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Spinors in three dimensions

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Abstract. A spinor calculus for three-dimensional riemannian manifolds is developed. The connection and curvature are written in spinor form and some examples of the usefulness of this formalism are also given.

Resumen. Se desarrolla un cálculo espinorial para variedades riemannianas de dimensión tres. La conexión y la curvatura se expresan en forma espinorial y se dan también algunos ejemplos de la utilidad de este formalismo.

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1. Introduction

The 2-spinor calculus employed in general relativity is simpler and more powerful than the tensor calculus. Many computations can be greatly abbreviated by using spinors instead of tensors and, in fact, spinors can be regarded as more basic than tensors (see, *e.g.*, Refs. [1,2]). Some advantages of the spinor calculus come from the fact that (in four dimensions) the basic spinors at a point form a (complex) two-dimensional vector space.

In the case of a riemannian manifold of dimension three, the basic spinors at a point also form a complex two-dimensional vector space (see, *e.g.*, Refs. [3,4]) and a spinor calculus very similar to that applied in general relativity can be developed. This formalism is useful in problems involving three-dimensional tensors or half-integral-spin fields and provides a very convenient way to compute the curvature of three-dimensional manifolds. This spinor calculus may be also useful in connection with homogeneous cosmological models.

In Sect. 2 the relationship between three-dimensional tensors and spinors is established and some basic algebraic properties of the spinors are given. In Sect. 3 the covariant derivative of spinors is defined and it is shown that the spinor connection is characterized by five independent quantities (spin-coefficients). In Sect. 4 the notion of spin-weight is introduced and in Sect. 5 it is shown that the curvature is represented by a totally symmetric four-index spinor together with the scalar curvature. In Sect. 6 the Maxwell equations and the Dirac equation are written regarding them as equations governing three-dimensional tensors or $SU(2)$ spinors. In this paper we restrict ourselves to spaces with a positive definite metric, but the formalism can be easily adapted to any signature. Lower-case Latin indices $i, j, \ldots$, run from 1 to 3 and capital Latin indices $A, B, \ldots$, run from 1 to 2; on each repeated index the summation convention applies.

2. SPINOR ALGEBRA

The components of a spinor, or a spinor field, will be denoted by symbols like $\psi_{AB\cdots D}$, where each of the indices $A, B, \ldots, D$ takes values from 1, 2. Under the rotation represented by a matrix $(U^A{}_B)$ belonging to SU(2), the components of a spinor transform according to

$$\psi'_{AB\cdots D} = U^G{}_A U^H{}_B \cdots U^J{}_D \psi_{GH\cdots J}. \tag{1}$$

The spinor indices will be raised and lowered by means of

$$\left(\varepsilon_{AB}\right) \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \equiv \left(\varepsilon^{AB}\right), \tag{2}$$

according to the convention

$$\psi_A = \varepsilon_{AB}\psi^B, \qquad \psi^B = \varepsilon^{AB}\psi_A. \tag{3}$$

Therefore, $\psi_A\phi^A = -\psi^A\phi_A$ and $\chi_{\ldots R \ldots}{}^{R}{}_{\ldots} = -\chi_{\ldots}{}^{R}{}_{\ldots R \ldots}$.

Denoting the elements of the Pauli matrices σ_i by $\sigma_i{}^A{}_B$ (where the superscript labels rows and the subscript labels columns) and following the convention (3), the elements of the matrix product $\varepsilon\sigma_i$, given by $\varepsilon_{AB}\sigma_i{}^B{}_C$, will be denoted by σ_{iAC}. From the explicit expression of the matrix products

$$\varepsilon\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varepsilon\sigma_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \varepsilon\sigma_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{4}$$

it follows that the elements σ_{iAB} are symmetric

$$\sigma_{iAB} = \sigma_{iBA} \tag{5}$$

and from the relation $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$, we have

$$\sigma_{iAB}\sigma_j{}^B{}_C + \sigma_{jAB}\sigma_i{}^B{}_C = 2\delta_{ij}\varepsilon_{AC}, \tag{6}$$

and therefore

$$\sigma_{iAB}\sigma_j{}^{AB} = -2\delta_{ij} \tag{7}$$

(note that $\varepsilon^A{}_B = \delta^A_B$). The elements σ_{iAB} also satisfy the relation [5]

$$\sigma_{iAB}\sigma_{jCD}\delta^{ij} = -\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{BC}\varepsilon_{AD}\right) \tag{8}$$

and from Eq. (4) we see that, under complex conjugation,

$$\overline{\sigma_{i11}} = -\sigma_{i22}, \qquad \overline{\sigma_{i12}} = \sigma_{i12}. \tag{9}$$

