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Debye potentials for isotropic elastic media

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla, 72000 Puebla, Pue., México

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Abstract. The equations of equilibrium for an isotropic elastic medium in the absence of body forces are solved in circular cylindrical coordinates by separation of variables making use of spin-weighted cylindrical functions. It is shown that the most general solution can be expressed in terms of three scalar potentials that satisfy the Laplace equation. A shorter derivation of the expression for the solution in terms of scalar potentials, using Wald's method of adjoint operators, is also given.

Resumen. Las ecuaciones de equilibrio para un medio elástico isótropo en ausencia de fuerzas de volumen se resuelven en coordenadas cilíndricas por separación de variables, haciendo uso de funciones cilíndricas con peso de espín. Se muestra que la solución más general puede expresarse en términos de tres potenciales escalares que satisfacen la ecuación de Laplace. Se da también una deducción más corta de la expresión para la solución en términos de potenciales escalares, usando el método de operadores adjuntos de Wald.

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1. Introduction

The equations governing vector, tensor, or spinor fields correspond to systems of partial differential equations whose solution is often an involved task due to the coupling of the components of the field, especially if noncartesian coordinates are employed. When the problem under consideration possesses spherical symmetry, it is convenient to use spherical coordinates; then, by expressing the system of equations in terms of quantities with a well-defined spin-weight [1-6], certain differential operators (denoted by $\eth$ and $\bar{\eth}$) that change the spin-weight in one unit arise naturally and the equations can be reduced to a set of ordinary differential equations making use of the spin-weighted spherical harmonics. A similar reduction can be obtained when a system of partial differential equations is written in circular cylindrical coordinates, making use of spin-weighted quantities and of the appropriate operators that raise or lower the spin-weight [7]. The reduction to a set of ordinary differential equations then follows from the existence of spin-weighted cylindrical functions that possess many properties analogous to those of the spin-weighted spherical harmonics.

In this paper the equations of equilibrium for an isotropic elastic medium in the absence of body forces are solved in circular cylindrical coordinates making use of spin-weighted quantities. It is shown that the most general solution of these equations can be written in terms of three scalar (Debye) potentials that satisfy Laplace's equation. It is also shown that the expression for the general solution in terms of scalar potentials can be easily derived using Wald's method of adjoint operators which allows the reduction of systems of homogeneous linear partial differential equations to simpler systems [8,9].

In Sect. 2 the basic concepts about spin-weighted quantities in cylindrical coordinates are summarized and in Sect. 3 these concepts are applied to solve the equations of equilibrium for an elastic medium in the absence of body forces by separation of variables. In Sect. 4 it is shown that the equations of equilibrium for an elastic medium can be reduced to Laplace's equation by means of the coordinate-free method of adjoint operators introduced by Wald [8].

2. SPIN-WEIGHTED CYLINDRICAL FUNCTIONS

Let $\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\}$ be the orthonormal basis induced by the circular cylindrical coordinates ρ, ϕ, z. A quantity η is said to have spin-weight s if under the rotation around $\hat{e}_z$ given by

$$\hat{e}'_\rho + i\hat{e}'_\phi = e^{i\theta}(\hat{e}_\rho + i\hat{e}_\phi), \tag{1}$$

transforms according to

$$\eta' = e^{is\theta}\eta. \tag{2}$$

An arbitrary vector field $\mathbf{F}$ can be written in the form

$$\mathbf{F} = \tfrac{1}{2}F_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\rho - i\hat{e}_\phi) + F_z\hat{e}_z, \tag{3}$$

where $F_\pm \equiv \mathbf{F}\cdot(\hat{e}_\rho \pm i\hat{e}_\phi)$ and $F_z \equiv \mathbf{F}\cdot\hat{e}_z$. It follows from Eqs. (1) and (2) that the components $F_\pm$ and F_z have spin-weight ± 1 and 0, respectively.

