled homogeneous linear partial differential equations allows one to obtain an expression for complete solutions of the system of equations in terms of scalar potentials. Wald's method, which in what follows will be called method of adjoint operators, reduces the problem of solving sets of coupled homogeneous linear partial

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differential equations to that of solving a simpler system that, in many cases of interest, consists of a single equation for a single unknown. Even though in most cases where the method of adjoint operators has been applied the corresponding expressions for the solutions in terms of potentials were obtained previously by direct integration or by means of some ansatz, the method of adjoint operators leads to such expressions very easily.

The aim of this paper is to present the method of adjoint operators in a general form, emphasizing the arbitrariness involved in the definition of the adjoint of a linear operator. As an example, the source-free Maxwell equations in anisotropic media are considered, obtaining the expressions found by Przeździecki and Hurd [2] for the electromagnetic field in a gyrotropic medium and showing that in a biaxial medium the electromagnetic field can be expressed in terms of two potentials that obey a system of two second-order differential equations or of a single scalar potential that satisfies a fourth-order differential equation.

2. Decoupled equation and potentials

Let f be a set of functions that satisfy a homogeneous system of linear partial differential equations given by

$$\mathcal{E}(f) = 0, \tag{1}$$

where $\mathcal{E}$ is a linear differential operator. By combining appropriately Eqs. (1) and their derivatives, one may be able to obtain a decoupled equation

$$\mathcal{O}(\chi) = 0, \tag{2}$$

where χ is a function, or a set of functions, which can be expressed as a linear combination of f and its derivatives and $\mathcal{O}$ is a linear differential operator. Then, there exists a linear operator $\mathcal{T}$ such that $\chi = \mathcal{T}(f)$ and the fact that Eq. (2) is obtained by linearly combining Eqs. (1) and their derivatives means that there exists a linear differential operator $\mathcal{S}$ (which may not be unique for a given decoupled equation) such that

$$\mathcal{O}(\chi) = \mathcal{S}\mathcal{E}(f), \tag{3}$$

hence

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T} \tag{4}$$

must hold as an operator identity, so that when both sides of Eq. (4) are applied to a solution f of Eq. (1) one gets the decoupled equation (2).

By defining the adjoint, $\mathcal{A}^\dagger$, of a linear differential operator $\mathcal{A}$ in such a way that $\mathcal{A}^\dagger$ is also a linear operator and

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger, \tag{5}$$

for any pair of linear operators $\mathcal{A}$ and $\mathcal{B}$ whose composition is well defined, from Eq. (4) it follows that

$$\mathcal{E}^\dagger \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger. \tag{6}$$

Hence, if ψ satisfies the equation

$$\mathcal{O}^\dagger(\psi) = 0, \tag{7}$$

then Eq. (6) implies that $\mathcal{S}^\dagger(\psi)$ satisfies $\mathcal{E}^\dagger(\mathcal{S}^\dagger(\psi)) = 0$; thus, if $\mathcal{E}^\dagger$ is proportional to $\mathcal{E}$, then $\mathcal{E}(\mathcal{S}^\dagger(\psi)) = 0$, which means that $\mathcal{S}^\dagger(\psi)$ is a solution of Eq. (1).

If the adjoint of a linear operator $\mathcal{A}$, that maps tensor or spinor fields into tensor or spinor fields, is defined as that linear operator $\mathcal{A}^\dagger$ such that [1]

$$g \cdot \mathcal{A}(f) - \left[\mathcal{A}^\dagger(g)\right] \cdot f = \nabla_\alpha s^\alpha \tag{8}$$

for every pair of tensor or spinor fields f and g for which the full contraction of g and $\mathcal{A}(f)$, denoted by $g \cdot \mathcal{A}(f)$, yields a scalar field, where s^α is some vector field (which depends on f and g), then it follows that Eq. (5) holds and that the source-free Maxwell equations (in flat or curved space-time) [1,3], the linearized Einstein vacuum field equations [1,4], the linearized Einstein-Maxwell equations [5,6], the linearized supergravity field equations for the spin-3/2 field [7], the linearized Yang-Mills equations [8] and the equations of equilibrium for an elastic medium [9] can be written in the form (1) with $\mathcal{E}$ being self-adjoint or anti-self-adjoint ($\mathcal{E}^\dagger = \pm\mathcal{E}$).

