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Some topological questions on metrics of D type

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ABSTRACT. In this work we present an interpretation of the topology of two limit cases of important exact solutions of Einstein's equations. The primary idea of this work was proposed by Plebański in 1975 [1] using a point of view different from that of W. Israel. It consists in looking for manifolds in which the 2-forms describing the electromagnetic field of the solution in the limit case are single valued. We found that the first manifold has two three-dimensional euclidean spaces with only one temporal axis. The second manifold has two four-dimensional minkowskian spaces. In both cases the two-spaces are joined by a wormhole.

RESUMEN. En este trabajo presentamos una interpretación topológica de dos casos límites de soluciones exactas de las ecuaciones de Einstein. La idea primaria de este trabajo fue propuesta por Plebański en 1975 desde un punto de vista diferente al de W. Israel. Este consiste en buscar variedades en las cuales las 2-formas que describen el campo electromagnético de la solución en el caso límite sean univaluadas. Nosotros encontramos que la primera variedad tiene dos espacios euclídeos tridimensionales con sólo un eje temporal. La segunda variedad tiene dos espacios de Minkowski. En ambos casos los dos espacios están unidos por un agujero de gusano.

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1. INTRODUCTION

We present a construction of topologies of two limit cases of exact solutions, different from that of W. Israel [2], who proposed to consider the Kerr singularity like the source of the field. Our point of view is to consider the Kerr singularity like a wormhole joining two "copies" of this sort of spaces, obtaining in this form a space without singularities but in a certain sense similar to some Riemann surface.

The objective of this work is to give to non-expert readers in general relativity the idea of how certain mathematical tools can be used to construct useful geometries.

This work is organized in the following way: in the first part we present the origin of the physical problem, showing the limit cases of the solutions which we will use and the formalism of the electromagnetic field in terms of differential forms. In the second part we give the mathematical background necessary to build surfaces in which the differential forms describing the electromagnetic field are well-defined. In the last part the construction is given in detail and the possible interpretations are explained.

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2. PRELIMINARY

It is known that in general relativity the electromagnetic fields are often represented by real of complex differential forms. Using complex forms one finds that they are not always single valued. In this work we present a way of finding manifolds in which two different fields are single valued and their possible physical consequences. Also we find that the structure of these manifolds is a type of wormhole.

The Einstein-Maxwell equations written in tensorial notation and units in which $G = c = 1$, are

$$\check{f}^{\mu\nu}_{;\nu} = 0, \qquad f^{\mu\nu}_{;\nu} = 0, \tag{2.1}$$

$$G_{\mu\nu} = 8\pi E_{\mu\nu} + \lambda g_{\mu\nu}, \tag{2.2}$$

where

$$4\pi E_{\mu\nu} = -f_{\mu\rho} f^{\rho}_{\nu} + g_{\mu\nu} F \tag{2.3}$$

and the duality operation is given by $\check{f}^{\mu\nu} = \frac{i}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\tau} f_{\rho\tau}$. The invariants of the electromagnetic field will be denoted by

$$F = \tfrac{1}{4} f_{\mu\nu} f^{\mu\nu}, \qquad \check{G} = \tfrac{1}{4} f_{\mu\nu} \check{f}^{\mu\nu}, \tag{2.5}$$

$$\mathcal{F} = F + \check{G};$$

or using orthogonal tetrads we have

$$f_{a'b'} = f_{\mu\nu} e^{\mu}_{a'} e^{\nu}_{b'},$$

$$g = \det(g_{\mu\nu}) = \det(g_{a'b'} e^{a'}_{\mu} e^{b'}_{\nu}), \tag{2.6}$$

$$\check{f}^{a'b'} = \tfrac{i}{2} \epsilon^{a'b'c'd'} f_{c'd'}. \tag{2.7}$$

It is known that Eqs. (2.1) are equivalent to the condition that the forms

$$f = \tfrac{1}{2} f_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = \tfrac{1}{2} \check{f}_{a'b'} e^{a'} \wedge e^{b'},$$

$$\check{f} = \tfrac{1}{2} \check{f}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = \tfrac{1}{2} \check{f}_{a'b'} e^{a'} \wedge e^{b'},$$

are closed. Using the fact that f is real and $\check{f}$ is pure imaginary we can introduce a two-complex form

$$w = f + \check{f} = \tfrac{1}{2}(f_{a'b'} + \check{f}_{a'b'}) e^{a'} \wedge e^{b'},$$

which replaces Eqs. (2.1) by the complex condition

$$dw = 0. \tag{2.8}$$

In 1975, Plebański [1] found a 6-parametric solution of type D and looked for the interpretation of the electromagnetic field. In the flat space limit, and using the transformations

$$\begin{aligned} x+iy &= \frac{1}{a}\left[(a^2-p^2)(a^2+q^2)\right]^{1/2}\exp(ia\sigma), \\ z &= \frac{pq}{a}, \qquad t=\tau-a^2\sigma, \\ q+ip &= \left[x^2+y^2+(z+ia)^2\right]^{1/2}, \\ \sigma &= \frac{1}{2ia}\ln\frac{x+iy}{x-iy}, \qquad \tau = t+\frac{a}{2i}\ln\frac{x+iy}{x-iy}, \end{aligned} \tag{2.9}$$

he found that the electromagnetic field is given by

$$\begin{aligned} w_k &= -d\left\{\frac{e+ig}{[x^2+y^2+(z+ia)^2]^{1/2}}\left[dt-i(z+ia)\frac{x\,dy-y\,dx}{x^2+y^2}\right]\right\}, \\ \mathcal{F} &= -\frac{1}{2}\left(\frac{e+ig}{x^2+y^2+(z+ia)^2}\right)^2. \end{aligned} \tag{2.10}$$

The presence of the complex numbers $[x^2+y^2+(z+ia)]^{1/2}$ in the field structure, presents an interesting problem. In order to make this root and hence the field uniquely defined over the euclidean space described by variables (x,y,z), one can propose to understand (2.10) as defined with the cut along the disc D: $x^2+y^2 \leq a^2$, $z=0$; then, if D is approached from the side $z>0$ to the side $z<0$, the electromagnetic field suffers a nontrivial jump $\Delta f^{\mu\nu}_{;\nu}$ along D. This interpretation assumes: computing $f^{\mu\nu}_{;\nu}$ one finds some δ–*like* currents along D [2]; if e is acompanied by a nontrivial g, we also find δ–*like* pseudocurrents along D. This interpretation however, cannot be considered as entirely satisfactory; *e.g.* assuming the cut along any surface D, topologically equivalent to D (*i.e.* a surface spanned on Kerr's circle) we will have as well nontrivial jumps $\Delta f^{\mu\nu}_{;\nu}$ on D' and some other distribution of δ–*like* currents on D'. On the other hand, nothing in the analytic structure of (2.10) indicates how to select preferentially the cut surface.