The operators ∂ and $\bar{\partial}$ acting on a quantity η with spin-weight s are defined by [7]

$$\begin{aligned} \partial\eta &\equiv -\rho^s\left(\partial_\rho + \frac{i}{\rho}\partial_\phi\right)(\rho^{-s}\eta), \\ \bar{\partial}\eta &\equiv -\rho^{-s}\left(\partial_\rho - \frac{i}{\rho}\partial_\phi\right)(\rho^{s}\eta). \end{aligned} \tag{4}$$

Then, $\partial\eta$ has spin-weight $s+1$ and $\bar{\partial}\eta$ has spin-weight $s-1$. A straightforward calculation shows that

$$\partial\bar{\partial}\eta = \bar{\partial}\partial\eta = \left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \frac{2is}{\rho^2}\partial_\phi - \frac{s^2}{\rho^2}\right)\eta. \tag{5}$$

In terms of ∂ and $\bar{\partial}$ the gradient of a function f with spin-weight 0 is given by

$$\nabla f = -\tfrac{1}{2}\bar{\partial}f(\hat{e}_\rho + i\hat{e}_\phi) - \tfrac{1}{2}\partial f(\hat{e}_\rho - i\hat{e}_\phi) + \partial_z f\hat{e}_z. \tag{6}$$

Similarly, by using Eqs. (4) one finds that the divergence and curl of a vector field $\mathbf{F}$ are

$$\begin{aligned}\nabla\cdot\mathbf{F} &= -\tfrac{1}{2}\eth F_- - \tfrac{1}{2}\bar{\eth}F_+ + \partial_z F_z,\\ \nabla\times\mathbf{F} &= \tfrac{1}{2i}(\bar{\eth}F_z + \partial_z F_-)(\hat{e}_\rho + i\hat{e}_\phi) - \tfrac{1}{2i}(\eth F_z + \partial_z F_+)(\hat{e}_\rho - i\hat{e}_\phi)\\ &\quad + \tfrac{1}{2i}(\eth F_- - \bar{\eth}F_+)\hat{e}_z.\end{aligned} \tag{7}$$

From Eqs. (6) and (7), using the fact that $\eth$ and $\bar{\eth}$ commute, one obtains

$$\nabla^2 f = \eth\bar{\eth}f + \partial_z^2 f, \tag{8}$$

and, making use of the identity $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$,

$$\begin{aligned}\nabla^2\mathbf{F} &= \tfrac{1}{2}(\bar{\eth}\eth F_- + \partial_z^2 F_-)(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}(\eth\bar{\eth}F_+ + \partial_z^2 F_+)(\hat{e}_\rho - i\hat{e}_\phi)\\ &\quad + (\bar{\eth}\eth F_z + \partial_z^2 F_z)\hat{e}_z.\end{aligned} \tag{9}$$

The cylindrical harmonics with spin-weight s, ${}_sF_{\alpha m}$, are defined by

$$\begin{aligned}\eth\bar{\eth}({}_sF_{\alpha m}) &= -\alpha^2\,{}_sF_{\alpha m},\\ -i\partial_\phi({}_sF_{\alpha m}) &= m\,{}_sF_{\alpha m},\end{aligned} \tag{10}$$

where α is a (real or complex) constant and m is an integer or a half-integer according to whether s is an integer or a half-integer. From Eqs. (5) and (10) it follows that, if α is different from zero,

$${}_sF_{\alpha m} = A\,{}_sJ_{\alpha m} + B\,{}_sN_{\alpha m}, \tag{11}$$

where A, B are arbitrary constants,

$$\begin{aligned}{}_sJ_{\alpha m}(\rho,\phi) &\equiv J_{m+s}(\alpha\rho)e^{im\phi},\\ {}_sN_{\alpha m}(\rho,\phi) &\equiv N_{m+s}(\alpha\rho)e^{im\phi},\end{aligned} \tag{12}$$

and J_ν, N_ν are Bessel functions. From Eqs. (4) and the recurrence relations for the Bessel functions one finds that

$$\begin{aligned}\eth({}_sZ_{\alpha m}) &= \alpha_{s+1}Z_{\alpha m},\\ \bar{\eth}({}_sZ_{\alpha m}) &= -\alpha_{s-1}Z_{\alpha m},\end{aligned} \tag{13}$$

where ${}_sF_{\alpha m}$ denotes ${}_sJ_{\alpha m}$ or ${}_sN_{\alpha m}$.

