

Noether's theorem and the invariants for dissipative and driven dissipative like systems

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ABSTRACT. Noether's theorem is used to obtain invariants for physical systems obeying equations of motion of the type $\ddot{x} + \lambda(t)\dot{x} + \omega^2(t)x = 0$ and $\ddot{x} + \lambda(t)\dot{x} + \omega^2(t)x = G(t)\partial V/\partial x$. The corresponding auxiliary equations are also obtained.

RESUMEN. Se usa el teorema de Noether para obtener invariantes de sistemas físicos que obedecen ecuaciones de movimiento del tipo $\ddot{x} + \lambda(t)\dot{x} + \omega^2(t)x = 0$ y $\ddot{x} + \lambda(t)\dot{x} + \omega^2(t)x = G(t)\partial V/\partial x$. También se obtienen las correspondientes ecuaciones auxiliares.

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1. INTRODUCTION

In this paper we obtain the constants of motion for systems described by the equation

$$\frac{d^2x}{dt^2} + \lambda(t)\frac{dx}{dt} + \omega^2(t)x = 0, \quad (1)$$

where $\lambda(t)$ and $\omega(t)$ are functions of t . Besides the mathematical interest of finding invariants, we know that the knowledge of certain functions of the coordinates and momenta which remain constant during the motion can be of great help in simplifying the equations of motion of the physical system and can lead to their solution.

Equation (1) arises in many physical problems, for example, the equation of motion of the pendulum seismograph, that of the LRC circuits, and damped harmonic oscillator. Another system characterized by Eq. (1) is a simple pendulum undergoing small amplitude oscillations but with accreting mass.

The fact that Eq. (1) describes the motion for both damped and non-damped systems has originated controversies about the physical meaning of the Lagrangian from which Eq. (1) may be derived [1,2,3]. This distinctness does not affect our calculations, because our results can be applied to all the problems described by Eq. (1). Of course, the physical

meaning of this equation and of the invariant emerges from the physical assignment of the parameters and variables of the mathematical equations.

We also communicate the extension of the results to non-linear systems governed by the Lagrangian

$$\mathcal{L} = \frac{F(t)}{2} [\dot{x}^2 - \omega^2(t)x^2 + 2G(t)V(\beta(t)x)], \tag{2}$$

where $G(t)$ and $\beta(t)$ are functions of time which must be determined.

One reports the mathematical invariant of the Lagrangian (2). Although the Lagrangian (2) can describe driven dissipative and non-dissipative systems, the physical interpretation of the constants of motion for both cases is subject of further study, when the treatment applied to concrete physical problems provide the physical meaning of the parameters and variables of the equations of motion, and accordingly of the invariant. Some of the results reported here are similar to those obtained by Pedrosa [3], however, our method for the obtention of the invariants is new.

The outline of the present paper is as follows. In Sect. 2 we obtain the invariant for the dissipative systems and in Sect. 3 the invariants for the driven dissipative like systems are obtained. Finally, in Sect. 4 we discuss the results.

2. OBTENTION OF THE INVARIANT FOR THE DISSIPATIVE SYSTEMS

We start with the Lagrangian

$$\mathcal{L} = \frac{1}{2}F(t) [\dot{x}^2 - \omega^2(t)x^2], \tag{3}$$

from which the equation of motion (1) is obtained when one sets

$$\lambda(t) = \frac{d}{dt} \ln F(t); \tag{4}$$

$F(t)$ is an arbitrary real function of t .

To find the invariant, Noether's theorem will be used formulated in the following way: let us apply symmetry transformations generated by

$$X = \xi(x, t) \frac{\partial}{\partial t} + \eta(x, t) \frac{\partial}{\partial x}, \tag{5}$$

leaving the action $S = \int \mathcal{L}(x, \dot{x}, t) dt$ invariant, then

$$\xi \frac{\partial \mathcal{L}}{\partial t} + \eta \frac{\partial \mathcal{L}}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi}) \frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{\xi} \mathcal{L} = f, \tag{6}$$

is satisfied and the quantity

$$I = (\xi \dot{x} - \eta) \frac{\partial \mathcal{L}}{\partial \dot{x}} - \xi \mathcal{L} + f \tag{7}$$

is a constant of motion [4].