If $t_{ij\cdots k}$ are the components of an n-index three-dimensional tensor with respect to an *orthonormal* basis, the components of its spinor equivalent are given by

$$t_{ABCD\cdots EF} \equiv \left(\tfrac{1}{\sqrt{2}}\sigma^i{}_{AB}\right)\left(\tfrac{1}{\sqrt{2}}\sigma^j{}_{CD}\right)\cdots\left(\tfrac{1}{\sqrt{2}}\sigma^k{}_{EF}\right)t_{ij\cdots k} \tag{10}$$

(the indices $i, j, \cdots$, are raised and lowered by means of the metric tensor δ_{ij}, and therefore $\sigma^i{}_{AB}$ coincides with σ_{iAB}). From Eq. (7) it follows that the inverse relation to (10) is

$$t_{ij\cdots k} = \left(-\tfrac{1}{\sqrt{2}}\sigma_i{}^{AB}\right)\left(-\tfrac{1}{\sqrt{2}}\sigma_j{}^{CD}\right)\cdots\left(-\tfrac{1}{\sqrt{2}}\sigma_k{}^{EF}\right)t_{ABCD\cdots EF}. \tag{11}$$

Equations (7) and (10) also imply that $t_{\ldots i\ldots}s^{\cdots i\cdots} = -t_{\ldots AB\ldots}s^{\cdots AB\cdots}$. Owing to Eq. (5), the components $t_{ABCD\cdots EF}$ are symmetric on AB, on $CD, \ldots$, and on EF. The components $t_{AB\cdots F}$ are totally symmetric if and only if the tensor $t_{ij\cdots k}$ is totally symmetric and trace-free (see *e.g.*, Ref. [5]).

If t_{ij} are the components of an anti-symmetric tensor ($t_{ij} = -t_{ji}$), the corresponding spinor components satisfy $t_{ABCD} = -t_{CDAB}$, therefore, making use of the identity

$$\psi_{\ldots A\cdots B\ldots} - \psi_{\ldots B\cdots A\ldots} = \psi_{\ldots}{}^C{}_{\ldots C\ldots}\varepsilon_{AB}, \tag{12}$$

one gets

$$\begin{aligned} t_{ABCD} &= \tfrac{1}{2}\big(t_{ABCD} - t_{CBAD}\big) + \tfrac{1}{2}\big(t_{ABCD} - t_{ADCB}\big) \\ &= \tfrac{1}{2}t^R{}_{BRD}\varepsilon_{AC} + \tfrac{1}{2}t_A{}^R{}_{CR}\varepsilon_{BD} \\ &= \tfrac{1}{2}t^R{}_{BRD}\varepsilon_{AC} + \tfrac{1}{2}t^R{}_{ARC}\varepsilon_{BD}, \end{aligned} \tag{13}$$

with $t^R{}_{BRD}$ being symmetric in B, D since

$$t^R{}_{BRD} = -t_{RD}{}^R{}_B = t^R{}_{DRB}. \tag{14}$$

In particular, the dual of a vector F_i, ${}^*F_{ij} \equiv \varepsilon_{ijk}F_k$, is an anti-symmetric tensor whose spinor components are

$${}^*F_{ABCD} = -\varepsilon_{ABCDEG}F^{EG}, \tag{15}$$

where

$$\begin{aligned} \varepsilon_{ABCDEG} &\equiv \tfrac{1}{\sqrt{2}}\sigma^i{}_{AB}\tfrac{1}{\sqrt{2}}\sigma^j{}_{CD}\tfrac{1}{\sqrt{2}}\sigma^k{}_{EG}\varepsilon_{ijk} \\ &= \tfrac{i}{2\sqrt{2}}\big(\varepsilon_{AC}\varepsilon_{BE}\varepsilon_{DG} + \varepsilon_{AC}\varepsilon_{BG}\varepsilon_{DE} + \varepsilon_{BD}\varepsilon_{AE}\varepsilon_{CG} + \varepsilon_{BD}\varepsilon_{AG}\varepsilon_{CE}\big) \end{aligned} \tag{16}$$

hence

$${}^*F^R{}_{BRD} = -\tfrac{i}{\sqrt{2}}\big(\varepsilon_{BE}\varepsilon_{DG} + \varepsilon_{BG}\varepsilon_{DE}\big)F^{EG} = -i\sqrt{2}F_{BD}, \tag{17}$$

and in view of Eq. (13) we conclude that

$$^{*}F_{ABCD} = -\tfrac{i}{\sqrt{2}}\big(F_{BD}\varepsilon_{AC} + F_{AC}\varepsilon_{BD}\big). \tag{18}$$