When $\alpha = 0$, the solution of Eqs. (10) is

$$
\begin{aligned}
{}_sF_{0m} &= A\rho^{m+s}e^{im\phi} + B\rho^{-m-s}e^{im\phi}, \qquad && (m+s \neq 0),\\
{}_sF_{0,-s} &= Ae^{-is\phi} + B(ln\rho)e^{-is\phi}, && (m=-s),
\end{aligned}
\tag{14}
$$

where A, B are arbitrary constants. It may be noticed that because of their behavior at $\rho = 0$ and when ρ goes to infinity, the spin-weighted cylindrical harmonics with $\alpha = 0$ [Eqs. (14)] cannot appear in the solution of some problems. The functions ${}_sF_{\alpha m}$ given by Eqs. (12) and (14) form a complete set for each value of s, in the sense that any function with spin-weight s can be expanded in a series in ${}_sF_{\alpha m}$.

3. Solution by separation of variables

The equations of equilibrium for an isotropic elastic medium in the absence of body forces are (see, *e.g.*, Ref. [10])

$$
(1-2\sigma)\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}) = 0, \tag{15}
$$

where σ denotes Poisson's ratio and $\mathbf{u}$ is the displacement vector. By expressing the vector field $\mathbf{u}$ in the form

$$
\mathbf{u} = \tfrac{1}{2}u_-(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}u_+(\hat{e}_\rho - i\hat{e}_\phi) + u_z\hat{e}_z, \tag{16}
$$

using Eqs. (6), (7) and (9) one finds that Eqs. (15) amount to

$$
\begin{aligned}
&(1-2\sigma)(\bar{\partial}\partial u_- + \partial_z^2 u_-) + \bar{\partial}\big(\tfrac{1}{2}\partial u_- + \tfrac{1}{2}\bar{\partial}u_+ - \partial_z u_z\big) = 0,\\
&(1-2\sigma)(\partial\bar{\partial} u_+ + \partial_z^2 u_+) + \partial\big(\tfrac{1}{2}\partial u_- + \tfrac{1}{2}\bar{\partial}u_+ - \partial_z u_z\big) = 0,\\
&(1-2\sigma)(\bar{\partial}\partial u_z + \partial_z^2 u_z) - \partial_z\big(\tfrac{1}{2}\partial u_- + \tfrac{1}{2}\bar{\partial}u_+ - \partial_z u_z\big) = 0.
\end{aligned}
\tag{17}
$$

We now look for separable solutions of Eqs. (17) of the form

$$
\begin{aligned}
u_- &= g_{-1}(z)\,{}_{-1}J_{\alpha m}(\rho,\phi) + G_{-1}(z)\,{}_{-1}N_{\alpha m}(\rho,\phi),\\
u_+ &= g_1(z)\,{}_1J_{\alpha m}(\rho,\phi) + G_1(z)\,{}_1N_{\alpha m}(\rho,\phi),\\
u_z &= g_0(z)\,{}_0J_{\alpha m}(\rho,\phi) + G_0(z)\,{}_0N_{\alpha m}(\rho,\phi),
\end{aligned}
\tag{18}
$$

where α is assumed to be different from zero and m is an integer. As shown in Ref. [7], a vector field of the form (18) is an eigenfunction of the operators corresponding to the z-component of the total angular momentum, J_3, and to the square of the linear momentum perpendicular to the z-axis, $P_1^2 + P_2^2$, with eigenvalues $m\hbar$ and $\hbar^2\alpha^2$, respectively.

Substituting Eqs. (18) into Eqs. (17), making use of Eqs. (10), (13) and of the linear independence of ${}_sJ_{\alpha m}$ and ${}_sN_{\alpha m}$, one finds that the functions $g_i(z)$ must satisfy the following system of ordinary differential equations:

$$(1-2\sigma)\left(\frac{d^2}{dz^2}g_{-1}-\alpha^2 g_{-1}\right)-\frac{1}{2}\alpha^2 g_{-1}+\frac{1}{2}\alpha^2 g_1+\alpha\frac{d}{dz}g_0=0, \tag{19a}$$

$$(1-2\sigma)\left(\frac{d^2}{dz^2}g_{1}-\alpha^2 g_{1}\right)+\frac{1}{2}\alpha^2 g_{-1}-\frac{1}{2}\alpha^2 g_1-\alpha\frac{d}{dz}g_0=0, \tag{19b}$$

$$(1-2\sigma)\left(\frac{d^2}{dz^2}g_{0}-\alpha^2 g_{0}\right)-\frac{1}{2}\alpha\frac{d}{dz}g_{-1}+\frac{1}{2}\alpha\frac{d}{dz}g_1+\frac{d^2}{dz^2}g_0=0; \tag{19c}$$

and that the functions $G_i(z)$ satisfy a system of equations identical to Eqs. (19), with G_i in place of $g_i (i=-1,1,0)$.