There is some alternative bolder interpretation of the electromagnetic field. We can state that although ds^2 is flat, the assumption that this flat space has open euclidean topology is an independent assumption. Abandoning this assumption, we can seek the topological structure of the flat space which corresponds adequately to the analytic structure of the electromagnetic field. (A fair example of the similar manner of proceeding, forms the well-known Kruskalization process with the standard Schwarzschild solution [4,5].)

Considering Eq. (2.9), it is natural to give the definition of a manifold with the following ranges of coordinates:

$$M_4 : \begin{cases} a \geq p \geq -a \\ \infty \geq q \geq -\infty \\ 2\pi \geq a\sigma \geq 0 \\ \infty \geq \tau \geq -\infty \end{cases} \tag{2.11}$$

(we identify points with $a\sigma = 0$ and $a\sigma = 2\pi$).

The structure of M_4 can be readily deduced from relations (2.9); the result is that M_4 is the product of the infinite time axis: $T = \{\infty \geq t \geq -\infty\}$ times the three-dimensional space S_3, which consists of two copies of open three-dimensional euclidean spaces E_1 and E_2 with some subset of points being identified. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be cartesian coordinates in E_1 and E_2, respectively; then we introduce the subset $E_1^{(\pm)}$ and $E_2^{(\pm)}$, defined respectively by $z_1 \geq 0$ or $z_1 \leq +0$ and $z_2 \geq +0$ or $z_2 \leq +0$. We identify now the points from a disc D_1 defined by

$$D_1: \quad \begin{matrix} z_1 = +0 & x_1^2 + y_1^2 \leq a^2 \\ z = -0 & x_2^2 + y_2^2 \leq a^2, \end{matrix}$$

i.e. on D_1 we have: $(x_1, y_1, z_1) = (x_2, y_2, z_2)$. Similarly, we identify the points from a disc D_2 defined by

$$D_2: \quad \begin{matrix} z_1 = -0 & x_1^2 + y_1^2 \leq a^2 \\ z_2 = +0 & x_2^2 + y_2^2 \leq a^2 \end{matrix}$$

i.e., on D_2 we have $(x_1, y_1, z_1) = (x_2, y_2, z_2)$. However if $x_1^2 + y_1^2 > a^2$ we identify the points $(x_1, y_1, z_1 = +0)$ and $(x_1, y_1, z_1 = -0)$ and similarly, if $x_2^2 + y_2^2 \geq a^2$, we identify the points $(x_2, y_2, z_2 = -0)$ with $(x_2, y_2, z_2 = +0)$. The construction of S_3 described above is symbolically visualized in Fig. 1, where arrows indicate basic identifications.

After making a loop around the Kerr circle in E_1, we do not find now any jump of the electromagnetic field: the new value of the field arising from the ramification point of $[x^2 + y^2 + (z + ia)^2]^{1/2}$ enters smoothly through the corresponding disk into E_2.

It should be noticed that with $g_0 = 0$ in the asymptotic points of $E_1(x_1^2 + y_1^2 + z_1^2 \to \infty)$ the studied field represents the field of an electric monopole of charge $= +e$ and a magnetic moment $+ea$. At the same time, from the point of view of E_2, the field represents asymptotically $(x_2^2 + y_2^2 + z_2^2 \to \infty)$ the field of an electric monopole with charge $-e$ and magnetic dipole of magnetic moment $-ea$. The first objective of this work is to give a description of this construction using more detailed mathematical terms and showing that the field is really single valued in this manifold. The second objective of this work is the generalization of this procedure for one limit case of the Plebański-Demianski solution in

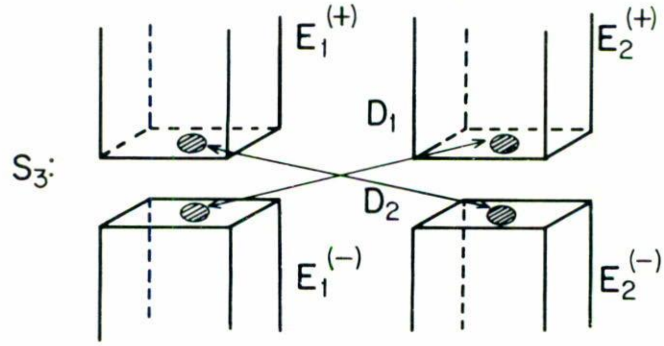

FIGURE 1. The construction of S_3.

the case in which the parameter b is considered. In 1976 Plebański and Demianski [3] got a seven-parametric solution, and studied the limit case to flat space. They found that the electromagnetic field and its invariant are given by

$$\begin{aligned} w_{P-D} &= d\left\{ \frac{(e+ig)}{\left(F^{(+)}F^{(-)}\right)^{1/2}} \left(G^{(-)}\,d\chi + iG^{(+)}\,d\phi\right) \right\}, \\ \mathcal{F} &= \frac{1}{2}(e+ig)^2 \left[\frac{2(a+ib)}{\left(F^{(+)}F^{(-)}\right)^{1/2}} \right]^4, \end{aligned} \tag{2.12}$$

with

$$\begin{aligned} F^{(\pm)} &= \left[(x^2+y^2)^{1/2} \pm (a+ib)\right]^2 + z^2 - t^2, \\ G^{(\pm)} &= (a+ib)^2 \pm (x^2+y^2+z^2-t^2) \end{aligned} \tag{2.13}$$

and

$$d\phi = \frac{x\,dy - y\,dx}{x^2+y^2}, \qquad d\chi = \frac{z\,dt - t\,dz}{z^2-t^2}. \tag{2.14}$$

When a and b are finite and positive , the electromagnetic field has singularities only when $F^{(-)}=0$, *i.e.*, in the set of points that fulfill the conditions

$$x^2+y^2=a^2 \qquad \text{and} \qquad z^2-t^2=b^2; \tag{2.15}$$

then the singular region consists of two circles:

$$\begin{aligned} x_1^2+y_1^2 &= a^2 \qquad \text{and} \qquad z_1 = +(t^2+b^2)^{1/2}; \\ x_2^2+y_2^2 &= a^2 \qquad \text{and} \qquad z_2 = -(t^2+b^2)^{1/2}. \end{aligned} \tag{2.16}$$

These results give the motivation to look for a manifold in which the electromagnetic field is single valued with the same spirit used above.

