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On the motion of the symmetric Lagrange's top

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ABSTRACT. This is a study on some aspects of the motion of a symmetric rigid body in the field of constant gravity. The emphasis we place is on those neglected features which remain unstressed in courses and treatises on the dynamics of the spinning top. The motion is naturally separated into two movements: a uniform rotation around the figure axis and a motion like that of a symmetric body having three equal inertia moments. The nutation motion is regarded as the model equivalent of a particle moving in a cubic potential well. Behavior near the bottom and the maximum of the potential are briefly considered. The Jacobi's theorem equating the rotation matrix of the top as the product of two rotation matrices of free torque asymmetric tops has been computed with care to resolve some contradictions which are found in the literature.

RESUMEN. Este trabajo constituye un estudio de diversas facetas del movimiento de un cuerpo rígido simétrico en el campo de gravedad constante. Se puso mayor énfasis en aquellas cuestiones olvidadas en los textos que tratan de la dinámica del trompo. En el movimiento de este cuerpo se distinguen dos componentes: una rotación uniforme alrededor del eje del trompo y un movimiento correspondiente a un cuerpo completamente simétrico, con tres momentos de inercia iguales. El movimiento de nutación es matemáticamente equivalente al de una partícula que se mueve en un potencial cúbico. El comportamiento en las cercanías del mínimo y del máximo del potencial también se describieron someramente. Se calculó con cuidado el teorema de Jacobi que iguala la matriz de rotación del trompo al producto de dos matrices de rotación de trompos asimétricos sin torcas, con el objeto de resolver algunas contradicciones que se encuentran en la literatura.

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1. INTRODUCTION

The mathematical description of the motion of a symmetrical top in the field of constant gravity is one of the few solved problems of rigid body dynamics, the solution to which was first obtained by Lagrange [1], and published in 1788. This problem is included in many advanced treatises on classical mechanics [2,3,4].

The more general motion of an asymmetric body with a fixed point in the constant gravitational field, however remains unraveled and is in Zihlin sense [5] not integrable. A numerical integration of it has been realized by Galgani *et al.* and by Chavoya *et al.* [6] and a bifurcation analysis of it has been described by Tatarinov [7].

Apart from the Lagrange symmetrical case, only the Euler's free torque top, and the Kovalevskaya top have a sufficient enough number of global constants of motion to be

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integrated [8] with no restriction on the values of the energy or of the angular momentum. Some other particular cases have been solved, all of which impose dynamic restrictions on the initial conditions.

The foregoing considerations reveal the importance of the particular case, called the Lagrange's top, an old subject mentioned in many places in the classical literature on mechanics.

Because it requires the use of unfamiliar elliptic functions of time, included only in selective treatments of the subject, the complete solution is found only in advanced treatises. A new glance at this subject however could well be justified when one finds that several of its dynamic aspects are either ignored or neglected in the usual treatment of the best known textbooks [2,3,4].

Our main objectives in this paper will be the following:

1 Reformulation of the problem into a more symmetric equivalent of three (instead of two) equal principal inertia moments [9].

2 Demonstration of the equivalence of the nutation motion with that of the dynamics of a particle in a cubic potential well. The consideration of vibrations near the minimum of this potential is obvious. Also interesting to note the possibility of a non periodic solution.

3 Understanding the relevance of the Jacobi's theorem in expressing the rotation matrix of the Lagrange top as the product of the matrices of two free torque asymmetric tops.

I feel the need for a complete treatment of the Jacobi's theorem exist based not only on the fact that many well known treatises on the subject barely mention it, but that in a partial treatment of the problem published in 1982 by Yamada and Shieh [10] a paradoxical result comes about: if one substitutes the Yamada and Shieh's expressions for the physical constants of the two free asymmetric tops into the total angular velocity, it becomes zero, *i.e.*, the Lagrange top does not move. This unexpected result comes about from an erroneous change of sign in four basic constants in the Yamada and Shieh's paper, which will be corrected in Sect. 6 of this paper.

Unless one seeks a broader treatment of this dynamic theorem and not a portion of it as Yamada and Shieh did, one will not have the proper prescription for finding the correct sign.