3. Covariant differentiation

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis and let ∂_i denote the directional derivative with respect to $\mathbf{e}_i (i.e., \partial_i f = \mathbf{e}_i \cdot \nabla f)$. As is known, in every riemannian manifold there exists a unique torsion-free connection (the riemannian or Levi-Civita connection) under which the metric is covariantly constant; this amounts to the conditions

$$\begin{aligned} [\partial_i, \partial_j] &= \nabla_i \partial_j - \nabla_j \partial_i, \\ \nabla_i(\partial_j \cdot \partial_k) &= \partial_j \cdot \nabla_i \partial_k + \partial_k \cdot \nabla_i \partial_j, \end{aligned} \tag{19}$$

where ∇_i denotes the covariant derivative with respect to ∂_i. Defining the (real-valued) functions $\Gamma^i{}_{jk}$ by

$$\nabla_i \partial_j = \Gamma^k{}_{ji} \partial_k, \tag{20}$$

and taking into account that $\partial_i \cdot \partial_j = \delta_{ij}$, Eqs. (19) are equivalent to

$$0 = \Gamma^m{}_{ki}\delta_{jm} + \Gamma^m{}_{ji}\delta_{km} \equiv \Gamma_{jki} + \Gamma_{kji}, \tag{21a}$$

$$[\partial_i, \partial_j] = (\Gamma^k{}_{ji} - \Gamma^k{}_{ij})\partial_k. \tag{21b}$$

Introducing the differential operators

$$\partial_{AB} \equiv \tfrac{1}{\sqrt{2}} \sigma^i{}_{AB} \partial_i, \tag{22}$$

which, according to Eq. (4), are given explicitly by

$$\partial_{11} = \tfrac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \quad \partial_{12} = \partial_{21} = -\tfrac{1}{\sqrt{2}}\partial_3, \quad \partial_{22} = -\tfrac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \tag{23}$$

and using Eq. (20) we obtain

$$\nabla_{AB}\partial_{CD} = -\Gamma^{EF}{}_{CDAB}\partial_{EF}, \tag{24}$$

where ∇_{AB} denotes the covariant derivative with respect to ∂_{AB} and

$$\Gamma_{EFCDAB} = \Big(\tfrac{1}{\sqrt{2}}\sigma^k{}_{EF}\Big)\Big(\tfrac{1}{\sqrt{2}}\sigma^j{}_{CD}\Big)\Big(\tfrac{1}{\sqrt{2}}\sigma^i{}_{AB}\Big)\Gamma_{kji}. \tag{25}$$

Since Γ_{ijk} is anti-symmetric in i, j, from Eqs. (13-14) it follows that

$$\Gamma_{ABCDEF} = -\Gamma_{BDEF}\varepsilon_{AC} - \Gamma_{ACEF}\varepsilon_{BD}, \tag{26}$$

where

$$\Gamma_{ABCD} \equiv -\tfrac{1}{2}\Gamma^R{}_{ARBCD} = \Gamma_{BACD} = \Gamma_{ABDC}, \tag{27}$$

which, following the terminology used in Ref. [2], will be called spin-coefficients. Substituting Eq. (26) into Eq. (24) we find

$$\nabla_{AB}\partial_{CD} = \Gamma^R{}_{CAB}\partial_{RD} + \Gamma^R{}_{DAB}\partial_{CR}. \tag{28}$$

The components of the covariant derivative of a spinor field $\psi_{FB\cdots}^{CD\cdots}$ with respect to ∂_{AB}, denoted by $\nabla_{AB}\psi_{FG\cdots}^{CD\cdots}$, are given by

$$\begin{aligned}\nabla_{AB}\psi_{FG\cdots}^{CD\cdots} = \partial_{AB}\psi_{FG\cdots}^{CD\cdots} &+ \Gamma^C{}_{RAB}\psi_{FG\cdots}^{RD\cdots} + \Gamma^D{}_{RAB}\psi_{FG\cdots}^{CR\cdots} \\ &+ \cdots - \Gamma^R{}_{FAB}\psi_{RG\cdots}^{CD\cdots} - \Gamma^R{}_{GAB}\psi_{FR\cdots}^{CD\cdots}.\end{aligned} \tag{29}$$

Owing to the symmetry of Γ_{ABCD} in A, B, the covariant differentiation commutes with the raising and lowering of indices.

Using Eqs. (9), (25) and (27) one finds that

$$\begin{aligned}&\overline{\Gamma_{1111}} = -\Gamma_{2222}, \qquad \overline{\Gamma_{1211}} = \Gamma_{1222},\\ &\overline{\Gamma_{2211}} = -\Gamma_{1122}, \qquad \overline{\Gamma_{1112}} = \Gamma_{2212},\\ &\overline{\Gamma_{1212}} = -\Gamma_{1212}.\end{aligned} \tag{30}$$

Therefore, the nine independent real coefficients Γ_{ijk} amount to the four complex quantities

$$\kappa \equiv \Gamma_{1111}, \quad \beta \equiv \Gamma_{1211}, \quad \rho \equiv \Gamma_{2211}, \quad \alpha \equiv \Gamma_{1112}, \tag{31a}$$

together with the pure imaginary quantity

$$\varepsilon \equiv \Gamma_{1212}. \tag{31b}$$

Equations (30) and (31a) give

$$\Gamma_{2222} = -\bar{\kappa}, \quad \Gamma_{1222} = \bar{\beta}, \quad \Gamma_{1122} = -\bar{\rho}, \quad \Gamma_{2212} = \bar{\alpha}. \tag{31c}$$