At this point we want to mention something about the limit transition $b \to \infty$ (acceleration $A = 0$). In order to realize the transition $b \to \infty$, we first have to translate the origin of the z coordinate:

$$z = z' + b. \tag{2.17}$$

Note that this transformation does not change the form of the Minkowski metric. Taking the limit in Eqs. (2.12) we have

$$\lim_{b\to\infty} w = -d\left\{\frac{e+ig}{[x^2+y^2+(z'+ia)^2]^{1/2}}\left(dt - i(z'+ia)\frac{x\,dy - y\,dx}{x^2+y^2)}\right)\right\}$$

and

$$\lim_{b\to\infty} \mathcal{F} = -\frac{1}{2}\left[\frac{e+ig}{[x^2+y^2+(z'+ia)^2}\right]^2,$$

which are just Eqs. (2.10).

3. Topological constructions

Spaces of identification

Definition: Let A and B be sets, their disjoint union being given by

$$A \coprod B = A \times \{1\} \cup B \times \{2\}.$$

The numbers 1 and 2 appearing in this definition are used only to distinguish between the points associated with A and the points associated with B, for example, if $x \in A \cup B$ then $(x,1), (x,2) \in A \coprod B$, *i.e.*, x is represented two times. The distinctive characteristics of the disjoint union can be seen in a stronger way when $A = B$.

$$A \coprod A = A \times \{1\} \cup A \times \{2\}.$$

This construction means: "Let X be the union of two copies of A", *i.e.* Let "$X = A \coprod A$". A partition of a set X is a collection of disjoint subsets of X whose union is X.

Definition: The space of identification, also called quotient space of a partition (or of an equivalence class) is the set of equivalence classes. When the equivalence class is denoted by $\sim$, the space of identification is denoted by $X/\sim$.

The topology of the identification space is given in the following way: let $[x] \in X/\sim$, if V is a neighborhood of x, then a neighborhood of $[x]$ is the collection of all classes $[y]$ such that $y \in V$. In other words, $V \subset X/\sim$ is open if and only if the set

$$\{x \in X \mid [x] \in U\}$$

is open in X. One of the more common ways of giving an equivalence relation is by means of an identification function. Let A and B be sets, and $A_0 \subset A$, consider a bijection $\varphi\colon A_0 \to B_0$ of A_0 over a subset $B_0 \subset B$. This situation which appears very often, determines an equivalence relation over $X = A \cup B$ in the following way:

$$x \sim y \text{ if and only if} \begin{cases} \text{i) } x = y; \text{ or} \\ \text{ii) } x \in A_0,\ y \in B_0 \text{ and } y = \varphi(x); \text{ or} \\ \text{iii) } y \in A_0,\ x \in B_0 \text{ and } x = \varphi(y). \end{cases}$$

The space of identification $X/\sim$ will be denoted in this case by $A \cup_\varphi B$ and will be called "A union B modulo the points identified by φ". If A and B are not disjoint and we want to identify one part of A with one part of B (not necessarily in $A \cap B$), the resulting space will be denoted by

$$A \coprod_\varphi B,$$

where φ is the function that gives the identification, *i.e.*, $A \coprod_\varphi B$, means $(A \times \{1\}) \cup_\psi (B \times \{2\})$, where $\psi(x,1) = (\varphi(x), 2)$ and where the definition domain of ψ is $A_0 \times \{1\}$, moreover $\psi : A_0 \times \{1\} \to B_0 \times \{2\}$ is a bijection between subsets of $A \times \{1\}$ and $B \times \{2\}$.

The charts $C_1 = (U_1, \psi_1)$ and $C_2 = (U_2, \psi_2)$ are compatible because the intersection of the sets U_1 and U_2 is empty, so the structures considered are differentiable manifolds. Moreover, these structures are two $\mathbf{R}^4$ spaces glued in an adequate way.

Finally consider the case in which we want to identify some points of $A \cup B$ with themselves in disjoint union, specifically the points of a given subset $A_0 \subset A \cap B$ then we have

$$A \coprod B/A_0 \equiv A \coprod_\varphi B, \tag{3.1}$$

with $\varphi\colon A_0 \to B_0 = A_0$, $\varphi(x) = x$. This kind of space is called "A and B made disjoints unless by A_0". It should be noted that $A \coprod B/A_0$ is *not* used in the sense of quotient set A/A_0 with $A_0 \subset A$.

An important example

We want to build a manifold in which $w = z^{1/2}$ can be realized as a single valued function. To do so, consider the following subset of $\boldsymbol{C}$:

$$\begin{aligned} P^\pm &= \{z \in \mathbf{C} \mid \pm \operatorname{Im} z > 0\}, & \overline{P^\pm} &= P^\pm \cup \mathbf{R} \cup \{\infty\}, \\ \mathbf{R}^\pm &= \{x \in \mathbf{R} \mid \pm x > 0\}, & \overline{\mathbf{R}^\pm} &= \{0\} \cup \mathbf{R}^\pm \cup \{\infty\}, \end{aligned} \tag{3.2}$$

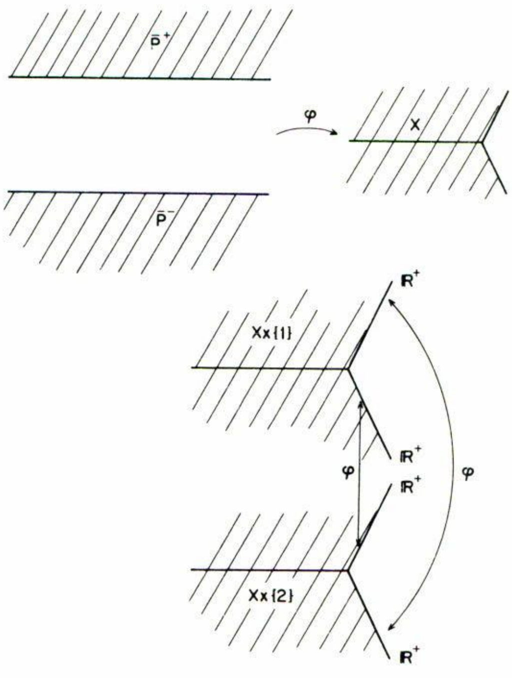

FIGURE 2. Joining of the positive semi-axes.

and consider $X = (\overline{P^+} \coprod \overline{P^-})/\mathbf{R}^-$, *i.e.* we join the closed semiplanes along the negative axis, now the only difference between X and $\hat{\mathbf{C}}$ (Riemann's sphere) being that X has two positive semi-axis $\mathbf{R}^+ \times \{1\}$ and $\mathbf{R}^+ \times \{2\}$.