Equation (8) also implies that $(\mathcal{A}^\dagger)^\dagger = \mathcal{A}$, therefore in the cases where $\mathcal{E}^\dagger$ is not proportional to $\mathcal{E}$, the above procedure can still be applied looking for a decoupled equation derived from the adjoint system $\mathcal{E}^\dagger(f) = 0$. Alternatively, owing to the freedom involved in the definition of the adjoint of a linear operator, at least in some cases, it is possible to make $\mathcal{E}$ self-adjoint or anti-self-adjoint by suitably defining the adjoint of a linear operator. For example, in the case of the Dirac equation written in the form

$$\mathcal{E}(\psi) \equiv i\hbar\gamma^\alpha \partial_\alpha \psi - mc\psi = 0, \tag{9}$$

following the conventions used in Ref. [10], one finds that if the adjoint of $\mathcal{E}$ is defined as that linear operator $\mathcal{E}^\dagger$ such that

$$\phi^{*\mathrm{t}}\gamma_0 \mathcal{E}(\psi) - [\mathcal{E}^\dagger(\phi)]^{*\mathrm{t}}\gamma_0\psi = \nabla_\alpha s^\alpha \tag{10}$$

for every pair of four-component spinors ϕ and ψ, where the superscript t denotes transposition, $*$ denotes complex conjugation and s^α is some vector field, then $\mathcal{E}$ is self-adjoint. As we shall show in the next section, by defining the adjoint in a form analogous to Eq. (10), the source-free Maxwell equations in a lossless medium can be written in terms of a self-adjoint operator. It must be pointed out that in order for $\mathcal{E}^\dagger$ to be proportional to $\mathcal{E}$, Eqs. (8) and (10) require that $\mathcal{E}(f)$ have the same number of components as f.

3. ELECTROMAGNETIC FIELDS IN ANISOTROPIC MEDIA

The propagation of a monochromatic electromagnetic wave in a medium characterized by the permittivity and permeability tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ is governed by the Maxwell equations

$$\begin{aligned} \nabla \cdot (\boldsymbol{\mu} \cdot \mathbf{H}) &= 0, & \nabla \times \mathbf{E} &= i\omega\boldsymbol{\mu} \cdot \mathbf{H}, \\ \nabla \cdot (\boldsymbol{\epsilon} \cdot \mathbf{E}) &= 0, & \nabla \times \mathbf{H} &= -i\omega\boldsymbol{\epsilon} \cdot \mathbf{E}, \end{aligned} \tag{11}$$

(it is assumed that the time dependence of the fields is given by a factor $e^{-i\omega t}$). The condition that dissipation of energy be absent is [11]

$$\boldsymbol{\epsilon}^{\mathrm{t}} = \boldsymbol{\epsilon}^*, \quad \boldsymbol{\mu}^{\mathrm{t}} = \boldsymbol{\mu}^*. \tag{12}$$

In order to have a system of equations where the number of unknowns is equal to the number of equations, the fields $\mathbf{E}$ and $\mathbf{H}$ can be expressed in terms of the electromagnetic potentials ϕ and $\mathbf{A}$ in the usual manner:

$$\mathbf{E} = -\nabla\phi + i\omega\mathbf{A}, \quad \boldsymbol{\mu} \cdot \mathbf{H} = \nabla \times \mathbf{A}. \tag{13}$$

Then Eqs. (11) amount to

$$\begin{aligned} &\nabla \cdot \left[\boldsymbol{\epsilon} \cdot (-\nabla\phi + i\omega\mathbf{A})\right] = 0, \\ &\nabla \times (\boldsymbol{\mu}^{-1} \cdot \nabla \times \mathbf{A}) = i\omega\boldsymbol{\epsilon} \cdot (\nabla\phi - i\omega\mathbf{A}), \end{aligned} \tag{14}$$

which is a system of four linear partial differential equations with four unknowns.

Equations (14) can be written in the form of Eq. (1) with f being the four-component column $\left[\begin{smallmatrix} \phi \\ c\mathbf{A} \end{smallmatrix}\right]$ and where $\mathcal{E}(f)$ is also a four-component column given by

$$\mathcal{E}\begin{bmatrix} \phi \\ c\mathbf{A} \end{bmatrix} \equiv \begin{bmatrix} c\nabla \cdot \boldsymbol{\epsilon} \cdot (-\nabla\phi + i\omega\mathbf{A}) \\ \nabla \times \boldsymbol{\mu}^{-1} \cdot \nabla \times \mathbf{A} - i\omega\boldsymbol{\epsilon} \cdot (\nabla\phi - i\omega\mathbf{A}) \end{bmatrix} \tag{15}$$