Introducing now the definitions

$$D \equiv -\partial_{12}, \quad \delta \equiv \partial_{11}, \quad \bar{\delta} \equiv -\partial_{22}, \tag{32a}$$

or, equivalently,

$$D = \tfrac{1}{\sqrt{2}}\partial_3, \quad \delta = \tfrac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \quad \bar{\delta} = \tfrac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \tag{32b}$$

from the relation $[\partial_{AB}, \partial_{CD}] = \nabla_{AB}\partial_{CD} - \nabla_{CD}\partial_{AB}$ (cf. Eq. (19)) and Eqs. (28), (27) and (31-32) one gets

$$\begin{aligned} [D, \delta] &= 2\alpha D + (2\varepsilon - \rho)\delta - \kappa\bar{\delta}, \\ [\delta, \bar{\delta}] &= 2(\bar{\rho} - \rho)D - 2\bar{\beta}\delta + 2\beta\bar{\delta}. \end{aligned} \tag{33}$$

4. SPIN-WEIGHT

A quantity η has spin-weight s if under the transformation given by the matrix

$$\left(U^A{}_B\right) = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \tag{34}$$

(which corresponds to a rotation through an angle θ about $\mathbf{e}_3$), it transforms according to

$$\eta \to e^{is\theta}\eta. \tag{35}$$

From Eqs. (1), (34) and (35) it follows that each component $\psi_{AB\cdots D}$ of a spinor has a spin-weight equal to one half of the difference between the number of the indices $A, B, \ldots, D$ taking the value 1 and the number of indices taking the value 2. Hence, the $2n + 1$ independent components of a totally symmetric $2n$-index spinor can be labeled by their spin-weight

$$\psi_n \equiv \psi_{11\cdots 1}, \quad \psi_{n-1} \equiv \psi_{21\cdots 1}, \quad \ldots, \psi_{-n} \equiv \psi_{22\cdots 2}, \tag{36}$$

Equations (9-10) imply that if $t_{ij\cdots k}$ is a *real* trace-free totally symmetric n-index tensor, then the spinor components $t_n \equiv t_{11\cdots 1}, t_{n-1} \equiv t_{21\cdots 1}, \ldots, t_{-n} \equiv t_{22\cdots 2}$, satisfy the relation

$$\overline{t_s} = (-1)^s t_{-s}. \tag{37}$$

In view of Eq. (32a), under the transformation (34) the operators D, δ and $\bar{\delta}$ transform according to

$$D \to D, \qquad \delta \to e^{i\theta}\delta, \qquad \bar{\delta} \to e^{-i\theta}\bar{\delta}, \tag{38}$$

and using Eqs. (33) one readily obtains that

$$\kappa \to e^{2i\theta}\kappa, \quad \rho \to \rho, \quad \alpha \to e^{i\theta}\alpha, \tag{39a}$$

$$\beta \to e^{i\theta}(\beta - \tfrac{i}{2}\delta\theta), \quad \varepsilon \to \varepsilon + \tfrac{i}{2}D\theta, \tag{39b}$$

which implies that κ, ρ and α (together with their complex conjugates) have a well-defined spin-weight. On the other hand, from Eqs. (35), (38) and (39b) it follows that if η has spin-weight s then $(D - 2s\varepsilon)\eta$, $(\delta + 2s\beta)\eta$ and $(\bar{\delta} - 2s\bar{\beta})\eta$ have spin-weight s, $s+1$ and $s-1$, respectively. (The operators $(D - 2s\varepsilon)$, $(\delta + 2s\beta)$ and $(\bar{\delta} - 2s\bar{\beta})$ are the analogues of the Geroch-Held-Penrose operators "thorn" and "edth" [6,2].)

Following the notation (36), the spinor components of the gradient of a scalar function f, $(\operatorname{grad} f)_{AB} \equiv \frac{1}{\sqrt{2}}\sigma^i{}_{AB}\partial_i f = \partial_{AB} f$, are given explicitly by [cf. Eqs. (10), (22) and (32)]

$$(\operatorname{grad} f)_{+1} = \delta f, \qquad (\operatorname{grad} f)_0 = -Df, \qquad (\operatorname{grad} f)_{-1} = -\bar{\delta} f. \tag{40}$$

Similarly, the spinor components of any vector field $\mathbf{F}$, $F_{AB} = \frac{1}{\sqrt{2}}\sigma^i{}_{AB}F_i$, are

$$\begin{aligned} F_{+1} = F_{11} &= \tfrac{1}{\sqrt{2}}(F_1 + iF_2), \\ F_0 = F_{12} &= -\tfrac{1}{\sqrt{2}}F_3, \\ F_{-1} = F_{22} &= -\tfrac{1}{\sqrt{2}}(F_1 - iF_2), \end{aligned} \tag{41}$$

where F_1, F_2, F_3 are the components of $\mathbf{F}$ with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. According to Eqs. (29), (31) and (32), the divergence of $\mathbf{F}$ is given by