Now we define $S = X \coprod_\varphi X$ where X is given above and φ is defined by

$$\varphi\colon \big(\overline{\mathbf{R}^+} \times \{1\}\big) \cup \big(\overline{\mathbf{R}^+} \times \{2\}\big) \to \big(\overline{\mathbf{R}^+} \times \{1\}\big) \cup \big(\overline{\mathbf{R}^+} \times \{2\}\big),$$

joining the positive semi-axes in the following way:

$$\varphi(x,1) = (x,2), \qquad \varphi(x,2) = (x,1).$$

This situation is illustrated in Fig. 2. A point of S like $((z,1),1), ((z,1),2)$ is a three-fold point and so on. (Actually they are equivalence classes of such threefold points). The last two coordinates of point are not so interesting as the values of z, but it is convenient to use them in order to know in which part of S we are working.

The first coordinate z, which represents the locus in the Riemann sphere, will be defined as the projection of the points

$$\pi(s) = \pi\big((z,j,k)\big) = z.$$

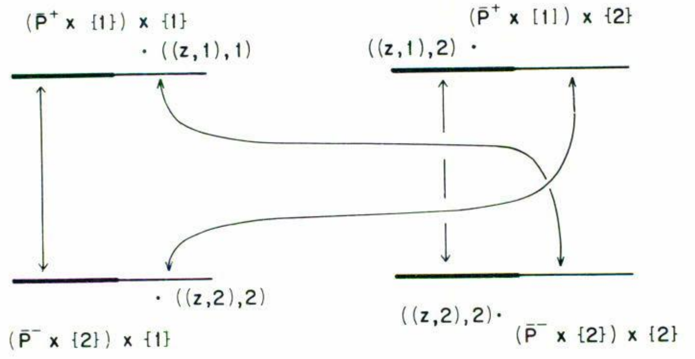

FIGURE 3. Decomposition of S.

To remember the position of each point we decompose S into its original elements (Fig. 3).

Now we can see the way in which $z^{1/2}$ can be realized as function in S, beginning with the branch of $z^{1/2}$ defined on $\boldsymbol{C} - \overline{\mathbf{R}^+}$ by

$$w = f(z) = r^{1/2} e^{\frac{i\theta}{2}},$$

with $z = re^{i\theta}$, $r > 0$, $0 < \theta < 2\pi$. Remember that $f(-1) = f(e^{i\pi}) = e^{\frac{i\pi}{2}} = i$ approaches the value $+1 = e^{\frac{0}{2}}$, when z approaches $+1 = e^0$ on $\overline{P^+}$, while f approaches the value $-1 = e^{\frac{2\pi i}{2}}$ when z approaches $+1 = e^{2\pi i}$ on $\overline{P^-}$. For this reason f admits a continuous extension on X and the existence of two positive semiaxes is used to solve the ambiguity $f(1) = \pm 1$; such extension can be written explicitly:

For each $x > 0$,

$$f(x, 1) = x^{1/2}, \qquad f(x, 2) = -x^{1/2}$$

with $x^{1/2} > 0$. Moreover $f(0) = 0$, $f(\infty) = \infty$, the image of $\mathbf{R}^+ \times \{1\}$ is $\mathbf{R}^+$ and the image of $\mathbf{R}^+ \times \{2\}$ is $\mathbf{R}^-$.

We need a function of S but we have one of X, so we consider the function we have, as the function on $X \times \{1\}$, *i.e.*,

$$\rho(s) = \rho((z, j), 1) = r^{\frac{1}{2}} e^{\frac{i\theta}{2}} \text{ on } X \times \{1\},$$

with $z = re^{i\theta}$, $0 \le \theta \le 2\pi$ and $j = 1$ or $j = 2$ depending on whether $z \in \overline{P^+}$ or $z \in \overline{P^-}$.

Consider the point $s = ((1, 2), 1)$, a neighborhood of s on S is partially contained on $X \times \{2\}$ (due to $[s] = [((1, 2), 1)] = [((1, 1,), 2)]$). The first part of that neighborhood will be projected by π on $\overline{P^-}$ and the second on $\overline{P^-}$. Then we have to define ρ on $X \times \{2\}$ in such a way that the point $z = +1$ viewed from P^+ in the copy 2 will be sent to $\omega = -1$. For this reason we define

$$\rho((z, j), 2) = r^{\frac{1}{2}} e^{\frac{i\theta}{2}} \text{ on } X \times \{2\},$$

with $z = re^{i\theta}$, $2\pi \le \theta \le 4\pi$ and $j = 1$ or $j = 2$. If we want to arrive at the conclusion that $\rho: S \to \mathbf{C}$ is well-defined we only need to check that the values of ρ on $\mathbf{R}^+ \times \{1\} \times \{1\}$ ($\theta = 0$) are in accordance with the values on $\mathbf{R}^+ \times \{2\} \times \{2\}$, ($\theta = 4\pi$), but this follows from

$$r^{\frac{1}{2}} e^{\frac{i(\theta + 4\pi)}{2}} = r^{\frac{1}{2}} e^{\frac{i\theta}{2}} e^{2\pi i} = r^{1/2} e^{\frac{i\theta}{2}},$$

and the relation between ρ and the square root is

$$\rho(s) = f(\pi(s)).$$

4. Manifolds that become w_K and w_{P_D} in single valued fields

a) For w_K the origin of the multivaluation is the root

$$f = \left[x^2 + y^2 + (z + ia)^2\right]^{1/2},$$

so that where this radical is single valued, so will also w_k be. Using the symmetry between x and y one defines $\rho^2 = x^2 + y^2$, so

$$f = \left[\rho^2 + (z + ia)^2\right]^{1/2} = \left[(\rho + iz - a)(\rho - iz + a)\right]^{1/2}.$$

Let

$$Z = \rho + iz, \quad Z - a = R_1 e^{i\alpha_1}, \quad \overline{Z} + a = r_2 e^{i\alpha_2},$$

in this way

$$f = \left[(Z - a)(\overline{Z} + a)\right]^{1/2} = \sqrt{r_1 r_2} \exp\left(\frac{i(\alpha_1 + \alpha_2)}{2}\right).$$