(the factors c are inserted for convenience). By defining the adjoint of a linear operator $\mathcal{A}$ that maps four-component columns into four-component columns as that linear operator $\mathcal{A}^\dagger$ such that

$$g^{*t}\eta\mathcal{A}(f) - \left[\mathcal{A}^\dagger(g)\right]^{*t}\eta f = \nabla \cdot \mathbf{s}, \tag{16}$$

where $\eta \equiv \mathrm{diag}(-1, 1, 1, 1)$ and $\mathbf{s}$ is some vector field, one finds that the operator $\mathcal{E}$, defined in Eq. (15), is self-adjoint if and only if conditions (12) hold. The adjoint of a linear operator $\mathcal{B}$ that maps four-component columns into two-component columns is the linear operator $\mathcal{B}^\dagger$ that maps two-component columns into four-component columns such that

$$g^{*t}\mathcal{B}(f) - \left[\mathcal{B}^\dagger(g)\right]^{*t}\eta f = \nabla \cdot \mathbf{t}, \tag{17}$$

where **t** is some vector field. Since $\mathcal{BA}$ maps four-component columns into two-component columns, using Eqs. (17) and (16) one finds that

$$\begin{aligned} g^{*t}(\mathcal{BA})(f) = g^{*t}\mathcal{B}(\mathcal{A}(f)) &= \left[\mathcal{B}^{\dagger}(g)\right]^{*t}\eta\mathcal{A}(f) + \nabla\cdot\mathbf{t} \\ &= \left[\mathcal{A}^{\dagger}(\mathcal{B}^{\dagger}(g))\right]^{*t}\eta f + \nabla\cdot\mathbf{s} + \nabla\cdot\mathbf{t} \\ &= \left[(\mathcal{A}^{\dagger}\mathcal{B}^{\dagger})(g)\right]^{*t}\eta f + \nabla\cdot(\mathbf{s}+\mathbf{t}), \end{aligned}$$

which, compared with Eq. (17), shows that $(\mathcal{BA})^{\dagger} = \mathcal{A}^{\dagger}\mathcal{B}^{\dagger}$. Similarly, one finds that if $\mathcal{C}$ is a linear operator that maps two-component columns into two-component columns and if $\mathcal{C}^{\dagger}$ is the linear operator defined by

$$g^{*t}\mathcal{C}(f) - \left[\mathcal{C}^{\dagger}(g)\right]^{*t} f = \nabla\cdot\mathbf{u}. \tag{18}$$

where **u** is some vector field, then $(\mathcal{CB})^{\dagger} = \mathcal{B}^{\dagger}\mathcal{C}^{\dagger}$. (It may be noticed that the matrix η appears in the definition of the adjoint of a linear operator only in those terms that contain four-component columns.)

a) Gyrotropic media

Following Ref. [2], we shall consider the propagation of electromagnetic waves in an anisotropic medium such that the permittivity and permeability tensors have the form

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon & -i\epsilon_g & 0 \\ i\epsilon_g & \epsilon & 0 \\ 0 & 0 & \epsilon_a \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu & -i\mu_g & 0 \\ i\mu_g & \mu & 0 \\ 0 & 0 & \mu_a \end{bmatrix}, \tag{19}$$

in an appropriate system of cartesian coordinates. If at least one of ϵ_g and μ_g is different from zero, the medium is said to be gyrotropic. Since the tensors (19) satisfy conditions (12), the linear operator $\mathcal{E}$ defined by Eq. (15) is self-adjoint. In order to obtain an operator identity of the form (4), it is convenient to introduce

$$K_0 \equiv c\nabla\cdot\boldsymbol{\epsilon}\cdot(-\nabla\phi + i\omega\mathbf{A}) = c\nabla\cdot\boldsymbol{\epsilon}\cdot\mathbf{E} \tag{20a}$$

$$\mathbf{K} \equiv \nabla\times\boldsymbol{\mu}^{-1}\cdot\nabla\times\mathbf{A} - i\omega\boldsymbol{\epsilon}\cdot(\nabla\phi - i\omega\mathbf{A}) = \nabla\times\mathbf{H} + i\omega\boldsymbol{\epsilon}\cdot\mathbf{E} \tag{20b}$$

(cf. Eqs. (14)). Then, Eq. (15) amounts to

$$\mathcal{E}\begin{bmatrix} \phi \\ c\mathbf{A} \end{bmatrix} = \begin{bmatrix} K_0 \\ \mathbf{K} \end{bmatrix}. \tag{21}$$

(Notice that $K_0 = 0$ and $\mathbf{K} = 0$ if and only if ϕ and $\mathbf{A}$ satisfy the Maxwell equations (14).)