Equations (19) can be rewritten in the form

$$\frac{d^2}{dz^2}M-\alpha^2 M=0, \tag{20a}$$

$$(1-2\sigma)\frac{d^2}{dz^2}H-2\alpha^2(1-\sigma)H-2\alpha\frac{d}{dz}g_0=0, \tag{20b}$$

$$2(1-\sigma)\frac{d^2}{dz^2}g_0-\alpha^2(1-2\sigma)g_0+\frac{1}{2}\alpha\frac{d}{dz}H=0, \tag{20c}$$

where

$$M\equiv g_1+g_{-1}, \qquad H\equiv g_1-g_{-1}. \tag{21}$$

The most general solution of Eq. (20a) is given by

$$M(z)=a_1e^{\alpha z}+a_2e^{-\alpha z}, \tag{22}$$

where a_1, a_2 are arbitrary constants, and by combining Eqs. (20b) and (20c) one finds that H obeys the decoupled equation

$$\frac{d^4}{dz^4}H-2\alpha^2\frac{d^2}{dz^2}H+\alpha^4 H=0, \tag{23}$$

whose general solution is

$$H(z)=b_1e^{\alpha z}+b_2e^{-\alpha z}+c_1ze^{\alpha z}+c_2ze^{-\alpha z}, \tag{24}$$

where b_1, b_2, c_1, c_2, are arbitrary constants. Substituting Eq. (24) into Eqs. (20b) and (20c) one obtains

$$\begin{aligned} g_0(z) &= \frac{1}{2}\left[-b_1 e^{\alpha z} + b_2 e^{-\alpha z} + \frac{3-4\sigma}{\alpha}(c_1 e^{\alpha z} + c_2 e^{-\alpha z}) - z(c_1 e^{\alpha z} - c_2 e^{-\alpha z})\right] \\ &= -\frac{1}{2\alpha}\frac{d}{dz}H + \frac{2(1-\sigma)}{\alpha}(c_1 e^{\alpha z} + c_2 e^{-\alpha z}). \end{aligned} \tag{25}$$

Hence, from Eqs. (21), (22) and (24) it follows that

$$\begin{aligned} g_{-1}(z) &= \tfrac{1}{2}\left[(a_1 - b_1)e^{\alpha z} + (a_2 - b_2)e^{-\alpha z} - z(c_1 e^{\alpha z} + c_2 e^{-\alpha z})\right], \\ g_1(z) &= \tfrac{1}{2}\left[(a_1 + b_1)e^{\alpha z} + (a_2 + b_2)e^{-\alpha z} + z(c_1 e^{\alpha z} + c_2 e^{-\alpha z})\right]. \end{aligned} \tag{26}$$

By analogy with Eqs. (25) and (26), the functions G_i are given by

$$\begin{aligned} G_{-1}(z) &= \tfrac{1}{2}\left[(a_3 - b_3)e^{\alpha z} + (a_4 - b_4)e^{-\alpha z} - z(c_3 e^{\alpha z} + c_4 e^{-\alpha z})\right], \\ G_1(z) &= \tfrac{1}{2}\left[(a_3 + b_3)e^{\alpha z} + (a_4 + b_4)e^{-\alpha z} + z(c_3 e^{\alpha z} + c_4 e^{-\alpha z})\right], \\ G_0(z) &= -\frac{1}{2\alpha}\frac{d}{dz}\left[b_3 e^{\alpha z} + b_4 e^{-\alpha z} + z(c_3 e^{\alpha z} + c_4 e^{-\alpha z})\right] \\ &\quad + \frac{2(1-\sigma)}{\alpha}(c_3 e^{\alpha z} + c_4 e^{-\alpha z}), \end{aligned} \tag{27}$$

where a_3, a_4, b_3, b_4, c_3, c_4, are arbitrary constants. Substituting Eqs. (25)-(27) into Eqs. (18) and making use of the relations (13) one finds that the spin-weighted components of the displacement vector $\mathbf{u}$ can be written as