Note that f is not an analytic function of Z due to the appearance of $\overline{Z}$. When one completes a cycle around C (Fig. 4) one has

$$\alpha_2 \to \alpha_2 + 2\pi,$$

$$\alpha_1 \to \alpha_1,$$

and then

$$f \to \sqrt{r_1 r_2} \exp\left[\frac{i(\alpha_1 + \alpha_2)}{2} + i\pi\right] = -f.$$

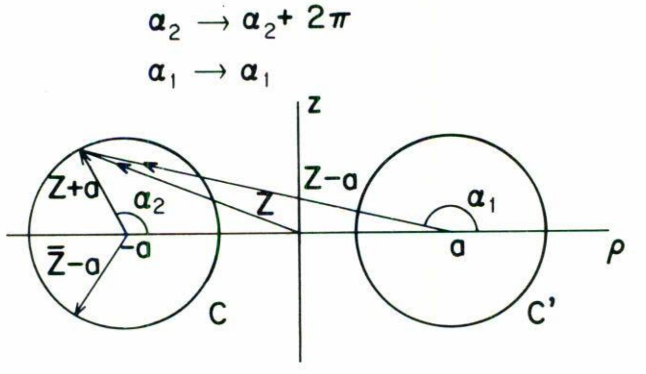

FIGURE 4. Circles around C.

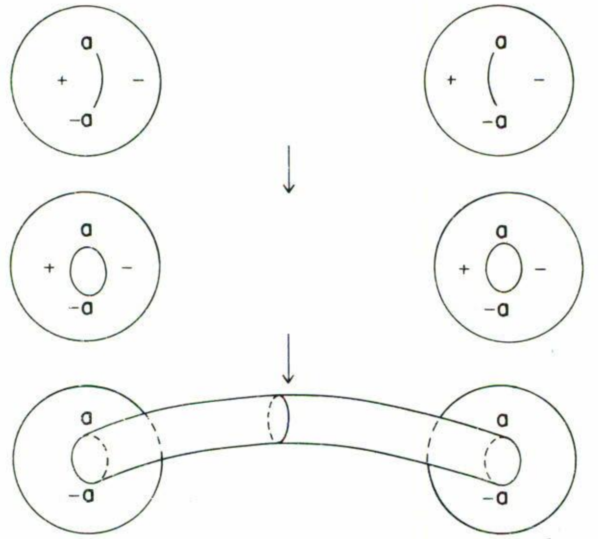

FIGURE 5. Joining points of ramification.

Seeing that $-a$ is a certain kind of branching point,* one can show the same when one gives a loop on the contour C'. The last paragraph suggests a construction of the following kind (Fig. 5): if one moves with $|\rho| < a$ and across the axis $z = 0$, one changes space, but if one crosses the axis with $|\rho| > a$, one does not have any change of space. It is clear that this surface is not the surface that we are looking for because this surface is two-dimensional and we need a three-dimensional surface, but this surface illustrates the physical situation and gives a motivation for the adequate surface.

*The concept of branching point is defined for analytical functions only.

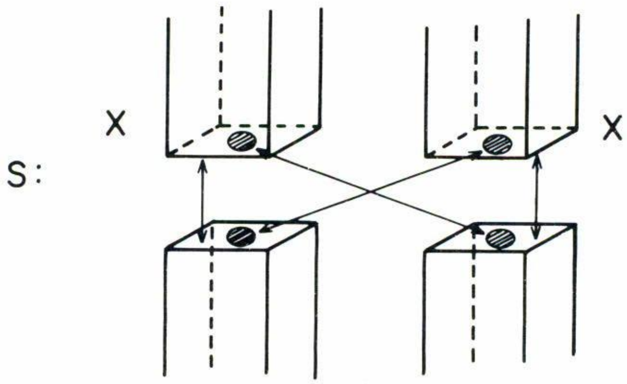

FIGURE 6. The function φ on S.

Using a similar notation, we take

$$A = B = \mathbf{R}^3, \qquad A_0 = \{v \in \mathbf{R}^3 | \rho^2 < a^2, z = 0\},$$

$$P^{\pm} = \{v \in \mathbf{R}^3 | \pm z > 0\}, \qquad P^0 = \{v \in \mathbf{R}^3 | z = 0\}$$

and

$$\overline{P^{\pm}} = P^{\pm} \cup P^0 \cup \{\infty\}.$$

Let

$$X = \overline{P^+} \coprod \overline{P^-} / P^0 - A_0$$

and $S = X \coprod_{\varphi} X$ with

$$\varphi : A_0 \times \{1\} \cup A_0 \times \{2\} \to A_0 \times \{1\} \cup A_0 \times \{2\};$$

identifying the disc in the following way:

$$\varphi(v, 1) = (v, 2),$$

$$\varphi(v, 2) = (v, 1),$$

this situation is represented by Fig. 6. Now define the function γ on S as

$$\gamma((v, i), 1) = \sqrt{r_1 r_2}\, e^{i/2(\alpha_1 + \alpha_2)} \text{ on } X \times \{1\},$$

with

$$v = (x, y, z), \quad \rho^2 = x^2 + y^2, \quad Z = \rho + iz,$$

$$Z - a = r_1 e^{i\alpha_1}, \quad \overline{Z} + a = r_2 e^{i\alpha_2},$$

and $i = 1$ or $i = 2$ for $v \in P^+$ or $v \in P^-$, respectively, and $0 \le \alpha_1, \alpha_2 \le 2\pi$.

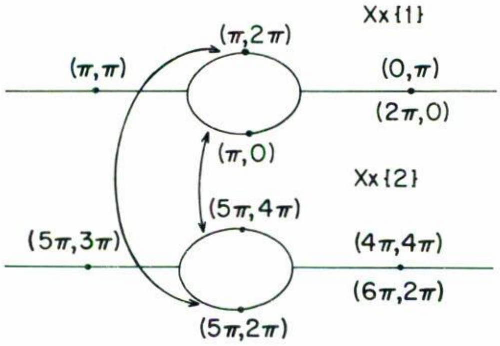

FIGURE 7. Transverse section of S.