As shown in Ref. [2], if Eq. (19) holds then there exists a decoupled system of equations for the components E_z and H_z. In fact, taking the curl of Eq. (20b) we get

$$\nabla \times \mathbf{K} = \nabla \times \nabla \times \mathbf{H} + i\omega \nabla \times \boldsymbol{\epsilon} \cdot \mathbf{E} = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} + i\omega \nabla \times \boldsymbol{\epsilon} \cdot \mathbf{E} \tag{22}$$

and using Eq. (19) we find that the z-component of Eq. (22) is

$$\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) - \frac{\partial^2 H_z}{\partial x^2} - \frac{\partial^2 H_z}{\partial y^2}$$
$$- \omega \epsilon_g \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) + i\omega\epsilon \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right), \tag{23}$$

where we have assumed that the components of the tensors (19) do not depend on x and y. (Since we are interested in obtaining operator identities, we are not assuming that ϕ and $\mathbf{A}$ satisfy Maxwell's equations.) On the other hand, the equation $\nabla \cdot \boldsymbol{\mu} \cdot \mathbf{H} = 0$, which follows from Eqs. (13), together with Eq. (19) yield

$$\mu \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) - i\mu_g \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) + \frac{\partial}{\partial z}(\mu_a H_z) = 0.$$

From Eqs. (20b) and (19) we have

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = K_z - i\omega\epsilon_a E_z,$$

therefore

$$\mu \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) - i\mu_g K_z - \omega\epsilon_a \mu_g E_z + \frac{\partial}{\partial z}(\mu_a H_z) = 0. \tag{24}$$

Similarly, from Eqs. (20a), (19) and $\nabla \times \mathbf{E} = i\omega \boldsymbol{\mu} \cdot \mathbf{H}$, which holds identically by virtue of Eqs. (1), we obtain

$$K_0 = c\epsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) + c\omega\mu_a \epsilon_g H_z + c\frac{\partial}{\partial z}(\epsilon_a E_z). \tag{25}$$

Substituting Eqs. (24–25) into Eq. (23) one finds

$$\nabla_t^2 H_z + \frac{\partial}{\partial z}\frac{1}{\mu}\frac{\partial}{\partial z}(\mu_a H_z) + k_m^2 H_z - \omega\tau_g \frac{\partial}{\partial z}(\epsilon_a E_z) - \omega \left(\frac{\partial}{\partial z}\frac{\mu_g}{\mu}\right)\epsilon_a E_z$$
$$= -\frac{\omega}{c}\frac{\epsilon_g}{\epsilon} K_0 + \frac{\partial K_x}{\partial y} - \frac{\partial K_y}{\partial x} + i\frac{\partial}{\partial z}\left(\frac{\mu_g}{\mu} K_z\right), \tag{26}$$

where, following Ref. [2], we have made use of the abbreviations

$$\nabla_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad k_m^2 \equiv \omega^2 \mu_a \frac{\epsilon^2 - \epsilon_g^2}{\epsilon}, \quad \tau_g \equiv \frac{\epsilon_g}{\epsilon} + \frac{\mu_g}{\mu}. \tag{27}$$

Taking now the curl of $\nabla \times \mathbf{E} - i\omega\boldsymbol{\mu} \cdot \mathbf{H} = 0$ and using Eq. (19) one obtains

$$\frac{\partial}{\partial z}(\nabla \cdot \mathbf{E}) - \nabla^2 E_z - i\omega \frac{\partial}{\partial x}(i\mu_g H_x + \mu H_y) + i\omega \frac{\partial}{\partial y}(\mu H_x - i\mu_g H_y) = 0.$$

