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Quantum phase space for an ideal relativistic gas in d–spatial dimensions II. Asymptotic formulas

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ABSTRACT. We derive asymptotic formulas for some of the statistical distributions for a quantum ideal relativistic gas in $(d+1)$–dimensions, such as the invariant phase space, the grand canonical density of states and the microcanonical density of cluster states. The exact formula for the invariant phase space was given in our previous paper, but those for the latter two densities of states are presented in this paper. For comparison, we also provide the thermodynamical description.

RESUMEN. Derivamos fórmulas asintóticas para algunas de las distribuciones estadísticas para un gas ideal cuántico relativista en $(d+1)$ dimensiones, tales como la del espacio fase invariante, la densidad de estados gran canónica y la densidad de estados de cúmulos microcanónica. En nuestro artículo previo se dio la fórmula exacta para el espacio fase invariante, pero en este trabajo se presentan las últimas dos densidades de estado. Para comparar, también damos la descripción termodinámica.

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1. INTRODUCTION

In a previous paper [1] we have presented the basic formulas for the d–spatial dimensional invariant phase-space (with Boltzmann (BO) statistics) integral and we have developed a formalism for the invariant phase space for a relativistic quantum ideal gas in d–spatial dimensions. We have derived the cluster decomposition for the grand canonical and canonical partition functions as well as for the microcanonical and grand microcanonical densities of states. These densities of states are expressed in terms of the ordinary relativistic (with BO statistics) phase-space integral in which appear(s) multiple mass(es) of the "cluster particle(s)". In this paper we derive additionally the basic formulas for the invariant grand canonical density of states, together with the microcanonical density of cluster states. Since most of the formulas look complicated at first sight, we derive asymptotic formulas for some of these quantities (the invariant phase space, the grand canonical density of states and the microcanonical density of cluster states), whereby the formulas become more suited for numerical calculation and for a better understanding of their physical (statistical-thermodynamical) meanings. For the sake of completeness, we provide the thermodynamical description using the standard method. We show, in particular, that the asymptotic formula for the average total number of particles N derived from the microcanonical density of cluster states for large number of cluster particles and the one for the average energy density ρ derived from the grand canonical density for large total energy P are consistent with the corresponding thermodynamical expressions.

We organize the paper as follows: In Sect. 2 we present the main results obtained in Ref. [1], *i.e.*, in Subsect. 2.1, the formula for the $(d+1)$–dimensional phase-space integral and in 2.2 the statistical-thermodynamical formulas for a quantum ideal relativistic gas: canonical partition function, grand canonical partition function, invariant microcanonical density of states, invariant grand microcanonical density of states and invariant canonical density of states together with microcanonical density of cluster states. In Sect. 3, we derive the asymptotic formulas for some of these quantities. In Sect. 4 we provide the thermodynamical description and compare with the asymptotic formulas. In Sect. 5 we summarize the necessary modifications if the "invariant momentum-space measure" [2] is employed instead of the "invariant phase-space measure" used throughout the paper.

2. Basic formulas

Let us summarize the main formulas derived in Ref. [1], keeping the same notations and the conventions.

2.1. Phase-space integral in d-spatial dimensions

The $(d+1)$–dimensional invariant N–particle phase-space integral $R_N^{(d)}(P, m_1, m_2, \dots, m_N)$ is defined by

$$R_N^{(d)}(P, m_1, m_2, \dots, m_N) = \int \delta^{(d+1)}\left(P - \sum_{i=1}^{N} p_i\right) \prod_{i=1}^{N} d\sigma^{(d)}(p_i, m_i), \tag{1}$$

where $d\sigma^{(d)}$ denotes the Touschek's invariant phase-space measure [2,3] in d–spatial dimensions (see Eq. (I.4) of Ref. [1]). For the Laplace transform of $R_N^{(d)}$ we have

$$\begin{aligned}\Phi_N^{(d)}(\beta, m_1, m_2, \dots, m_N) &= \int \exp(-\beta P) R_N^{(d)}(P, m_1, m_2, \dots, m_N)\, d^{(d+1)}P \\ &= \prod_{i=1}^{N} \phi^{(d)}(\beta, m_i),\end{aligned} \tag{2}$$

where

$$\begin{aligned}\phi^{(d)}(\beta, m_i) &= \int \exp(-\beta p)\, d\sigma^{(d)}(p, m_i) \\ &= C_d (m_i\beta)^{-d''} K_{d'}(m_i\beta).\end{aligned} \tag{3}$$