Any neighborhood of the point $s = ((v,1),1)$ with v on the disc is partially contained in $X \times \{1\}$ and partially in $X \times \{2\}$, so we have to define γ on $X \times \{2\}$ in such a way that the points viewed from P^- on copy 2 will be sent to the same values which are sent by γ on copy 1. So consider

$$\gamma((v,j),2) = \sqrt{r_1 r_2}\, e^{i/2(\alpha_1+\alpha_2)} \text{ on } X \times \{2\},$$

with the same notation used before, but now $2\pi \leq \alpha_2 \leq 4\pi$ and $4\pi \leq \alpha_1 \leq 6\pi$. Figure 7 shows a transverse section with the values of α_1 and α_2.

To convince ourselves that γ is well-defined one only needs to note that when one "passes" from one copy to the other through the discs, the sum $\alpha_1 + \alpha_2$ is such that $e^{\frac{i}{2}(\alpha_1+\alpha_2)}$ has always the same value.

b) Working with the same idea but now for ω_{P-D}, we have that the origin of the multivaluation comes from the root

$$f = \left(F^{(+)}F^{(-)}\right)^{1/2},$$

with

$$F^{(\pm)} = \left[(x^2+y^2)^{1/2} \pm (a-ib)\right]^2 + z^2 - t^2.$$

Note that

$$F^{(\pm)} = \left[\rho \pm (a+ib)\right]^2 + z^2 - t^2 = \left[\rho \pm (a+ib) + i(z^2-t^2)^{1/2}\right] \times \left[\rho \pm (a+ib) - i(z^2-t^2)^{1/2}\right].^*$$

*We have used this in the region where we are working, $z^2 > t^2$; see Eq. (4.13) in Ref. [3].

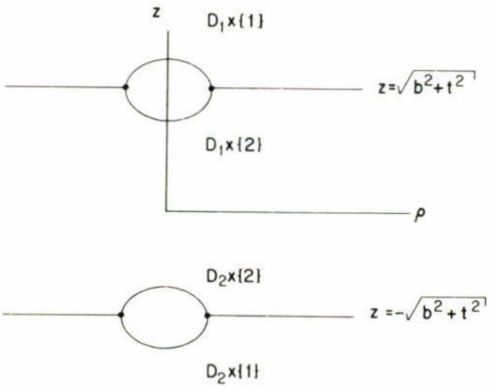

FIGURE 8. Transverse section of X.

Defining $Z = \rho + i(z^2 - t^2)^{1/2}$, $\overline{Z} = \rho - i(z^2 - t^2)^{1/2}$, we find $F^{(\pm)} = [Z \pm (a+ib)][\overline{Z} \pm (a+ib)]$. Let us take $Z - a - ib = \gamma_1 e^{i\alpha_1}$ and $\overline{Z} - a - ib = \gamma_1 e^{i\alpha_2}$; for $F^{(-)}$ one has

$$F^{(-)} = \gamma_1\gamma_2 e^{i(\alpha_1+\alpha_2)}.$$

Analogously, $Z + a + ib = \gamma_3 e^{i\alpha_3}$ and $Z + a + ib = \gamma_4 e^{i\alpha_4}$, so $F^{(+)} = \gamma_3\gamma_4 e^{i(\alpha_3+\alpha_4)}$ and $f = (F^{(+)}F^{(-)})^{1/2} = \sqrt{\gamma_1\gamma_2\gamma_3\gamma_4}\, e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)/2}$. Let $A = \{v \in \mathbf{R}^4 \mid |z| > \sqrt{b^2+t^2}\}$; $A_0 = \{v \in \mathbf{R}^4 \mid |z| < \sqrt{b^2+t^2}\}$; $H = \{v \in \mathbf{R}^4 \mid z^2 - t^2 = b^2\}$ and $R = \{v \in H \mid \rho^2 < a^2\}$; with $v = (x,y,z,t)$ we build $X = A \coprod A_0/(H-R)$. We try to illustrate this construction giving a transverse section in Fig. 8.

On the other hand, let $S = X \coprod_\varphi X$, where the identification function is given by

$$\varphi\colon (R \times \{1\}) \cup (R \times \{2\}) \to (R \times \{1\}) \cup (R \times \{2\}),$$

$$\varphi(v,1) = (v,2),^*$$

$$\varphi(v,2) = (v,1).$$

Define $\sigma(s)$ with $s \in S$ as

$$\sigma(s) = \sigma((v,i),j) = f(v) = \sqrt{\gamma_1\gamma_2\gamma_3\gamma_4}\, e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)/2},$$

with

$$0 \le \alpha_i \le 2\pi \text{ if } j = 1 \text{ (copy 1)}$$

and

$$2\pi \le \alpha_i \le 4\pi \text{ if } j = 2 \text{ (copy 2)}.$$

*Here the subindices 1 or 2 correspond to A or A_0, respectively.

To convince ourselves that this is a function, we only have to check that σ takes the same values on the points of identification. We will show this for points of the type $s = [[((v,1),1)]] = [[((v,2),2)]]$; the proof for points of the type $s = [[((v,2),1)]] = [[((v,1),2)]]$ is exactly the same.

We know that the multivaluation of σ comes from the exponential function, so we should study it for a point $v \in \mathbf{R}$; we have $\alpha_1 = \arg(Z - a - ib) = \arg(\rho - a + i(z^2 - t^2) - ib)$, but $v \in R \Rightarrow z^2 - t^2 = b^2$, $\rho < a$, so $\alpha_1 = \arg(\rho - a) = n\pi$ ($n = 1$ on copy 1 and $n = 3$ on copy 2):

$$\alpha_2 = \arg(\overline{Z} - a - ib) = \arg(\rho - a - 2ib),$$

$$\alpha_3 = \arg(Z + a + ib) = \arg(\rho + a + 2ib),$$

$$\alpha_4 = \arg(\overline{Z} + a + ib) = \arg(\rho + a) = m\pi$$

($m = 0$ on copy 1 and $m = 2$ on copy 2). Denoting with a prime the argument on copy 2 and without a prime on copy 1, we have

$$\alpha_1' = \alpha_1 + 2\pi,$$

$$\alpha_2' = \alpha_2 + 2\pi,$$

$$\alpha_3' = \alpha_3 + 2\pi,$$

$$\alpha_4' = \alpha_4 + 2\pi,$$

$$\Rightarrow \frac{1}{2}\sum_i \alpha_i' = \frac{1}{2}\sum_i \alpha_i + 4\pi \Rightarrow \exp\left(\frac{1}{2}\sum_i \alpha_i'\right) = \exp\left(\frac{1}{2}\sum_i \alpha_i\right),$$