Actually this Jacobi theorem, published posthumously in 1882 has deserved mention in many publications. Leimanis [8] included at least 14 references to it, and many others are to be found in new editions of old treatises reprinted by Dover [11]. A new difficulty is that most of these treatises and publications were written in what is now obsolete mathematical jargon. Such writing describe the motion using notations and geometrical constructions which are nowadays neither remembered nor familiar. A new approach to the subject. I believe, could spark new interest in teachers, scholars and researchers. The subject as is found in most textbooks and treatises appears as fragments and many different sources must be consulted before one finds material enough to fully grasp the dynamics early described by unwieldy elliptic functions then replaced by the less cumbersome geometrical constructions which now can again be described by mathematical elliptic functions, easily computed and plotted with present day computers.

Furthermore, the matrix treatment of the Jacobi theorem in this paper produces new results relating the herpolodes of the free bodies with the angular velocities of the Lagrange's symmetric top.

This study could be also useful for the treatment of the general nonsymmetric case, following a perturbation approach [6].

It could be unreasonable to claim great originality in considering the dynamics of the top. Suffice is to say that most readers, I believe, will find in this paper diverse new results presented in a somewhat unordinary yet compact manner, that may seem interesting and practical if one wishes to know how fast the symmetric top moves.

The description of the motion of the Lagrange top is usually given in terms of Euler angles, where ψ is the angle of rotation around the symmetry axis of the top, φ, the procession angle around the vertical gravity, and θ, the angle between these two directions.

The Lagrangian of the system in these coordinates is

$$\mathcal{L} = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - Q\cos\theta, \tag{1.1}$$

where I_1 is the value of the two symmetrical principal inertia momenta and I_3 is the inertia moment with respect to the symmetry axis. Q is equal to the product of the weight of the top times the distance from the mass center to the fixed point. And a dot above a symbol denotes its time derivative.

The corresponding Hamiltonian is easily obtained as

$$\mathcal{H} = \frac{1}{2I_1}\left[p_\theta^2 + \frac{1}{\sin^2\theta}(p_\varphi - p_\psi\cos\theta)^2\right] + \frac{1}{2I_3}p_\psi^2 + Q\cos\theta, \tag{1.2}$$

where p_φ, p_θ, p_ψ are the canonical momenta in Euler coordinates.

Solution comes forthwith observing from the Lagrangian or the Hamiltonian that Euler angles φ and ψ are cyclic coordinates, and that the corresponding momenta p_φ, and p_ψ are constants of motion. Energy is also conserved and the problem is then reduced to quadratures [1].

One finds the solution in terms of

$$z = \cos\theta, \tag{1.3}$$

$$\dot{z}^2 = -\frac{2Q}{I_1}(1-z^2)z + \left(\frac{2E}{I_1} - \frac{p_\psi^2}{I_1 I_3}\right)(1-z^2) - \left(\frac{p_\varphi}{I_1} - \frac{p_\psi}{I_1}z\right)^2, \tag{1.4}$$

where E is the constant value of the Hamiltonian H, and derivatives of the cyclic variables are functions of z:

$$\dot{\varphi} = \frac{1}{I_1}\frac{p_\varphi - zp_\psi}{1-z^2}, \tag{1.5}$$

$$\dot{\psi} = \frac{1}{I_1}\frac{-p_\varphi + zp_\psi}{1-z^2}z + \frac{1}{I_3}p_\psi. \tag{1.6}$$

2. Dynamic equivalence with a spherical top and with a particle in a cubic potential

The Lagrange top has been shown to have a dynamic equivalence with the top having spherical symmetry when the three principal momenta of inertia have the same value. This symmetry represents an important simplification and an increase in the symmetry of the problem which is evident throughout this study. One finds reference to it in Whittaker treatise [2], it is not taken into account however in other important books on the subject. It was briefly mentioned by Piña [9] in a novel presentation of parameterizing the rotation matrix.