$$\begin{aligned} u_- &= \bar{\partial}(-i\psi_1 + \psi_2 + z\psi_3), \\ u_+ &= \partial(i\psi_1 + \psi_2 + z\psi_3), \\ u_z &= 4(1-\sigma)\psi_3 - \partial_z(\psi_2 + z\psi_3), \end{aligned} \tag{28}$$

where

$$\begin{aligned} \psi_1 &\equiv \frac{1}{2i\alpha}\left[(a_1 e^{\alpha z} + a_2 e^{-\alpha z})\,{}_0J_{\alpha m} + (a_3 e^{\alpha z} + a_4 e^{-\alpha z})\,{}_0N_{\alpha m}\right], \\ \psi_2 &\equiv \frac{1}{2\alpha}\left[(b_1 e^{\alpha z} + b_2 e^{-\alpha z})\,{}_0J_{\alpha m} + (b_3 e^{\alpha z} + b_4 e^{-\alpha z})\,{}_0N_{\alpha m}\right], \\ \psi_3 &\equiv \frac{1}{2\alpha}\left[(c_1 e^{\alpha z} + c_2 e^{-\alpha z})\,{}_0J_{\alpha m} + (c_3 e^{\alpha z} + c_4 e^{-\alpha z})\,{}_0N_{\alpha m}\right], \end{aligned} \tag{29}$$

which satisfy Laplace's equation

$$\nabla^2\psi_1 = 0, \quad \nabla^2\psi_2 = 0, \quad \nabla^2\psi_3 = 0. \tag{30}$$

According to Eqs. (6) and (7), Eqs. (28) amount to the simple expression

$$\mathbf{u} = \nabla \times (\psi_1\hat{e}_z) - \nabla(\psi_2 + z\psi_3) + 4(1-\sigma)\psi_3\hat{e}_z. \tag{31}$$

In the case where $\alpha = 0$, in place of Eqs. (18), one has to consider combinations of the functions given by Eqs. (14). A straightforward but somewhat lengthy calculation shows that the corresponding solutions can also be expressed in the form (28), where ψ_1, ψ_2, ψ_3, are solutions of the Laplace equation. However, an important difference is that, when $\alpha = 0$, the harmonic functions ψ_1, ψ_2, ψ_3 are not independent and, by contrast with expressions (29), are not separable solutions of Laplace's equation. For example, in the case where $\alpha = 0$ and $m = 0$, according to Eqs. (14), the solution of Eqs. (17) has the form

$$\begin{aligned} u_- &= g_{-1}(z)\rho^{-1} + G_{-1}(z)\rho, \\ u_+ &= g_1(z)\rho + G_1(z)\rho^{-1}, \\ u_z &= g_0(z) + G_0(z)\ln\rho. \end{aligned} \tag{32}$$

Substituting Eqs. (32) into Eqs. (17) and solving the ordinary differential equations thus obtained, one finds that

$$\begin{aligned} u_- &= \big[(b_1 - a_1)z + b_2 - a_2\big]\rho + \left[(b_3 - a_3)z + (b_4 - a_4) - \frac{c_3 z^2}{1-2\sigma}\right]\rho^{-1}, \\ u_+ &= \big[(a_1 + b_1)z + a_2 + b_2\big]\rho + \left[(a_3 + b_3)z + (a_4 + b_4) - \frac{c_3 z^2}{1-2\sigma}\right]\rho^{-1}, \\ u_z &= 2c_2 + 2c_4 z - \frac{b_1 z^2}{2(1-\sigma)} + 2(c_1 + c_3 z)\ln\rho, \end{aligned} \tag{33}$$

where the a_i, b_i, c_i are arbitrary constants. The components (33) can be written in the form (28) with

$$\psi_1 = i\left\{a_1\left(\frac{z\rho^2}{2}-\frac{z^3}{3}\right)+a_2\left(\frac{\rho^2}{2}-z^2\right)+(a_3 z+a_4)\ln\rho\right\},$$

$$\begin{aligned}\psi_2 = &-\frac{(3-4\sigma)}{4(1-\sigma)}\left[b_1\left(\frac{z\rho^2}{2}-\frac{z^3}{3}\right)+b_3 z\ln\rho\right]-\frac{z}{2(1-\sigma)}(c_2+c_1\ln\rho)\\ &-b_2\left(\frac{\rho^2}{2}-z^2\right)-b_4\ln\rho,\end{aligned} \tag{34}$$

$$\begin{aligned}\psi_3 = &\frac{1}{2(1-\sigma)}\left\{-\frac{b_1}{2}\left(\frac{\rho^2}{2}-z^2\right)+c_2+\left(c_1-\frac{b_3}{2}\right)\ln\rho\right\}\\ &+\frac{z}{1-2\sigma}(b_2+c_4+c_3\ln\rho).\end{aligned}$$