(For the definition of d', d'', C_d and $K(x)$, see Eqs. (I.9), (I.10) and (I.11) of Ref. [1], respectively.)

In the β–rest frame, one can rewrite Eq. (2) as

$$\Phi_N^{(d)}(\beta, m_1, m_2, \ldots, m_N) = \frac{S_d}{\beta} \int_0^\infty K_1(\beta P) R_N^{(d)}(P, m_1, m_2, \ldots, m_N) P^{d-1}\, dP. \quad (4)$$

(For the definition of S_d, see Eq. (I.13) of Ref. [1].) By inverting Eq. (4), we get a rigorous formula for $R_N^{(d)}$, which for $c > 0$, reads as

$$R_N^{(d)}(P, m_1, m_2, \ldots, m_N) = \frac{1}{(\pi S_d P^{d-2})} \frac{1}{i} \int_{c-i\infty}^{c+i\infty} d\beta\, \beta^2 I_1(\beta P) \Phi_N^{(d)}(\beta, m_1, m_2, \ldots, m_N), \quad (5)$$

where $I_1(z)$ is the modified Bessel function of order 1.

2.2. *Quantum relativistic ideal gas in d-spatial dimensions*

Canonical partition function:

$$Z_N^{(d)}(\gamma) = \sum_{\{n,N\}} \prod_{k=1}^{N} \frac{1}{n_k!} \left[C_d (m\beta)^{-d''} \left\{ \frac{(-\gamma)^{k-1}}{k^{d'}} \right\} K_{d'}(km\beta) \right]^{n_k}, \quad (6)$$

where $\{n, N\}$ is the partition number of N satisfying $\sum_{k=1}^{N} kn_k = N$, $\gamma = 1$ for Fermi-Dirac (FD) statistics, and $\gamma = -1$ for Bose-Einstein (BE) statistics. In the BO case ($\gamma = 0$), only one partition survives, namely, $\{n, N\} = \{N, 0, \ldots, 0\}$.

Grand canonical partition function:

$$\begin{aligned} Z^{(d)}(z, \gamma) &= \sum_{N=0}^{\infty} Z_N^{(d)}(\gamma) z^N \\ &= \exp\left\{ C_d (m\beta)^{-d''} \sum_{k=1}^{\infty} \left[\frac{(-\gamma)^{k-1} z^k}{k^{d'}} \right] K_{d'}(km\beta) \right\}. \end{aligned} \quad (7)$$

Invariant microcanonical density of states:

$$\begin{aligned} \sigma_N^{(d)}(Q, \omega_d, \gamma) = \sum_{\{n,N\}} G^{(d)}(\{n, N\}, \gamma) \int \delta^{(d+1)} \left(Q - \sum_{k=1}^{N} P_k \right) \\ \times \prod_{k=1}^{N} R_N^{(d)}(P_k, km)\, d^{(d+1)} P_k, \end{aligned} \quad (8)$$

where

$$G^{(d)}\left[\{n,N\},\gamma\right]=\prod_{k=1}^{N}\frac{1}{n_k!}\left[\frac{(-\gamma)^{k-1}}{k^{(d+1)}}\right]^{n_k}. \tag{9}$$