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla^i F_i = -\nabla^{AB} F_{AB} \\ &= (\bar{\delta} - 2\bar{\beta} + 2\bar{\alpha})F_{+1} - 2(D + \rho + \bar{\rho})F_0 - (\delta - 2\beta + 2\alpha)F_{-1}. \end{aligned} \tag{42}$$

Using Eq. (16) one finds that the spinor components of curl $\mathbf{F}$ are $(\operatorname{curl} \mathbf{F})_{AB} = \varepsilon_{ABCDEG} \times \nabla^{CD} F^{EG} = \frac{i}{\sqrt{2}}(\nabla^R{}_A F_{BR} + \nabla^R{}_B F_{AR}) \equiv i\sqrt{2}\nabla^R{}_{(A}F_{B)R}$, where the parenthesis denotes symmetrization on the indices enclosed, therefore

$$\begin{aligned} (\operatorname{curl} \mathbf{F})_{+1} &= i\sqrt{2}\big\{(D - 2\varepsilon + \rho)F_{+1} + (\delta + 2\alpha)F_0 - \kappa F_{-1}\big\}, \\ (\operatorname{curl} \mathbf{F})_0 &= i\sqrt{2}\big\{\tfrac{1}{2}(\bar{\delta} - 2\bar{\beta})F_{+1} + (\rho - \bar{\rho})F_0 + \tfrac{1}{2}(\delta - 2\beta)F_{-1}\big\}, \\ (\operatorname{curl} \mathbf{F})_{-1} &= i\sqrt{2}\big\{\bar{\kappa}F_{+1} + (\bar{\delta} + 2\bar{\alpha})F_0 - (D + 2\varepsilon + \bar{\rho})F_{-1}\big\}. \end{aligned} \tag{43}$$

Substituting Eq. (40) into Eq. (43) one readily obtains that curl grad $= 0$ amounts to the commutation relations (33).

In the case of a trace-free symmetric 2-index tensor field, t_{ij}, the spinor components of the vector field $(\operatorname{div} t)_i \equiv \nabla^j t_{ij}$ are given explicitly by

$$\begin{aligned} (\operatorname{div} t)_{+1} &= (\bar{\delta} - 4\bar{\beta} + 2\bar{\alpha})t_{+2} - 2(D - 2\varepsilon + \rho + 2\bar{\rho})t_{+1} - (\delta + 6\alpha)t_0 + 2\kappa t_{-1}, \\ (\operatorname{div} t)_0 &= \bar{\kappa}t_{+2} + (\bar{\delta} - 2\bar{\beta} + 4\bar{\alpha})t_{+1} - (2D + 3\rho + 3\bar{\rho})t_0 \\ &\quad - (\delta - 2\beta + 4\alpha)t_{-1} + \kappa t_{-2}, \\ (\operatorname{div} t)_{-1} &= 2\bar{\kappa}t_{+1} + (\bar{\delta} + 6\bar{\alpha})t_0 - 2(D + 2\varepsilon + 2\rho + \bar{\rho})t_{-1} - (\delta - 4\beta + 2\alpha)t_{-2}. \end{aligned} \tag{44}$$

As in the case of the spinor formalism applied in general relativity [6,2], we can introduce the map $'$ defined by the matrix

$$\left(U^A{}_B\right) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \tag{45}$$

which belongs to SU(2) and corresponds essentially to interchange the basis spinors. The operators D, δ and $\bar{\delta}$ transform according to

$$D' = -D, \qquad \delta' = \bar{\delta}, \qquad \bar{\delta}' = \delta, \tag{46}$$

[cf. Eqs. (32)], from which it is clear that the matrix (45) represents a rotation through 180° about $\mathbf{e}_1$. Then, from Eqs. (33) and (46) it is easy to see that

$$\kappa' = -\bar{\kappa}, \quad \beta' = \bar{\beta}, \quad \rho' = -\bar{\rho}, \quad \alpha' = \bar{\alpha}, \quad \varepsilon' = \varepsilon \ (= -\bar{\varepsilon}). \tag{47}$$

The components of a totally symmetric $2n$-index spinor [Eq. (36)] transform according to

$$\psi'_s = i^{2n}\psi_{-s}. \tag{48}$$

Note that under the prime operation Eqs. (40) and (42-44) are mapped into themselves.