then

$$\sigma\big(((v,1),1)\big) = \sigma\big(((v,2),2)\big),$$

and σ is a function, being $S = X \coprod_\varphi X$ (with σ and X defined as above) the manifold where ω_{P-D} is single valued. If in the case of w_K one defines

$$\psi_j: X \times \{j\} \times T - \{[((v,1),j)] \mid v \in A_0\} \times T \to \mathbf{R}^4,$$

$$([(v,i),j)], t) \to (v,t)$$

and

$$\psi_j^{-1}: \mathbf{R}^4 \to X \times \{j\} \times T - \{[((v,1),j)] \mid v \in A_0\} \times T$$

$$(v,t) \to \begin{cases} \big([((v,1),j)],t\big) & \text{if } z > 0 \\ \\ \big([((v,2),j)],t\big) & \text{if } z \leq 0 \end{cases}$$

with $v = (x, y, z)$, one finds that the manifold constructed is differentiable and that each one of its two charts are homeomorphic to $\mathbf{R}^4$; physically we have the requirement for the single valuedness of the 2-form w_K in the existence of two spaces $\mathbf{R}^3$ and only one temporal axis T.

In the case of w_{P-D} one defines

$$\psi_j\colon X \times \{j\} - \{[((v,1,j)] \mid v \in R\} \to \mathbf{R}^4,$$

$$[[((v,i),j)]] \to v$$

and

$$\psi_j^{-1}\colon \mathbf{R}^4 \to X \times \{j\} - \{[[((v,1),j)]] \mid v \in R\},$$

$$v \to \begin{cases} [[((v,1),j)]] & \text{if } |z| > \sqrt{b^2+t^2} \\ \\ [[((v,2),j)]] & \text{if } |z| \leq \sqrt{b^2+t^2}; \end{cases}$$

then, this manifold is differentiable and each one of its charts is homeomorphic to $\mathbf{R}^4$, but in contrast to the earlier case, we need two temporal axes for the single valuedness of w_{P-D}.

The charts $C_1 = (U_1, \psi_1)$ and $C_2 = (U_2, \psi_2)$ are compatible because the intersection of the sets U_1 and U_2 is empty, so the structures considered are differentiable manifolds. Moreover, these structures are two $\mathbf{R}^4$ spaces glued in an adequate way.

5. Limit transition

Now we consider the limit case of acceleration equal to zero [Eq. (2.30)], *i.e.*, after the translation $z = z' + b$ we will take $b \to \infty$. By definition we had

$$A = \left\{v \in \mathbf{R}^4 \mid |z| > \sqrt{b^2+t^2}\right\}, \quad A_0 = \left\{v \in \mathbf{R}^4 \mid |z| < \sqrt{b^2+t^2}\right\},$$

and

$$H = \left\{v \in \mathbf{R}^4 \mid z^2 - t^2 = b^2\right\},$$

$$R = \left\{v \in H \mid \rho^2 < a^2\right\}.$$

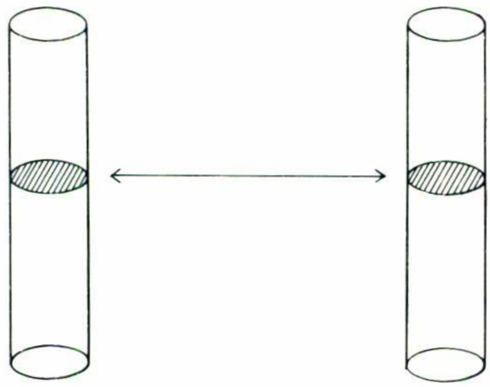

FIGURE 9. Singularity for the w_K space.

After the transition these sets are transformed into the sets

$$A \to A_\infty = \{v \in \mathbf{R}^4 \mid z > 0\},$$

$$A_0 \to A_{0\infty} = \{v \in \mathbf{R}^4 \mid z < 0\},$$

$$H \to H_\infty = \{v \in \mathbf{R}^4 \mid z = 0\},$$

$$R \to R_\infty = \{v \in H_\infty \mid \rho^2 < a^2\}.$$

These new sets can be written in the form

$$A_{0\infty} = P^- \times T, \quad T = \{t \in \mathbf{R} \mid -\infty < t < \infty\},$$

$$A_\infty = P^+ \times T, \quad H_\infty = P^0 \times T, \quad R_\infty = A_0 \times T,$$

where $R^\pm$, P^0 and A_0 are the sets defined for the case w_k; moreover the expressions above are actually M_4 (Eq. 2.11) but written in another way. The reader should remember that $M_4 = S_3 \times T$ and S_3 is composed of two cartesian spaces. In the last expressions we only have one cartesian space $\times T$, but this is due to the fact that we are considering these sets on one copy only; when we take into account the two copies we get the two spaces of S_3. Moreover under these conditions it is easy to show that $w_{P-D} \to w_K$.

The earlier problem can be seen from another point of view: consider the case w_K in a three-dimensional form on the space x–y–t. The singularity is represented by a cylinder on each copy and from this point of view the earlier process is the identification of discs having the same t as is illustrated in Fig. 9.

Considering the same idea for the case w_{P-D}, the singularity is homeomorphic to a two-dimensional manifold, which can be represented by Fig. 10. Note that the effect of the limit transition is to "straighten" one of the surfaces and "send" the other one to infinity. From this point of view the identification is illustrated in Fig. 11.

Finally we want to mention something about the geodesics for the limit case, of the Plebański and Demianski metric studied. Consider x and y constant such that $x^2+y^2 < a^2$;

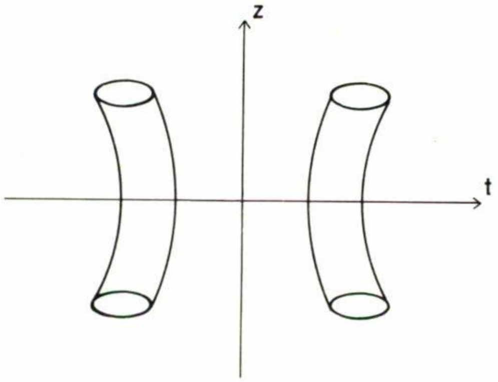

FIGURE 10. Singularity for the w_{P-D} space.