Invariant grand microcanonical density of states ($z=1$):

$$\sigma^{(d)}(Q^2,\omega_d^2,Q\omega_d,\gamma)=\sum_{N=0}^{\infty}\frac{1}{N!}R_N^{(d),\mathrm{eff}}(Q,m), \tag{10}$$

where

$$R_N^{(d),\mathrm{eff}}(Q,m)=\sum_{k_1,k_2,\dots}^{\infty}\prod_{i=1}^{N}\left[\frac{(-\gamma)^{k_i-1}}{k_i^{d+1}}\right]R_N^{(d)}(Q,k_1m,k_2m,\dots,k_Nm)\qquad (N\text{ masses}). \tag{11}$$

Invariant grand canonical density of states:

$$\sigma^{(d)}(Q^2,\omega_d^2,Q\omega_d,z,\gamma)=\sum_{N=0}^{\infty}\sigma_N^{(d)}(Q,\omega_d,\gamma)z^N;\qquad Q^2=M^2. \tag{12}$$

This formula can be rewritten as follows (hereafter we omit the arguments ω_d^2 and $Q\omega_d$ in $\sigma^{(d)}$ for brevity):

$$\sigma^{(d)}(Q^2,z,\gamma)=\sum_{n_1n_2\dots}\prod_{j=1}^{\infty}\frac{1}{n_j!\,j^{n_j}}\rho^{(d)}_{(n_1,n_2,\dots)}(Q^2,z,\gamma), \tag{13}$$

where

$$\rho^{(d)}_{(n_1,n_2,\dots)}(Q^2,z,\gamma)=\int\delta^{(d+1)}\left(Q-\sum_{j=1}^{\infty}\sum_{i_j=1}^{n_j}Q_{i_j}^{(j)}\right)$$
$$\prod_{j=1}^{\infty}\prod_{i_j=1}^{n_j}\sigma_j^{(d)}\left(Q_{i_j}^{(j)2},z,\gamma\right)\,d^{(d+1)}Q_{i_j}^{(j)}, \tag{14}$$

with

$$\sigma_j^{(d)}\left(Q_{i_j}^{(j)2},z,\gamma\right)=2g\left(Q_{i_j}^{(j)}\omega_d\right)\left[\frac{(-\gamma)^{j-1}z^j}{j^{(d+1)}}\right]$$
$$\times\,\theta\left(Q_{i_j0}^{(j)}\right)\delta\left[Q_{i_j}^{(j)2}-(jm)^2\right]. \tag{15}$$

$\rho^{(d)}_{(n_1,n_2,\ldots)}(Q^2,z,\gamma)$ is the microcanonical density of cluster states, *i.e.*, the phase-space integral for a number n_1 of 1 identical particle clusters, n_2 of 2 identical particle clusters,..., n_j of j identical particle clusters, and $n = \sum_{j=1}^{\infty} n_j$ is the total number of clusters. Although Eq. (13) has not been presented in Ref. [1], its validity can be straightforwardly proven by Laplace transforming it, *i.e.*, by going into the corresponding canonical partition function

$$Z^{(d)}(z,\gamma) = \int \exp(-\beta Q)\sigma^{(d)}(Q^2,z,\gamma)\, d^{(d+1)}Q. \tag{16}$$

Both $\sigma^{(d)}(Q^2,z,\gamma)$ and $Z^{(d)}(z,\gamma)$ describe an ensemble with varying number of particles.

3. Derivation of asymptotic formulas

3.1. d-dimensional phase-space integral

For the approximate evaluation of the phase-space integral $R_N^{(d)}$ in Eq. (1) for large N (P/N = const.), it is possible to apply the well-known method of Lurçat and Mazur [4], which is essentially an extension of Khinchin's method [5], based on the central limit theorem of statistics. In order to apply here this method to the $(d+1)$–dimensional case, we introduce the distribution function

$$U_N^{(d)}(P,\beta,m_1,m_2,\ldots,m_N) = \frac{R_N^{(d)}(\beta,m_1,m_2,\ldots,m_N)\exp(-\beta P)}{\Phi_N^{(d)}(\beta,m_1,m_2,\ldots,m_N)}. \tag{17}$$