5. CURVATURE

The curvature tensor, defined by

$$\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)F_k = -R^m{}_{kij}F_m, \tag{49}$$

possesses the symmetries $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$, which imply that the components $R_{ijk\ell}$ can be written in the form

$$R_{ijk\ell} = -\varepsilon_{ijm}\varepsilon_{k\ell n}G_{mn}, \tag{50}$$

where G_{ij} is some tensor field (the minus sign is introduced for later convenience); this means that $R_{ijk\ell}$ is (minus) the double dual of G_{ij}. According to Eq. (50) the components of the Ricci tensor, $R_{ij} \equiv R^k{}_{ikj}$, are $R_{ij} = G_{ij} - G^m{}_m\delta_{ij}$ and the scalar curvature is given by $R \equiv R^i{}_i = -2G^m{}_m$, therefore $G_{ij} = R_{ij} - \frac{1}{2}R\delta_{ij}$. Denoting by Φ_{ij} the components of the trace-free part of the Ricci tensor, we have

$$G_{ij} = \Phi_{ij} - \tfrac{R}{6}\delta_{ij}. \tag{51}$$

The spinor equivalent of Eq. (50) is

$$\begin{aligned} R_{ABCDEFHI} = \tfrac{1}{2}\big(&\varepsilon_{AC}\varepsilon_{EH}G_{BDFI} + \varepsilon_{AC}\varepsilon_{FI}G_{BDEH} \\ &+ \varepsilon_{BD}\varepsilon_{EH}G_{ACFI} + \varepsilon_{BD}\varepsilon_{FI}G_{ACEH}\big), \end{aligned} \tag{52}$$

where G_{ABCD} are the spinor components of G_{ij} (cf. Eq. (18)).

Applying the decomposition (13) to the left-hand side of Eq. (49) one gets

$$\left(\varepsilon_{AC}\Box_{BD}+\varepsilon_{BD}\Box_{AC}\right)F_{EG}=R^{HI}{}_{EGABCD}F_{HI}, \tag{53}$$

where

$$\Box_{AB}\equiv\nabla^{R}{}_{(A}\nabla_{B)R}. \tag{54}$$

Then, Eqs. (52-53) yield

$$\Box_{AB}\,\psi_C=-\tfrac{1}{2}G_{DCAB}\psi^{D}, \tag{55}$$

which, in view of Eqs. (51) and (8), is equivalent to

$$\Box_{AB}\,\psi_C=-\tfrac{1}{2}\Phi_{ABCD}\psi^{D}-\tfrac{R}{24}(\varepsilon_{BC}\psi_A+\varepsilon_{AC}\psi_B), \tag{56}$$

where Φ_{ABCD} are the spinor components of Φ_{ij}, which are totally symmetric. Expanding the left-hand side of Eq. (56), making use of Eqs. (54) and (29), one obtains

$$\begin{aligned}-\tfrac{1}{2}\Phi_{ABCD}-\tfrac{R}{24}(\varepsilon_{AD}\varepsilon_{BC}+\varepsilon_{AC}\varepsilon_{BD})&=\partial^{R}{}_{(A}\Gamma_{|DC|B)R}-\Gamma^{S}{}_{R}{}^{R}{}_{(A}\Gamma_{|DC|B)S}\\&\quad-\Gamma^{S}{}_{(A}{}^{R}{}_{B)}\Gamma_{DCSR}-\Gamma^{S}{}_{C}{}^{R}{}_{(A}\Gamma_{|DS|B)R},\end{aligned} \tag{57}$$

where the indices between bars are excluded from the symmetrization. Equation (57) leads to the explicit expressions

$$-\tfrac{1}{2}\Phi_{+2}=\left(D-4\varepsilon+\rho+\bar{\rho}\right)\kappa+\left(\delta+2\beta+2\alpha\right)\alpha, \tag{58a}$$

$$-\tfrac{1}{2}\Phi_{+1}=\left(D-2\varepsilon+\rho\right)\beta+\left(\delta+2\alpha\right)\varepsilon-\left(\bar{\alpha}+\bar{\beta}\right)\kappa+\alpha\rho, \tag{58b}$$

$$-\Phi_{+1}=\left(\bar{\delta}-4\bar{\beta}\right)\kappa-\left(\delta+2\alpha\right)\bar{\rho}+2\alpha\rho, \tag{58c}$$

$$-\tfrac{1}{2}\Phi_{0}-\tfrac{R}{12}=(D+\rho)\rho+\left(\delta-2\beta+2\alpha\right)\bar{\alpha}+\kappa\bar{\kappa}, \tag{58d}$$

$$-\Phi_{0}+\tfrac{R}{12}=\bar{\delta}\beta+\delta\bar{\beta}-4\beta\bar{\beta}+2\varepsilon\left(\rho-\bar{\rho}\right)+\kappa\bar{\kappa}-\rho\bar{\rho}, \tag{58e}$$

together with the complex conjugates of Eqs. (58a-d), taking into account that $\bar{D}=D$, $\bar{\varepsilon}=-\varepsilon$ and $\overline{\Phi_s}=(-1)^s\Phi_{-s}$.