In Eq. (6), f, η, ξ , are functions of x and t , and $\dot{f}, \dot{\eta}, \dot{\xi}$ their total time derivatives respectively, for instance,

$$\dot{f}(x, t) = \frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x}. \tag{8}$$

The substitution of the Lagrangian (3) in Eq. (6) leads to an expression of \dot{f} and setting the coefficients of \dot{x}^3 and \dot{x}^2 equal to zero we obtain the following equations for ξ, η , and f

$$\xi(x, t) = \xi(t), \tag{9}$$

$$\eta(x, t) = \frac{1}{2}(\dot{\xi} - \lambda(t)\xi)x + \psi(t), \tag{10}$$

$$\begin{aligned} \dot{f}(x, t) = & \left[-\frac{1}{2}\xi\dot{F}(t)\omega^2 - \frac{1}{2}\dot{\xi}F(t)\omega^2 - \xi F(t)\omega\dot{\omega} - \frac{1}{2}(\dot{\xi} - \lambda(t)\xi)\dot{F}(t)\omega^2 \right] x^2 \\ & - F(t)\omega^2\psi x + \frac{1}{2}F(t)(\ddot{\xi} - \dot{\lambda}(t)\xi - \lambda(t)\dot{\xi})x\dot{x} + F(t)\dot{\psi}\dot{x}. \end{aligned} \tag{11}$$

A more general expression of the function $f(x, t)$ consistent with (11) is

$$f(x, t) = a(t)x^2 + b(t)x + \phi(k(t)x), \tag{12}$$

where a, b and k are function of time only and must be determined.

Calculating the time derivative of the function $f(x, t)$ from (12) and comparing each term with (11) we obtain the relations

$$b(t) = F(t)\dot{\psi} \tag{13}$$

and

$$\dot{b}(t) = -F(t)\omega^2\psi, \tag{14}$$

from the coefficients of \dot{x} and x , respectively.

Eqs. (13) and (14) provide immediately a differential equation for $\psi(t)$

$$\ddot{\psi}(t) + \lambda(t)\dot{\psi}(t) + \omega^2(t)\psi(t) = 0. \tag{15}$$

This means that the function $\psi(t)$ must be a solution of the equation of motion.

From the coefficients of $x\dot{x}$ and x^2 we obtain

$$2a(t) + \frac{k^2}{y} \frac{d\phi}{dy} = \frac{F(t)}{2} [\ddot{\xi} - \dot{\lambda}(t)\xi - \lambda(t)\dot{\xi}], \tag{16}$$

$$\dot{a}(t) + \frac{k\dot{k}}{y} \frac{d\phi}{dy} = -\frac{d}{dt} \left[\frac{1}{2}\xi F(t)\omega^2 \right] - \frac{1}{2} [\dot{\xi} - \lambda(t)\xi] F(t)\omega^2, \tag{17}$$

where $y = k(t)x$.

After carrying out some simple algebra with Eqs. (16) and (17) one finds that the function $\xi(t)$ must satisfy the differential equation

$$\ddot{\xi} + 4\Omega^2\dot{\xi} + 4\dot{\Omega}\Omega\xi = 2F(t)^{-1}k^2 \frac{d}{dt} \left[\frac{1}{y} \frac{d\phi}{dy} \right], \tag{18}$$

where

$$\Omega^2(t) = \omega^2(t) - \frac{\lambda^2(t)}{4} - \frac{\dot{\lambda}(t)}{2} \tag{19}$$

is the reduced actual frequency.

A first integral of Eq. (18) with the condition $F(t) = k^2(t)\xi(t)$ is

$$\xi\ddot{\xi} - \frac{1}{2}\dot{\xi}^2 + 2\Omega^2\xi^2 = \frac{2}{y} \frac{d\phi}{dy}, \tag{20}$$

but Eq. (20) transforms into

$$\ddot{\sigma}(t) + \lambda(t)\dot{\sigma}(t) + \omega^2(t)\sigma(t) = \frac{F^{-2}(t)}{\sigma^2 x} \frac{d\phi}{dy}, \tag{21}$$

with $\sigma^2(t) = \xi(t)/F(t)$.