Taking into account Eqs. (1), (2) and (3), one can rewrite this equation as

$$U_N^{(d)}(P,\beta,m_1,m_2,\ldots,m_N) = \int \delta^{(d+1)}\left(P - \sum_{i=1}^{N} p_i\right)\prod_{i=1}^{N} u_i^{(d)}(p_i,\beta,m_i), \tag{18}$$

where

$$u_i^{(d)}(p_i,\beta,m_i) = \frac{d\sigma^{(d)}(p_i,m_i)\exp(-\beta p_i)}{\phi^{(d)}(\beta,m_i)}. \tag{19}$$

It is clear from Eq. (18) that $U_N^{(d)}(P,\beta,m_1,m_2,\ldots,m_N)$ can be considered as the distribution function for the sum of N independent random variables. According to the central limit theorem of probability [5], one has the following formula valid for $N \to \infty$

$$U_N^{(d)}(P,\beta,m_1,m_2,\ldots,m_N) = (2\pi)^{-d'}(\det B)^{-1/2} + O(1/N). \tag{20}$$

Thus the method of Ref. [4] allows one to derive from Eqs. (17) and (20) the following asymptotic formula

$$R_N^{(d)}(P,m_1,m_2,\ldots,m_N) = \Phi_N^{(d)}(\beta,m_1,m_2,\ldots,m_N)\exp(\beta P_0) \times (2\pi)^{-d'}(\det B)^{-1/2} + O(1/N), \tag{21}$$

where the statistical inverse temperature β is the solution of

$$P_0 = -\frac{\partial}{\partial\beta}\left[\ln \Phi_N^{(d)}(\beta, m_1, m_2, \ldots, m_N)\right]$$

$$= \beta^{-1}\left[dN + \sum_{i=1}^{N} S(m_i\beta)\right] \tag{22}$$

with

$$S(m_i\beta) = \frac{m_i\beta K_{d'-1}(m_i\beta)}{K_{d'}(m_i\beta)}. \tag{23}$$

The dispersion matrix $B_{\mu\nu}$ is

$$B_{\mu\nu} = \frac{\partial^2}{\partial\beta_\mu\partial\beta_\nu}\left[\ln \Phi_N^{(d)}(\beta, m_1, m_2, \ldots, m_N)\right]\left(\frac{P_0}{\beta}\right)^d. \tag{24}$$

More explicitly Eq. (21) reads

$$R_N^{(d)}(P, m_1, m_2, \ldots, m_N) = \Phi_N^{(d)}(\beta, m_1, m_2, \ldots, m_N)\exp(\beta P_0)$$

$$\times (2\pi)^{-d'}\beta^{(d+2)/2}P_0^{-d/2}[a(\beta)]^{-1/2} + O(1/N), \tag{25}$$

where

$$a(\beta) = d(d+1)N - d\beta P_0 + \sum_{i=1}^{N}(m_i\beta)^2 - \sum_{i=1}^{N}\left[S(m_i\beta)\right]^2. \tag{26}$$

This formula is more suited for the numerical evaluation of the phase-space integral for large N.

3.2 Grand canonical density of states

Performing a similar calculation which led to Eq. (5), we derive from Eq. (16)

$$\sigma^{(d)}(P^2, z, \gamma) = \frac{1}{\pi S_d P^{(d-2)}}\frac{1}{i}\int_{c-i\infty}^{c+i\infty} d\beta\, \beta^2 I_1(\beta P) Z^{(d)}(z, \gamma); \qquad P^2 = M^2. \tag{27}$$