As is known, the Bianchi identities $\nabla_i R_{jk\ell m}+\nabla_m R_{jki\ell}+\nabla_\ell R_{jkmi}=0$, imply the contracted Bianchi identities $\nabla^j G_{ij}=0$, which are equivalent to

$$\nabla^{AB}\Phi_{ABCD}+\tfrac{1}{6}\partial_{CD}R=0. \tag{59}$$

Using Eq. (50) one finds that, in the present case, the Bianchi identities follow from the contracted Bianchi identities. In terms of the notation of (31-32) and (36), the Bianchi identities are [cf. Eqs. (40) and (44)]

$$\begin{aligned}\big(\bar{\delta} - 4\bar{\beta} + 2\bar{\alpha}\big)\Phi_{+2} - 2\big(D - 2\varepsilon + \rho + 2\bar{\rho}\big)\Phi_{+1} - (\delta + 6\alpha)\Phi_0& \\ +2\kappa\Phi_{-1} - \tfrac{1}{6}\delta R = 0,& \qquad (60a)\end{aligned}$$

$$\begin{aligned}\bar{\kappa}\Phi_{+2} + \big(\bar{\delta} - 2\bar{\beta} + 4\bar{\alpha}\big)\Phi_{+1} - \big(2D + 3\rho + 3\bar{\rho}\big)\Phi_0& \\ -\big(\delta - 2\beta + 4\alpha\big)\Phi_{-1} + \kappa\Phi_{-2} + \tfrac{1}{6}DR = 0,& \qquad (60b)\end{aligned}$$

$$\begin{aligned}2\bar{\kappa}\Phi_{+1} + \big(\bar{\delta} + 6\bar{\alpha}\big)\Phi_0 - 2\big(D + 2\varepsilon + 2\rho + \bar{\rho}\big)\Phi_{-1}& \\ -\big(\delta - 4\beta + 2\alpha\big)\Phi_{-2} + \tfrac{1}{6}\bar{\delta}R = 0.& \qquad (60c)\end{aligned}$$

Note that Eq. (60c) can be obtained from Eq. (60*a*) by complex conjugation or through the prime operation.

6. EXAMPLES

a) Dirac's equation

The Dirac equation can be written in the standard form

$$\begin{aligned} i\hbar\frac{\partial u}{\partial t} &= -i\hbar c\sigma_j\frac{\partial v}{\partial x^j} + mc^2 u, \\ i\hbar\frac{\partial v}{\partial t} &= -i\hbar c\sigma_j\frac{\partial u}{\partial x^j} - mc^2 v, \end{aligned} \qquad (61)$$

where $\psi = \left[\begin{smallmatrix} u \\ v\end{smallmatrix}\right]$ is a four-component Dirac spinor and the x^i are cartesian coordinates. Recalling that the elements of the Pauli matrices σ_j are ${\sigma_j}^A{}_B$ and using Eq. (22) it is easy to see that Eq. (61) corresponds to the covariant expression

$$\begin{aligned} \frac{1}{c}\frac{\partial u^A}{\partial t} &= -\sqrt{2}\,\nabla^A{}_B v^B - \frac{imc}{\hbar}u^A, \\ \frac{1}{c}\frac{\partial v^A}{\partial t} &= -\sqrt{2}\,\nabla^A{}_B u^B + \frac{imc}{\hbar}v^A, \end{aligned} \qquad (62)$$

(cf. Ref. [7]) which is equivalent to

$$\begin{aligned}
\frac{1}{c}\frac{\partial u^1}{\partial t} &= -\sqrt{2}\,\big(D+\varepsilon+\bar{\rho}\big)v^1 - \sqrt{2}\,\big(\bar{\delta}-\bar{\beta}+\bar{\alpha}\big)v^2 - \frac{imc}{\hbar}u^1,\\
\frac{1}{c}\frac{\partial u^2}{\partial t} &= -\sqrt{2}\,\big(\delta-\beta+\alpha\big)v^1 + \sqrt{2}\,\big(D-\varepsilon+\rho\big)v^2 - \frac{imc}{\hbar}u^2,\\
\frac{1}{c}\frac{\partial v^1}{\partial t} &= -\sqrt{2}\,\big(D+\varepsilon+\bar{\rho}\big)u^1 - \sqrt{2}\,\big(\bar{\delta}-\bar{\beta}+\bar{\alpha}\big)u^2 + \frac{imc}{\hbar}v^1,\\
\frac{1}{c}\frac{\partial v^2}{\partial t} &= -\sqrt{2}\,\big(\delta-\beta+\alpha\big)u^1 + \sqrt{2}\,\big(D-\varepsilon+\rho\big)u^2 + \frac{imc}{\hbar}v^2.
\end{aligned} \tag{63}$$

From Eqs. (63) one can readily obtain the form of the Dirac equation in any orthogonal coordinate system (see below) or in an arbitrary system of coordinates. (For the case of orthogonal coordinates, alternative procedures are given in Refs. [8-10]).