Hence, from Eqs. (20*b*) and (24–25) it follows that

$$\nabla_t^2 E_z + \frac{\partial}{\partial z}\frac{1}{\epsilon}\frac{\partial}{\partial z}(\epsilon_a E_z) + k_e^2 E_z + \omega\tau_g \frac{\partial}{\partial z}(\mu_a H_z) + \omega \left(\frac{\partial}{\partial z}\frac{\epsilon_g}{\epsilon}\right) \mu_a H_z = \frac{1}{c}\frac{\partial}{\partial z}\frac{K_0}{\epsilon} - \frac{ik_e^2}{\omega\epsilon_a} K_z, \tag{28}$$

where

$$k_e^2 \equiv \omega^2 \epsilon_a \frac{\mu^2 - \mu_g^2}{\mu}. \tag{29}$$

Thus, identities (26) and (28) show that if ϕ and $\mathbf{A}$ satisfy the Maxwell equations (14) (*i.e.* $K_0 = 0$ and $\mathbf{K} = 0$) then E_z and H_z obey the system of equations

$$\begin{aligned} \nabla_t^2 E_z + \frac{\partial}{\partial z}\frac{1}{\epsilon}\frac{\partial}{\partial z}(\epsilon_a E_z) + k_e^2 E_z + \omega\tau_g \frac{\partial}{\partial z}(\mu_a H_z) + \omega \left(\frac{\partial}{\partial z}\frac{\epsilon_g}{\epsilon}\right) \mu_a H_z &= 0 \\ \nabla_t^2 H_z + \frac{\partial}{\partial z}\frac{1}{\mu}\frac{\partial}{\partial z}(\mu_a H_z) + k_m^2 H_z - \omega\tau_g \frac{\partial}{\partial z}(\epsilon_a E_z) - \omega \left(\frac{\partial}{\partial z}\frac{\mu_g}{\mu}\right) \epsilon_a E_z &= 0 \end{aligned} \tag{30}$$

which was obtained in Ref. [2]. This system of equations is of the form (2) with

$$\chi \equiv \begin{bmatrix} \epsilon_a E_z \\ \mu_a H_z \end{bmatrix} \tag{31}$$

and

$$\mathcal{O} \equiv \begin{bmatrix} \nabla_t^2 + \epsilon_a \frac{\partial}{\partial z}\frac{1}{\epsilon}\frac{\partial}{\partial z} + k_e^2 & \omega\epsilon_a\tau_g \frac{\partial}{\partial z} + \omega\epsilon_a \left(\frac{\partial}{\partial z}\frac{\epsilon_g}{\epsilon}\right) \\ -\omega\mu_a\tau_g \frac{\partial}{\partial z} - \omega\mu_a \left(\frac{\partial}{\partial z}\frac{\mu_g}{\mu}\right) & \nabla_t^2 + \mu_a \frac{\partial}{\partial z}\frac{1}{\mu}\frac{\partial}{\partial z} + k_m^2 \end{bmatrix}. \tag{32}$$

By using the definitions (31–32) we see that the identities (26) and (28) amount to

$$\mathcal{O}(\chi) = \begin{bmatrix} \dfrac{\epsilon_a}{c}\dfrac{\partial}{\partial z}\dfrac{K_0}{\epsilon} - \dfrac{ik_e^2}{\omega}K_z \\ -\mu_a\dfrac{\omega}{c}\dfrac{\epsilon_g}{\epsilon}K_0 + \mu_a\dfrac{\partial K_x}{\partial y} - \mu_a\dfrac{\partial K_y}{\partial x} + i\mu_a\dfrac{\partial}{\partial z}\left(\dfrac{\mu_g}{\mu}K_z\right) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\epsilon_a}{c}\dfrac{\partial}{\partial z}\dfrac{1}{\epsilon} & 0 & 0 & -\dfrac{i}{\omega}k_e^2 \\ -\mu_a\dfrac{\omega}{c}\dfrac{\epsilon_g}{\epsilon} & \mu_a\dfrac{\partial}{\partial y} & -\mu_a\dfrac{\partial}{\partial x} & i\mu_a\dfrac{\partial}{\partial z}\dfrac{\mu_g}{\mu} \end{bmatrix} \mathcal{E}\begin{bmatrix} \phi \\ c\mathbf{A} \end{bmatrix},$$

where we have made use of Eq. (21); this last equation is of the form (3) with

$$\mathcal{S} = \begin{bmatrix} \dfrac{\epsilon_a}{c}\dfrac{\partial}{\partial z}\dfrac{1}{\epsilon} & 0 & 0 & -\dfrac{i}{\omega}k_e^2 \\ -\mu_a\dfrac{\omega}{c}\dfrac{\epsilon_g}{\epsilon} & \mu_a\dfrac{\partial}{\partial y} & -\mu_a\dfrac{\partial}{\partial x} & i\mu_a\dfrac{\partial}{\partial z}\dfrac{\mu_g}{\mu} \end{bmatrix} \tag{31}$$