Let us estimate $\sigma^{(d)}(P^2, z, \gamma)$ in the thermodynamical limit using (see Eq. (7))

$$Z^{(d)}(z, \gamma) = \exp\left\{gV_d\left[\frac{m}{2\pi\beta}\right]^{d/2} F_{d'}^{(d')}(z, m\beta)\right\}, \tag{28}$$

where we use the notation

$$F_{\mu}^{(i)}(x,y) = \left(\frac{2y}{\pi}\right)^{1/2} \sum_{k=1}^{\infty} \left[\frac{(-\gamma)^{k-1}x^k}{k^\mu}\right] K_i(ky). \tag{29}$$

Since the modified Bessel function behaves asymptotically as

$$I_1(\beta P) \underset{P\to\infty}{\sim} \frac{\exp(P\beta)}{(2\pi P\beta)^{1/2}}, \tag{30}$$

one can write the integral in Eq. (27) as

$$(\cdots)\exp\left\{P\beta + P\rho^{-1}g\left[\frac{m}{2\pi\beta}\right]^{d/2} F_{d'}^{(d')}(z, m\beta)\right\}, \tag{31}$$

where $(\cdots)$ means a factor slowly varying in β. In the thermodynamical limit the grand canonical energy density $\rho = P/V_d$ and z are fixed. This exponent is strongly peaked at the point $\beta = \beta_s$ on the real β–axis, and we find

$$\rho = g\left[\frac{m}{2\pi\beta_s}\right]^{d/2} \left[\frac{d}{\beta_s} F_{d'}^{(d')}(z, m\beta_s) + mF_{d'-1}^{(d'-1)}(z, m\beta_s)\right]. \tag{32}$$

To get the steepest path we choose $c = \beta_s$. Then the saddle point method gives the following asymptotic formula at $\beta = \beta_s$

$$\sigma^{(d)}(P^2, z, \gamma) = \frac{1}{\pi S_d P^{(d-2)}} \beta P^{-(d-3/2)} \left[P\frac{\partial^2}{\partial\beta^2}[f(\beta,z)]\right]^{-1/2} \exp[Pf(\beta,z)], \tag{33}$$

where

$$f(\beta, z) = \beta + \rho^{-1}g\left[\frac{m}{2\pi\beta}\right]^{d/2} F_{d'}^{(d')}(z, m\beta), \tag{34}$$

and

$$\frac{\partial^2}{\partial\beta^2}[f(\beta,z)] = \rho^{-1}g\left[\frac{m}{2\pi\beta}\right]^{d/2} \left\{\frac{d(d+1)}{\beta^2} F_{d'}^{(d')}(z, m\beta) + \frac{m(2d-1)}{\beta} F_{d'-1}^{(d'-1)}(z, m\beta) + m^2 F_{d'-2}^{(d'-2)}(z, m\beta)\right\}. \tag{35}$$

The asymptotic formula (33) together with Eqs. (34) and (35) is suited for the numerical evaluation of the invariant grand canonical density of states (defined by Eq. (13)) for large value of P with $\rho = P/V_d$ and z fixed.

3.3. Microcanonical density of cluster states

For a better understanding of the Eqs. (13) and (14), let us derive still another approximate formulas. Proceeding in a similar way as in the case of ordinary phase space in Subsect. 3.1, *i.e.*, using the method of Lurçat and Mazur [4], we obtain for large n_i

$$\rho^{(d)}_{(n_1,n_2\cdots)}(P^2,z,\gamma) \sim (2\pi)^{d'} P_0^{-d/2} \beta_s^{(d+2)/2}$$

$$\times \exp(\beta_s P_0)[a(\beta_s)]^{-1/2} \prod_{j=1}^{\infty} \left[Z_j^{(d)}(\beta_s,z)\right]^{n_j} + O(1/n_j), \qquad (36)$$

where

$$a(\beta) = d(d+1)n_j - d\beta P_0 + \sum_{j=1}^{\infty} \left(jm\beta\right)^2 n_j - \sum_{j=1}^{\infty} \left[S(jm\beta)\right]^2 n_j, \qquad (37)$$

and

$$Z_j^{(d)}(\beta_s,z) = C_d(m\beta)^{-d''} \frac{(-\gamma)^{j-1} z^j}{j^{d'-1}} K_{d'}(jm\beta). \qquad (38)$$

In Eq. (36), β_s is the solution of the equation

$$P_0\beta - dn = \sum_{j=1}^{\infty} S(jm\beta)n_j. \qquad (39)$$

The term $j = 1$ with $n_1 = n = N$ reproduces the classical Boltzmann case, already treated in Subsect. 3.2.