b) Maxwell's equations

The Maxwell equations in vacuum written in terms of the complex vector field $\mathbf{F} \equiv \mathbf{E}+i\mathbf{B}$ are

$$\operatorname{curl}\mathbf{F} - \frac{i}{c}\frac{\partial \mathbf{F}}{\partial t} = i\frac{4\pi}{c}\mathbf{J}, \quad \operatorname{div}\mathbf{F} = 4\pi\rho, \tag{64}$$

where $\mathbf{J}$ and ρ are the current density and the charge density. The spinor form of Eqs. (64) is

$$\sqrt{2}\,\nabla^C{}_{(A}F_{B)C} - \frac{1}{c}\frac{\partial}{\partial t}F_{AB} = \frac{4\pi}{c}J_{AB}, \quad \nabla^{AB}F_{AB} = -4\pi\rho, \tag{65}$$

or, equivalently, (cf.Eqs. [42-43])

$$\begin{aligned}
&\sqrt{2}\,\big\{(D-2\varepsilon+\rho)F_{+1} + (\delta+2\alpha)F_0 - \kappa F_{-1}\big\} - \frac{1}{c}\frac{\partial}{\partial t}F_{+1} = \frac{4\pi}{c}J_{+1},\\
&\sqrt{2}\,\big\{\tfrac{1}{2}(\bar{\delta}-2\bar{\beta})F_{+1} + (\rho-\bar{\rho})F_0 + \tfrac{1}{2}(\delta-2\beta)F_{-1}\big\} - \frac{1}{c}\frac{\partial}{\partial t}F_0 = \frac{4\pi}{c}J_0,\\
&\sqrt{2}\,\big\{\bar{\kappa}F_{+1} + (\bar{\delta}+2\bar{\alpha})F_0 - (D+2\varepsilon+\bar{\rho})F_{-1}\big\} - \frac{1}{c}\frac{\partial}{\partial t}F_{-1} = \frac{4\pi}{c}J_{-1},\\
&(\bar{\delta}-2\bar{\beta}+2\bar{\alpha})F_{+1} - 2(D+\rho+\bar{\rho})F_0 - (\delta-2\beta+2\alpha)F_{-1} = 4\pi\rho.
\end{aligned} \tag{66}$$

The foregoing explicit expressions are further simplified if the triad D, δ, $\bar{\delta}$, is chosen in such a way that several spin-coefficients vanish or some components of the spinor field under consideration vanish. In the case of flat space, it may be useful to take the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as that induced by an orthogonal coordinate system. The simplest example corresponds to cartesian coordinates; by choosing $\partial_i = \partial/\partial x^i$ one gets $D = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}$ and $\delta = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Then from Eqs. (33) it is clear that all the spin-coefficients vanish.

In the case of the spherical coordinates, starting from the orthonormal basis $\partial_1 = \frac{1}{r}\frac{\partial}{\partial \theta}$, $\partial_2 = \frac{1}{r \operatorname{sen}\theta}\frac{\partial}{\partial \phi}$, $\partial_3 = \frac{\partial}{\partial r}$, one has $D = \frac{1}{\sqrt{2}}\frac{\partial}{\partial r}$ and $\delta = \frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial \theta} + \frac{i}{\operatorname{sen}\theta}\frac{\partial}{\partial \phi}\right)$. From Eqs. (33) one readily obtains that the only non-vanishing spin-coefficients are $\beta = -\frac{1}{2\sqrt{2}\,r}\cot\theta$ and $\rho = \frac{1}{\sqrt{2}\,r}$; therefore the operators $(\delta + 2s\beta)$ and $(\bar{\delta} - 2s\bar{\beta})$ are equal to $-\frac{1}{\sqrt{2}\,r}\eth$ and $-\frac{1}{\sqrt{2}r}\bar{\eth}$, where $\eth$ and $\bar{\eth}$ are the operators related with the spin-weighted spherical harmonics defined in Ref. [11]. Similarly, taking $\partial_1 = \frac{\partial}{\partial \rho}$, $\partial_2 = \frac{1}{\rho}\frac{\partial}{\partial \phi}$, $\partial_3 = \frac{\partial}{\partial z}$, where ρ, ϕ, z are circular cylindrical coordinates, one finds that $D = \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}$, $\delta = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \rho} + \frac{i}{\rho}\frac{\partial}{\partial \phi}\right)$, and the only non-vanishing spin-coefficient is $\beta = -\frac{1}{2\sqrt{2}\,\rho}$. Thus, in this case, the operators $(\delta + 2s\beta)$ and $(\bar{\delta} - 2s\bar{\beta})$ amount to $-\frac{1}{\sqrt{2}}\eth$ and $-\frac{1}{\sqrt{2}}\bar{\eth}$, where $\eth$ and $\bar{\eth}$ are the operators related with the spin- weighted cylindrical harmonics [9].

The spinor formalism presented here is also useful in solving the Helmholtz equation for spin-2 fields in orthogonal curvilinear coordinates [12].

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