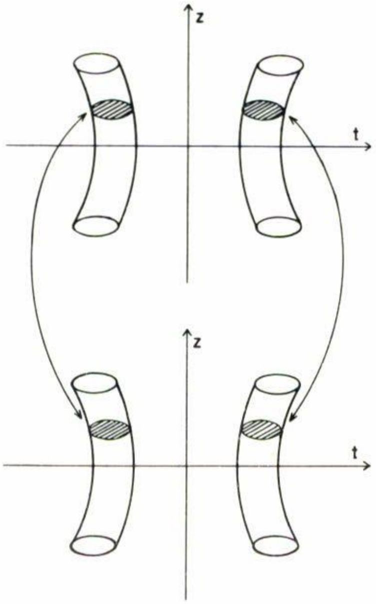

FIGURE 11. Identification of spaces w_{P-D}.

in this case on the plane t–z we have four regions determined by the asymptotes of the hyperbola $z^2 - t^2 = b^2$ (Fig. 12); as we have considered x and y constant, the metric is now $ds^2 = dt^2$ and for light rays we have

$$dz = \pm dt.$$

Then if in the last diagram the geodesic with increasing t is inserted, one finds the diagram shown in Fig. 13, where all rays leaving the region I arrive at region II or at region III, and rays leaving the regions II and III arrive at region IV. The last observation suggests the interpretation that regions like a white-black hole, from the point of view of

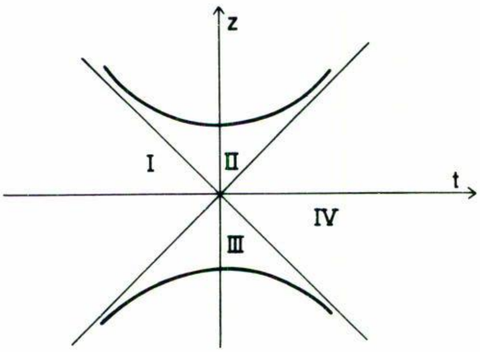

FIGURE 12. The w_{P-D} in the limit case.

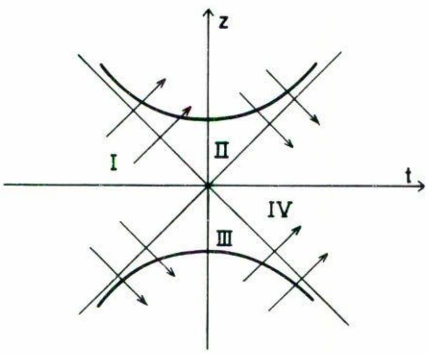

FIGURE 13. Geodesics in the limit case.

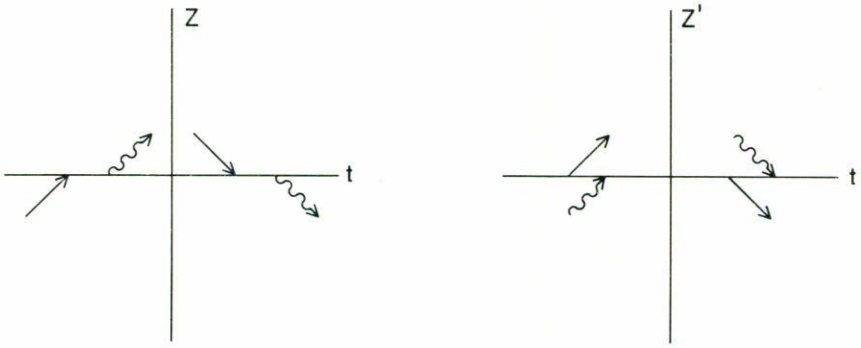

FIGURE 14. Geodesic $\nearrow$ arrives at $z = 0$ to emerge on the other diagram, as the wavy geodesic does.

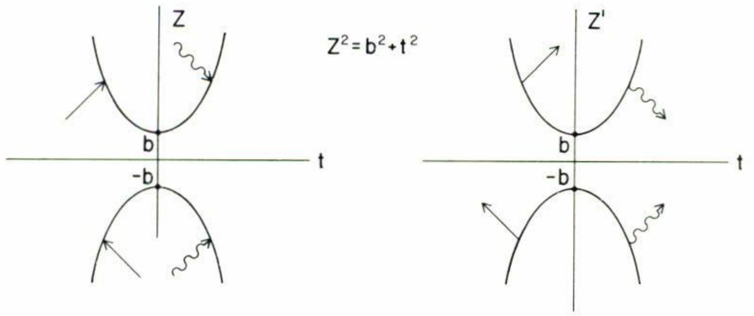

FIGURE 15. Geodesic ↗ arrives at $z^2 = b^2 + t^2$ to emerge on the other diagram, as the wavy geodesic does.

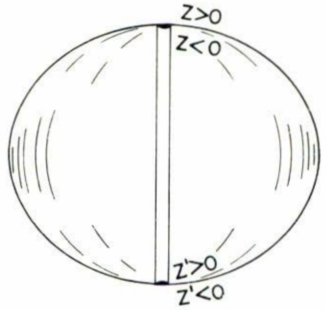

FIGURE 16. A possible whole space time.

an observer in universe I are regions having the behavior of a black hole, but from the point of view of an observer in region IV they have the behaviour of white holes.

5. DISCUSSION

In Fig. 14 it is schematically shown how the two spaces $\mathbf{R}^3$ are joined to build the needed space in order to have a single valued 2-form w_K. One sees that when an observer travelling in the upper part of the left space approaches the circle $\rho^2 = a^2$, he suddenly goes off through the other dimension into the lower part of the right space. The same phenomenon occurs with an observer travelling in the lower part of the left space approaching the region $\rho^2 < a^2$; he appears in the upper part of the right space. If we drop the x and y coordinates making them constant but such that $x^2 + y^2 < a^2$, the light rays arriving at the $z = 0$ axis will suddenly disappear, to emerge in other z–t diagram as shown in Figs. 15 and 16. The solution of Eq. (1) gives us a local behavior of the topology of the whole space. Nevertheless one could think about many possibilities of the whole space with the same local behavior. Of course one can imagine a whole space with two $\mathbf{R}^3$ and only one time (or two times in the second case) connected by a "worm" but without any other communication between them, so that it is possible to think in a torus-like topology where the two spaces could be joined as it is shown in Fig. 16. The worm is a way of communicating between the $R^3 - R^3$ spaces corresponding to an extra dimension. The external surfaces of the torus correspond to the $z > 0$, $z' < 0$ surfaces, and the internal surfaces to the $z < 0$, $z' > 0$ ones. This whole space was more in agreement with a universe without big bang. For the second case one had an analogous analysis but with the singularity in $z^2 = b^2 + t^2$.

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