Using Eqs. (13), (36) and the Stirling formula

$$n! \sim (2\pi n)^{1/2} \exp(-n) n^n, \qquad (40)$$

the logarithm of the product in Eq. (13) can be approximated by

$$(\cdots)\left\{ \sum_{j=1}^{\infty} \left[-n_j \ln j - n_j \ln n_j + n_j + n_j \ln Z_j^{(d)}(\beta_s,z)\right] + \beta_s P_0 \right\}, \qquad (41)$$

where $(\cdots)$ means slowly changing pieces in n_j. Thus requiring that the first derivative of the exponent vanishes at the point $n = \overline{n_j}$, we find

$$\overline{n_j} = j^{-1} Z_j^{(d)}(\beta_s,z). \qquad (42)$$

Consequently we have

$$\overline{n} = \sum_{j=1}^{\infty} \overline{n_j} = \sum_{j=1}^{\infty} j^{-1} Z_j^{(d)}(\beta_s, z),$$

$$= gV_d \left[\frac{m}{2\pi\beta_s}\right]^{d/2} F_{d'}^{(d')}(z, m\beta_s), \tag{43}$$

$$\overline{N} = \sum_{j=1}^{\infty} j\overline{n_j} = \sum_{j=1}^{\infty} Z_j^{(d)}(\beta_s, z)$$

$$= gV_d \left[\frac{m}{2\pi\beta_s}\right]^{d/2} F_{d'-1}^{(d')}(z, m\beta_s). \tag{44}$$

For given P_0 and V_d, the solution of Eq. (39) determines the inverse temperature β_s and the dominant average multiplicities $\overline{n_j}$, $\overline{n}$, and $\overline{N}$.

4. CONVENTIONAL THERMODYNAMICAL DESCRIPTION

For the sake of completeness and also for the consistency check, let us rederive the thermodynamical formulas for $(d+1)$-dimensions, adopting the standard technique. The average number of particles in the ensemble is given by

$$\overline{N} = z\frac{\partial}{\partial z}\left[\ln Z_j^{(d)}(z, \gamma)\right]$$

$$= gV_d \left[\frac{m}{2\pi\beta}\right]^{d/2} F_{d'-1}^{(d')}(z, m\beta). \tag{45}$$

This formula in its form coincides with Eq. (44). The average total energy over the ensemble is given by

$$\overline{E} = -\frac{\partial}{\partial \beta}\left[\ln Z^{(d)}(z, \gamma)\right]$$

$$= gV_d \left[\frac{m}{2\pi\beta}\right]^{d/2} \left[\frac{d}{\beta} F_{d'}^{(d')}(z, m\beta) + mF_{d'-1}^{(d'-1)}(z, m\beta)\right]. \tag{46}$$

This is consistent with the formula (32) obtained in Subsect. 3.2. The pressure p is defined as

$$p\beta = \lim_{V_d \to \infty} \frac{1}{V_d}\left[\ln Z^{(d)}(z, \gamma)\right]$$

$$= gV_d \left[\frac{m}{2\pi\beta}\right]^{d/2} F_{d'}^{(d')}(z, m\beta). \tag{47}$$

Notice that the dimensionality of p is $[(\text{length})^{-(d+1)}]$. From Eqs. (45) and (47) one can derive the virial expansion for the ideal relativistic quantum gas in $(d+1)$–dimensions:

$$\frac{pV_d\beta}{\overline{N}} = 1 - \sum_{k\geq 1} \frac{k}{k+1}\beta_k v^k; \qquad v = \frac{\overline{N}}{V_d}. \tag{48}$$

For the explicit expression of the virial coefficients β_k, which is identical to the one for a classical interacting gas, see, *e.g.*, Ref. [6]. The case $d = 3$ in this Sect. correctly reproduces all the familiar formulas in the literature [2,4,7].