Besides, using Eq. (20), Eq. (16) gives the following expression for the function $a(t)$

$$a(t) = \frac{1}{2}F^2(t) [\dot{\sigma}^2(t) - \omega^2(t)\sigma^2(t)]. \tag{22}$$

Finally, with the aim of Eqs. (13) and (22) one obtains the function $f(x, t)$, whose substitution in (7) leads to the invariant

$$I = \frac{1}{2}F^2(t) [\sigma\dot{x} - \dot{\sigma}x]^2 + \phi(x/\sigma). \tag{23}$$

3. OBTENTION OF THE INVARIANT FOR THE DRIVEN DISSIPATIVE LIKE SYSTEMS

In order to obtain the constant of motion we will use the same treatment given in the last section.

In this case, the functions ξ , η , and \dot{f} can be expressed as

$$\xi = \xi(t), \tag{24}$$

$$\eta(x, t) = \frac{1}{2}(\dot{\xi} - \lambda(t)\xi)x + \psi(t), \tag{25}$$

$$\begin{aligned} \dot{f}(x, t) = & \left[-\frac{1}{2}\xi\dot{F}\omega^2 - \xi F\omega\dot{\omega} - \frac{1}{2}F\dot{\xi}\omega^2 - \frac{1}{2}(\dot{\xi} - \lambda\xi)F\omega^2 \right] x^2 \\ & - \psi F\omega^2 x + \frac{F}{2}(\ddot{\xi} - \dot{\lambda}\xi - \lambda\dot{\xi})x\dot{x} + \dot{\psi}F\dot{x} + \frac{d}{dt} [\xi FG] V \\ & + [\xi FG\dot{\beta} + \frac{1}{2}(\dot{\xi} - \lambda\xi)FG\beta] \frac{\partial V}{\partial u} x + \psi FG\beta \frac{\partial V}{\partial u}. \end{aligned} \tag{26}$$

Since the potential $G(t)V(\beta(t)x)$ is arbitrary we can choose it so that the coefficients of V , $\frac{\partial V}{\partial u}x$, and $\frac{\partial V}{\partial u}$ vanish separately, so that the relations

$$\frac{d}{dt}[\xi FG] = 0, \quad (27)$$

$$\xi FG\dot{\beta} + \frac{1}{2}(\dot{\xi} - \lambda\xi)FG\beta = 0, \quad (28)$$

$$\psi FG\beta = 0 \quad (29)$$

must hold.

The set of Eqs. (27)–(29) determine the functions $G(t)$, $\beta(t)$ and $\psi(t)$ completely

$$G(t) = \frac{F(t)^{-1}}{\xi}, \quad (30)$$

$$\beta(t) = \left[\frac{F(t)}{\xi}\right]^{1/2}, \quad (31)$$

$$\psi(t) = 0. \quad (32)$$

The results allow us to write Eq. (26) as

$$\dot{f} = \left[-\frac{1}{2}\xi\dot{F}\omega^2 - \xi F\omega\dot{\omega} - \frac{1}{2}F\dot{\xi}\omega^2 - \frac{1}{2}(\dot{\xi} - \lambda\xi)F\omega^2\right]x^2 + \frac{F}{2}[\ddot{\xi} - \dot{\lambda}\xi - \lambda\dot{\xi}]x\dot{x}. \quad (33)$$

The more general expression for the function $f(x, t)$ consistent with (33) is now

$$f(x, t) = a(t)x^2 + \phi(k(t)x), \quad (34)$$

where $a(t)$ and $k(t)$ are functions of t only.

In this way one has reduced the problem to that analyzed in Sect. 2, but with the condition $b(t) = 0$. The expressions for $a(t)$ and $k(t)$ are the same as in that case, and with the help of (30) and (31) we conclude that the Lagrangian (2) which admits an Ermakov-Lewis invariant must be of the form

$$\mathcal{L} = \frac{F(t)}{2} \left[\dot{x}^2 - \omega^2(t)x^2 + \frac{2F(t)^{-2}}{\sigma^2} V(x/\sigma) \right], \quad (35)$$

with $\sigma^2(t) = \xi(t)/F(t)$.