According to Eqs. (18) and (17) the adjoints of the operators (32–33) are given by

$$\mathcal{O}^\dagger = \begin{bmatrix} \nabla_t^2 + \dfrac{\partial}{\partial z}\dfrac{1}{\epsilon}\dfrac{\partial}{\partial z}\epsilon_a + k_e^2 & \omega\dfrac{\partial}{\partial z}\mu_a\tau_g - \omega\mu_a\left(\dfrac{\partial}{\partial z}\dfrac{\mu_g}{\mu}\right) \\ -\omega\dfrac{\partial}{\partial z}\epsilon_a\tau_g + \omega\epsilon_a\left(\dfrac{\partial}{\partial z}\dfrac{\epsilon_g}{\epsilon}\right) & \nabla_t^2 + \dfrac{\partial}{\partial z}\dfrac{1}{\mu}\dfrac{\partial}{\partial z}\mu_a + k_m^2 \end{bmatrix} \tag{34}$$

and

$$\mathcal{S}^\dagger = \begin{bmatrix} \dfrac{1}{c\epsilon}\dfrac{\partial}{\partial z}\epsilon_a & \mu_a\dfrac{\omega}{c}\dfrac{\epsilon_g}{\epsilon} \\ 0 & -\dfrac{\partial}{\partial y}\mu_a \\ 0 & \dfrac{\partial}{\partial x}\mu_a \\ \dfrac{i}{\omega}k_e^2 & i\dfrac{\mu_g}{\mu}\dfrac{\partial}{\partial z}\mu_a \end{bmatrix}. \tag{35}$$

In order to get agreement with the expressions found in Ref. [2], the components of the potential ψ are written as

$$\psi = \begin{bmatrix} -\dfrac{c}{\epsilon_a}u \\ -\dfrac{c}{\mu_a}v \end{bmatrix},$$

then condition $\mathcal{O}^\dagger(\psi) = 0$ amounts to

$$\begin{aligned}
\left(\nabla_t^2 + \epsilon_a \frac{\partial}{\partial z}\frac{1}{\epsilon}\frac{\partial}{\partial z} + k_e^2\right) u &= -\omega\epsilon_a\tau_g \frac{\partial v}{\partial z} - \omega\epsilon_a \left(\frac{\partial}{\partial z}\frac{\epsilon_g}{\epsilon}\right) v, \\
\left(\nabla_t^2 + \mu_a \frac{\partial}{\partial z}\frac{1}{\mu}\frac{\partial}{\partial z} + k_m^2\right) v &= \omega\mu_a\tau_g \frac{\partial u}{\partial z} + \omega\mu_a \left(\frac{\partial}{\partial z}\frac{\mu_g}{\mu}\right) u,
\end{aligned} \tag{36}$$

and the solution $\left[\begin{smallmatrix}\phi \\ c\mathbf{A}\end{smallmatrix}\right] = \mathcal{S}^\dagger(\psi)$, generated by the potential ψ, is given explicitly by

$$\begin{aligned}
\phi &= -\frac{1}{\epsilon}\frac{\partial u}{\partial z} - \omega\frac{\epsilon_g}{\epsilon} v, & A_x &= \frac{\partial v}{\partial y}, \\
A_y &= -\frac{\partial v}{\partial x}, & A_z &= -\frac{ik_e^2}{\omega\epsilon_a} u - i\frac{\mu_g}{\mu}\frac{\partial v}{\partial z}.
\end{aligned} \tag{37}$$

In the case where the cartesian components of the tensors (19) are constant, the scalar potentials u and v can be expressed in terms of a single scalar potential [2]. In this case, the equations governing the potentials u and v can be written in the form $\mathcal{E}_2(f) = 0$ with

$$\mathcal{E}_2 \equiv \begin{bmatrix} \frac{1}{\mu_a}\left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right) & \omega\tau_g \frac{\partial}{\partial z} \\ -\omega\tau_g\frac{\partial}{\partial z} & \frac{1}{\epsilon_a}\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) \end{bmatrix} \tag{38}$$

and

$$f = \begin{bmatrix} \mu_a u \\ \epsilon_a v \end{bmatrix}. \tag{39}$$

It is easy to see that $\mathcal{E}_2$ is self-adjoint and that from $\mathcal{E}_2(f) = 0$ one can derive a decoupled equation for $\mu_a u$ or for $\epsilon_a v$. By defining the functions