5. Invariant momentum space measure

Throughout the paper we have adopted the Touschek's "invariant phase space measure" for the phase-space integration. If one employs, however, the Srivastava-Sudarshan's "invariant momentum-space measure" [2,8] for the phase-space integration, the following changes should be performed in the main (and subsequent) formulas

$$K_{d'}(m_i\beta) \to K_{d''}(m_i\beta) \qquad \text{in Eq. (3)},$$

$$C_d \to gB(2\pi m_i^2)^{d''} \qquad \text{in Eq. (3)},$$

$$K_{d'}(km_i\beta) \to K_{d''}(km_i\beta) \qquad \text{in Eq. (6) and (7)},$$

$$\frac{1}{k^{d+1}} \to \frac{1}{k^d} \qquad \text{in Eq. (9)},$$

$$\frac{1}{k_i^{d+1}} \to \frac{1}{k_i^d} \qquad \text{in Eq. (11)},$$

$$\frac{2g(\omega_d Q_{i_j}^{(j)})}{j^{d+1}} \to \frac{gB}{j^d} \qquad \text{in Eq. (15)},$$

$$d \to d-1 \qquad \text{in Eqs. (22) and (39)},$$

$$d' \to d'' \qquad \text{in Eq. (23)},$$

$$d(d+1)N - d\beta P_0 + \cdots \to (d-1)^2 N - (d-2)\beta P_0 + \cdots \qquad \text{in Eq. (26)}$$

In Subsect. 3.2 the parameter V_d is replaced by B using the relation

$$\frac{2mV_d}{(2\pi)^d} \to B, \tag{49}$$

and the following changes should take place:

$$F_{d'}^{(d')}(z, m\beta_{(s)}) \rightarrow F_{d'}^{(d'')}(z, m\beta_{(s)})$$

in Eqs. (28), (31), (34), (43) and (47);

$$F_{d'-1}^{(d')}(z, m\beta_{(s)}) \rightarrow F_{d'-1}^{(d'')}(z, m\beta_{(s)})$$

in Eqs. (44) and (45);

$$\frac{d}{\beta_s} F_{d'}^{(d')}(z, m\beta_{(s)}) + m F_{d'-1}^{(d'-1)}(z, m\beta_{(s)}) \rightarrow \frac{(d-1)}{\beta_s} F_{d'}^{(d'')}(z, m\beta_{(s)}) + m F_{d'-1}^{(d''-1)}(z, m\beta_{(s)})$$

in Eqs. (32) and (46);

$$\frac{d(d+1)}{\beta^2} F_{d'}^{(d')}(z, m\beta) + \frac{m(2d-1)}{\beta} F_{d'-1}^{(d'-1)}(z, m\beta) + m^2 F_{d'-2}^{(d'-2)}(z, m\beta)$$

$$\rightarrow \frac{d(d-1)}{\beta^2} F_{d'}^{(d'')}(z, m\beta) + \frac{m(2d-3)}{\beta} F_{d'-1}^{(d''-1)}(z, m\beta) + m^2 F_{d'-2}^{(d''-2)}(z, m\beta)$$

in Eq. (35);

$$d(d+1)n_j - d\beta P_0 + \cdots \rightarrow (d-1)^2 n_j - (d-2)\beta P_0 + \cdots$$

in Eq. (37); and

$$K_{d'}(jm\beta) \rightarrow K_{d''}(jm\beta)$$

in Eq. (38).

We do not quote here the results for the less important formulas, which can easily be evaluated using these replacements.

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