An expression similar to Eq. (31) is given in Ref. [10] (p. 26), where it is shown that the solutions of Eq. (15) can be written in terms of four arbitrary harmonic functions.

4. Debye potentials via the method of adjoint operators

The equations of equilibrium for an isotropic elastic medium (15) can be written in the form

$$\mathcal{E}(\mathbf{u}) = 0, \tag{35}$$

where $\mathcal{E}$ is the partial differential operator that maps vector fields into vector fields given by

$$\mathcal{E}(\mathbf{u}) \equiv (1-2\sigma)\nabla^2\mathbf{u}+\nabla(\nabla\cdot\mathbf{u}). \tag{36}$$

By defining the adjoint of a linear operator $\mathcal{A}$ that maps n-index tensor fields into m-index tensor fields as that linear operator $\mathcal{A}^\dagger$ that maps m-index tensor fields into n-index tensor fields such that [8]

$$g^{\alpha\beta\cdots}[\mathcal{A}(f_{\mu\nu\cdots})]_{\alpha\beta\cdots}-[\mathcal{A}^\dagger(g^{\alpha\beta\cdots})]^{\mu\nu\cdots}f_{\mu\nu\cdots} = \nabla_\alpha s^\alpha, \tag{37}$$

where s^α is some vector field, it follows that

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger, \quad (\mathcal{A}+\mathcal{B})^\dagger = \mathcal{A}^\dagger+\mathcal{B}^\dagger, \quad (\mathcal{A}^\dagger)^\dagger = \mathcal{A}, \tag{38}$$

and

$$\mathrm{grad}^\dagger = -\,\mathrm{div}, \quad \mathrm{div}^\dagger = -\,\mathrm{grad}, \quad \mathrm{curl}^\dagger = \mathrm{curl}\,. \tag{39}$$

Therefore, by expressing the operator $\mathcal{E}$ [Eq. (36)] in the form $\mathcal{E} = 2(1-\sigma)\,\mathrm{grad\,div}-(1-2\sigma)\,\mathrm{curl\,curl}$, from Eqs. (38) and (39) it follows that $\mathcal{E}$ is self-adjoint: $\mathcal{E}^\dagger = \mathcal{E}$.

If there exist linear operators $\mathcal{O}$, $\mathcal{T}$, $\mathcal{S}$, such that

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}, \tag{40}$$

then, using Eqs. (38) and the fact that $\mathcal{E}^\dagger = \mathcal{E}$, Eq. (40) implies that

$$\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger. \tag{41}$$

Hence, if ψ satisfies the condition

$$\mathcal{O}^\dagger(\psi) = 0, \tag{42}$$

from Eq. (41) it follows that $\mathbf{u} = \mathcal{S}^\dagger(\psi)$ satisfies Eq. (35).

In order to find operators $\mathcal{O}$, $\mathcal{T}$, $\mathcal{S}$, satisfying Eq. (40) it is convenient to introduce the vector field

$$\mathbf{K} \equiv \mathcal{E}(\mathbf{u}) = (1 - 2\sigma)\nabla^2\mathbf{u} + \nabla(\nabla \cdot \mathbf{u}). \tag{43}$$

Thus, $\mathbf{K} = 0$ if and only if $\mathbf{u}$ satisfies Eq. (35). Taking the curl of Eq. (43) one finds that $\nabla \times \mathbf{K} = \nabla^2[(1 - 2\sigma)\nabla \times \mathbf{u}]$, therefore

$$\hat{e}_z \cdot \nabla \times \mathbf{K} = \nabla^2[(1 - 2\sigma)\hat{e}_z \cdot \nabla \times \mathbf{u}], \tag{44}$$

which, recalling that $\mathbf{K} = \mathcal{E}(\mathbf{u})$, is an operator identity of the form (40) with

$$\mathcal{S} = \hat{e}_z \cdot \text{curl}, \quad \mathcal{O} = \nabla^2, \quad \mathcal{T} = (1 - 2\sigma)\hat{e}_z \cdot \text{curl}. \tag{45}$$