$$\begin{aligned}
r &\equiv \left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right) u + \omega\tau_g\epsilon_a \frac{\partial v}{\partial z}, \\
s &\equiv -\omega\tau_g\mu_a\frac{\partial u}{\partial z} + \left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) v,
\end{aligned} \tag{40}$$

applying $\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right)$ to the first of these equations and using the second one, we obtain the identity

$$\left[\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right)\left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right) + \omega^2\tau_g^2\epsilon_a\mu_a\frac{\partial^2}{\partial z^2}\right] u$$
$$= \left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) r - \omega\tau_g\epsilon_a\frac{\partial s}{\partial z},$$

which is of the form (3) with

$$\mathcal{O}_2 \equiv \left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right)\left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right) + \omega^2\tau_g^2\epsilon_a\mu_a\frac{\partial^2}{\partial z^2} \tag{41}$$

and

$$\mathcal{S}_2 \equiv \left[\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2, \quad -\omega\tau_g\epsilon_a\frac{\partial}{\partial z}\right]. \tag{42}$$

The adjoints of the operators (41–42) are given by

$$\mathcal{O}_2^\dagger = \left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right)\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) + \omega^2\tau_g^2\epsilon_a\mu_a\frac{\partial^2}{\partial z^2} \tag{43}$$

and

$$\mathcal{S}_2^\dagger = \begin{bmatrix} \nabla_t^2 + \dfrac{\mu_a}{\mu}\dfrac{\partial^2}{\partial z^2} + k_m^2 \\ \omega\tau_g\epsilon_a\dfrac{\partial}{\partial z} \end{bmatrix}, \tag{44}$$

therefore, if the potential V satisfies

$$\left[\left(\nabla_t^2 + \frac{\epsilon_a}{\epsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right)\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) + \omega^2\tau_g^2\epsilon_a\mu_a\frac{\partial^2}{\partial z^2}\right] V = 0 \tag{45}$$

then $\left[\begin{smallmatrix}\mu_a u \\ \epsilon_e v\end{smallmatrix}\right] = \mathcal{S}_2^\dagger(V)$, *i.e.*,

$$\begin{aligned} u &= \frac{1}{\mu_a}\left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right) V, \\ v &= \omega\tau_g\frac{\partial V}{\partial z}, \end{aligned} \tag{46}$$

satisfy Eqs. (36).

b) Biaxial media

In the case of a biaxial medium there exists a system of cartesian coordinates where the permittivity tensor has the form

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \tag{47}$$

with $\epsilon_1 \neq \epsilon_2, \epsilon_2 \neq \epsilon_3$, $\epsilon_3 \neq \epsilon_1$. We shall restrict ourselves to the most important case where the permeability μ is a scalar and where $\epsilon_1, \epsilon_2, \epsilon_3$ are constant. Taking the curl of $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$ and using Eq. (20*b*) we obtain

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = i\omega\mu(\mathbf{K} - i\omega\boldsymbol{\epsilon} \cdot \mathbf{E}). \tag{48}$$

In view of Eq. (47), the z-component of Eq. (48) is

$$\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) - \frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} = i\omega\mu K_z + \omega^2\mu\epsilon_3 E_z. \tag{49}$$

On the other hand, from Eqs. (20*a*) and (47) it follows that

$$\epsilon_1 \frac{\partial E_x}{\partial x} + \epsilon_2 \frac{\partial E_y}{\partial y} + \epsilon_3 \frac{\partial E_z}{\partial z} = \frac{1}{c} K_0$$

therefore,

$$\frac{\partial E_x}{\partial x} = -\frac{\epsilon_2}{\epsilon_1}\frac{\partial E_y}{\partial y} - \frac{\epsilon_3}{\epsilon_1}\frac{\partial E_z}{\partial z} + \frac{1}{c\epsilon_1}K_0. \tag{50}$$

Substituting Eq. (50) into Eq. (49) we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\epsilon_3}{\epsilon_1}\frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon_3\right)E_z + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}\frac{\partial^2 E_y}{\partial y \partial z} = \frac{1}{c\epsilon_1}\frac{\partial K_0}{\partial z} - i\omega\mu K_z. \tag{51}$$

In an entirely similar manner, the y-component of Eq. (48) and Eqs. (47) and (50) give