Moreover using (32), (34) and (35) in Eq. (7), the invariant follows immediately

$$I = \frac{1}{2}F^2(t) [\sigma\dot{x} - \dot{\sigma}x]^2 + \phi\left(\frac{x}{\sigma}\right) - V\left(\frac{x}{\sigma}\right). \quad (36)$$

σ is solution of the auxiliary equation

$$\ddot{\sigma}(t) + \lambda(t)\dot{\sigma}(t) + \omega^2(t)\sigma(t) = \frac{F^{-2}(t) d\phi}{\sigma^2 x dy}, \tag{37}$$

where $y = x/\sigma$. Eq. (37) appears as a consequence of the formalism used to obtain the invariant.

We have thus used Noether’s theorem to obtain the invariant for systems characterized by the Lagrangian (35). In Eq. (2) we have proposed a Lagrangian with a more general potential but we have found that such potential must be restricted to the form given in (35) in order to have Eq. (36) as a Noether invariant.

Eq. (36) is the more general expression of the constant of motion. If $F(t) = 1$, the results obtained by Lutzky emerge immediately when

$$\phi(x/\sigma) = \frac{1}{2} \left(\frac{x}{\sigma}\right)^2 \quad \text{and} \quad V\left(\frac{x}{\sigma}\right) = 0.$$

On the other hand with the values

$$\phi\left(\frac{x}{\sigma}\right) = \int^{\frac{x}{\sigma}} g(u) du \quad \text{and} \quad V\left(\frac{x}{\sigma}\right) = \int^{\frac{x}{\sigma}} f(\eta) d\eta,$$

Eqs. (36) and (37) transform into

$$I = \frac{1}{2}F(t)^2[\sigma\dot{x} - \dot{\sigma}x]^2 + \int^{\frac{x}{\sigma}} f(\eta) d\eta + \int^{\frac{x}{\sigma}} g(u) du \tag{38}$$

and

$$\ddot{\sigma} + \lambda(t)\dot{\sigma} + \omega^2(t)\sigma = \frac{F^{-2}(t)}{\sigma^2 x} g\left(\frac{x}{\sigma}\right). \tag{39}$$

Eqs. (38) and (39) generalize the results obtained by Kaushal, *et al.* [5] and J.R. Ray, *et al.* [10]. The conditions $\lambda(t) = f(\eta) = 0$ lead to the results obtained by E. González-Acosta, *et al.* [9].

4. CONCLUSIONS

In this paper we have applied Noether’s theorem to the Lagrangian given by Eq. (3) and we have obtained the invariant

$$I = \frac{1}{2}F^2(t)[\sigma\dot{x} - \dot{\sigma}x]^2 + \phi(x/\sigma). \tag{40}$$

These results generalize those obtained by other authors [4,6] who have obtained the invariants of the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 - \omega^2(t)x^2). \tag{41}$$

Equation (40) is a general representation for the invariant, but in the case of $F(t) = 1$ and $\lambda(t) = 0$, the results obtained by Lutzky [4] follow immediately when $\phi(x/\sigma) = \frac{1}{2}(x/\sigma)^2$.

If $\phi(x/\sigma) = \int^{x/\sigma} g(u) du$, Eqs. (21) and (23) transform into Eqs. (39) and (38) respectively which generalize the particular results obtained by Kaushal *et al.* for the damped harmonic oscillator [5].

We have used the Noether's formalism because it seems to be more powerful than Ermakov's method, since it is consistent and it does not need the use of auxiliary devices to obtain the constants of motion. Of course, the physical interpretation of Eqs. (40) and (38) is still open, because of the physical assignment of the parameters and variables of Eqs. (1) and (2), which would allow to associate Eq. (40) to a conservation law only if the invariant 'I', represents a physical quantity directly. Likewise, such physical meaning of the equations is also bound to the selection of the transformations represented in Eq. (5). The establishment of this physical interpretation, and the generalization of (40) and (38) to other systems are topics of further study.

The physical interpretation of Eqs. (36) and (37) has also a direct dependence on the physical problem to be analyzed and with the transformations ξ and η of Eq. (5). A very rough analysis shows that for the point transformation $x = \sigma(t)y$ [9], Eq. (36) contains the kinetic and potential energy affected by an inflationary term σ^2 , whereas Eq. (37) describes the behavior of the scale factor $\sigma(t)$.

Transformation (5) can be also considered as a generator of a symmetry algebra [11], and Eq. (1) can be seen as a spectral equation in the construction of soliton solutions of non linear equations [12]. This opens the possibilities for the applications of our methods to find invariants to this topic.

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