It is easy to see that $\mathcal{S}^\dagger = -\hat{e}_z \times \text{grad}$, and $\mathcal{O}^\dagger = \nabla^2$. Therefore, according to the preceding paragraph,

$$\mathbf{u} = \mathcal{S}^\dagger(\psi_1) = -\hat{e}_z \times \nabla\psi_1 = \nabla \times (\psi_1\hat{e}_z), \tag{46}$$

is a solution of Eq. (35) provided that $\mathcal{O}^\dagger(\psi_1) = 0$, *i.e.*, $\nabla^2\psi_1 = 0$.

Taking now the divergence of Eq. (43) one obtains the identity

$$\nabla \cdot \mathbf{K} = \nabla^2[2(1 - \sigma)\nabla \cdot \mathbf{u}], \tag{47}$$

which is of the form (40) with

$$\mathcal{S} = \text{div}, \quad \mathcal{O} = \nabla^2, \quad \mathcal{T} = 2(1 - \sigma)\,\text{div}. \tag{48}$$

Making use of Eqs. (39) one finds that $\mathcal{S}^\dagger = -\,\text{grad}$, $\mathcal{O}^\dagger = \nabla^2$, and therefore

$$\mathbf{u} = \mathcal{S}^\dagger(\psi_2) = -\nabla\psi_2 \tag{49}$$

satisfies Eq. (35) provided that $\nabla^2\psi_2 = 0$.

Finally, from Eqs. (47) and (43) it follows that

$$\begin{aligned}\nabla^2[z2(1-\sigma)\nabla\cdot\mathbf{u}] &= z\nabla\cdot\mathbf{K}+4(1-\sigma)\hat{e}_z\cdot\nabla(\nabla\cdot\mathbf{u})\\ &= z\nabla\cdot\mathbf{K}+4(1-\sigma)\hat{e}_z\cdot[\mathbf{K}-(1-2\sigma)\nabla^2\mathbf{u}],\end{aligned}$$

hence,

$$z\nabla\cdot\mathbf{K}+4(1-\sigma)\hat{e}_z\cdot\mathbf{K}=\nabla^2[2(1-\sigma)z\nabla\cdot\mathbf{u}+4(1-\sigma)(1-2\sigma)\hat{e}_z\cdot\mathbf{u}], \qquad (50)$$

which is of the form (40) with

$$\begin{aligned}&\mathcal{S}=z\,\mathrm{div}+4(1-\sigma)\hat{e}_z\cdot, \quad \mathcal{O}=\nabla^2,\\ &\mathcal{T}=2(1-\sigma)[z\,\mathrm{div}+2(1-2\sigma)\hat{e}_z\cdot\ ].\end{aligned} \qquad (51)$$

In this case one finds that $\mathcal{S}^\dagger=-\nabla(z\cdot\ )+4(1-\sigma)\hat{e}_z$ and $\mathcal{O}^\dagger=\nabla^2$; thus, if $\nabla^2\psi_3=0$ then

$$\mathbf{u}=\mathcal{S}^\dagger(\psi_3)=-\nabla(z\psi_3)+4(1-\sigma)\psi_3\hat{e}_z \qquad (52)$$

satisfies Eq. (35).

By adding the solutions (46), (49) and (52) one obtains precisely the general solution of Eq. (35) given by Eq. (31).

5. Concluding remarks

The results of Sect. 3 show the usefulness of the spin-weighted quantities in the solution of systems of partial differential equations governing nonscalar fields. Some additional examples making use of the spin-weighted cylindrical harmonics are given in Ref. [7].

Section 4 provides examples of the application of Wald's method of adjoint operators which, in many cases, is the simplest procedure to solve sets of homogeneous linear partial differential equations. The operator identities derived in Sect. 4 are not the only ones that can be obtained from Eq. (43); the identities considered here are those that yield expressions equivalent to Eq. (31), which is adapted to the circular cylindrical coordinates.

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