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon_2\right)E_y + \frac{\epsilon_3 - \epsilon_1}{\epsilon_1}\frac{\partial^2 E_z}{\partial y \partial z} = \frac{1}{c\epsilon_1}\frac{\partial K_0}{\partial y} - i\omega\mu K_y. \tag{52}$$

Equations (51–52) show that E_z and E_y obey a decoupled system of equations and that the operators

$$\mathcal{O} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\epsilon_3}{\epsilon_1}\frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon_3 & \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}\frac{\partial^2}{\partial y \partial z} \\ \frac{\epsilon_3 - \epsilon_1}{\epsilon_1}\frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon_2 \end{bmatrix}, \tag{53}$$

$$\mathcal{S} = \begin{bmatrix} \frac{1}{c\epsilon}\frac{\partial}{\partial z} & 0 & 0 & -i\omega\mu \\ \frac{1}{c\epsilon}\frac{\partial}{\partial y} & 0 & -i\omega\mu & 0 \end{bmatrix}, \tag{54}$$

satisfy the identity $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$ [Eq. (4)], where $\mathcal{T}$ is the linear operator given by

$$\mathcal{T}\begin{bmatrix} \phi \\ c\mathbf{A} \end{bmatrix} = \begin{bmatrix} E_z \\ E_y \end{bmatrix}.$$

It is easy to see that

$$\mathcal{O}^\dagger = \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\epsilon_3}{\epsilon_1}\frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon_3 & \frac{\epsilon_3 - \epsilon_1}{\epsilon_1}\frac{\partial^2}{\partial y \partial z} \\ \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}\frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon_2 \end{bmatrix} \tag{55}$$

and

$$\mathcal{S}^\dagger = \begin{bmatrix} \frac{1}{c\epsilon_1}\frac{\partial}{\partial z} & \frac{1}{c\epsilon_1}\frac{\partial}{\partial y} \\ 0 & 0 \\ 0 & i\omega\mu \\ i\omega\mu & 0 \end{bmatrix}. \tag{56}$$

Therefore, taking $\psi = \left[\begin{smallmatrix} cu \\ cv \end{smallmatrix}\right]$, one concludes that

$$\phi = \frac{1}{\epsilon_1}\frac{\partial u}{\partial z} + \frac{1}{\epsilon_1}\frac{\partial v}{\partial y}, \quad A_x = 0, \quad A_y = i\omega\mu v, \quad A_z = i\omega\mu u, \tag{57}$$

satisfy Maxwell's equations provided that the scalar potentials u and v fulfill

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\epsilon_3}{\epsilon_1}\frac{\partial^2}{\partial z^2} + \omega^2\mu\epsilon_3\right) u + \frac{\epsilon_3 - \epsilon_1}{\epsilon_1}\frac{\partial^2 v}{\partial y \partial z} &= 0, \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon_2\right) v + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}\frac{\partial^2 u}{\partial y \partial z} &= 0. \end{aligned} \tag{58}$$

As in the case of gyrotropic media, the potentials u and v can be expressed in terms of a single potential that obeys a fourth-order partial differential equation. Following the procedure presented above, one finds that

$$\begin{aligned} u &= \frac{\epsilon_1}{\epsilon_2 - \epsilon_1}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon_2\right) V, \\ v &= -\frac{\partial^2 V}{\partial y \partial z} \end{aligned} \tag{59}$$

satisfy Eqs. (58) provided that

$$\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\epsilon_3}{\epsilon_1}\frac{\partial^2}{\partial z^2}+\omega^2\mu\epsilon_3\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial y^2}+\omega^2\mu\epsilon_2\right)\right.$$
$$\left.-\frac{(\epsilon_3-\epsilon_1)(\epsilon_2-\epsilon_1)}{\epsilon_1^2}\frac{\partial^4}{\partial^2 y\partial^2 z}\right]V=0. \quad (60)$$

4. CONCLUDING REMARKS

We have shown that in the cases of the Dirac equation and of the Maxwell equations in an anisotropic medium it is possible to define the adjoint of a linear operator in such a way that the corresponding system of equations is self-adjoint. Then, any decoupled set of equations derived from the original system, by means of linear operations, leads to an expression for the complete solutions of the system of equations in terms of potentials. In the case of the Maxwell equations, the method of adjoint operators yields in a straightforward manner expressions for the electromagnetic potentials (*cf.* also Refs. [1,3]). An example of the usefulness of the expressions for the electromagnetic field in a gyrotropic medium in terms of potentials can be found in Ref. [12].

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