It is convenient to transform angle ψ by subtracting from it a rotation around the symmetry axis of the top with constant angular velocity, to obtain the new angle

$$\sigma = \psi - tp_\psi\left(\frac{1}{I_3} - \frac{1}{I_1}\right). \tag{2.1}$$

This change is brought about by a time dependent canonical transformation generated by the function

$$F_3(p_\psi, p_\theta, p_\varphi; \sigma, \Theta, \Phi; t) = -\frac{1}{2}tp_\psi^2\left(\frac{1}{I_3} - \frac{1}{I_1}\right) - p_\psi\sigma - p_\varphi\Phi - p_\theta\Theta. \tag{2.2}$$

Capital letters for φ and θ in this transformation are reminders that both should be considered as new variables despite the fact they have not themselves changed. We use F_3 to generate the transformation

$$\varphi = -\frac{\partial F_3}{\partial p_\varphi} = \Phi, \qquad \theta = -\frac{\partial F_3}{\partial p_\theta} = \Theta. \tag{2.3}$$

The angle ψ satisfies (2.1) according to

$$\psi = -\frac{\partial F_3}{\partial p_\psi}. \tag{2.4}$$

The canonical momenta become the same as

$$p_\Phi = -\frac{\partial F_3}{\partial \Phi} = p_\varphi, \quad p_\Theta = -\frac{\partial F_3}{\partial \Theta} = p_\theta, \quad p_\sigma = -\frac{\partial F_3}{\partial \sigma} = p_\psi. \tag{2.5}$$

In the following we will conserve the lower case notation for θ and φ.

The time dependence of the transformation produces a new Hamiltonian which is also a constant of motion:

$$K = H + \frac{\partial F_3}{\partial t} = \frac{1}{2I_1}\left[p_\theta^2 + \frac{1}{\sin^2\theta}\left(p_\varphi^2 - 2p_\varphi p_\sigma\cos\theta + p_\sigma^2\right)\right] + Q\cos\theta, \tag{2.6}$$

with two very important results.

The new Hamiltonian K is not a function of the inertia moment I_3. The only effect of I_3 in the dynamics is through this canonical transformation associated to a constant angular velocity of magnitude

$$p_\psi \left(\frac{1}{I_3} - \frac{1}{I_1} \right) \tag{2.7}$$

around the symmetry axis. And which modifies the energy with a term equal to

$$E - D = \frac{1}{2} p_\psi^2 \left(\frac{1}{I_3} - \frac{1}{I_1} \right), \tag{2.8}$$

where D is the constant value of the new Hamiltonian K.

The second important fact is that angles φ and σ behave symmetrically in the new Hamiltonian (2.6) which becomes invariant with respect to interchange of the canonical momenta p_φ and p_σ.

The coordinates can be integrated from the system

$$\dot{z}^2 = \frac{2D}{I_1}(1 - z^2) - \frac{2Q}{I_1}(1 - z^2)z - \left(\frac{p_\varphi^2}{I_1^2} - \frac{2 p_\varphi p_\sigma z}{I_1^2} - \frac{p_\sigma^2}{I_1^2} \right), \tag{2.9}$$

$$\dot{\varphi} = \frac{1}{I_1} \frac{p_\varphi - p_\sigma z}{1 - z^2}, \tag{2.10}$$

$$\dot{\sigma} = \frac{1}{I_1} \frac{p_\sigma - p_\varphi z}{1 - z^2}, \tag{2.11}$$

to be compared with the less symmetric expressions (1.4–6). Note that to a given z, solution to the first equation, there are two different possible solutions that correspond to the interchange of p_φ with p_σ (and φ with σ). This symmetry explains the double behavior of the top motion referred to as the fast and the slow tops in Goldstein's [3] treatment of this problem.

The z motion is generally analyzed [2,3,4] by considering the properties of the polynomial which is equal to the square of the z-velocity in Eqs. (1.4) or (2.9).

To improve our pedagogical approach to this problem let us write the differential Eqs. (2.9–11) in terms of a dimensionless time

$$\tau = t/T, \tag{2.12}$$

measured in terms of the time unit T such that

$$T^2 = \frac{I_1}{2Q} \tag{2.13}$$

and introduce the dimensionless constants

$$a = T\frac{p_\sigma}{I_1}, \qquad b = T\frac{p_\varphi}{I_1}, \qquad c = \frac{D}{Q}. \tag{2.14}$$

Eq. (2.9) becomes

$$\frac{1}{2}\left(\frac{dz}{d\tau}\right)^2 - \frac{1}{2}\left[(1-z^2)(c-z) + 2zab - a^2 - b^2\right] = 0, \tag{2.15}$$

which can be considered as the energy equation of a particle moving along coordinate z in the cubic potential well

$$V(z) = -\tfrac{1}{2}\left[(1-z^2)(c-z) + 2zab - a^2 - b^2\right]. \tag{2.16}$$

Eqs. (2.10) and (2.11) are also simplified into

$$\begin{aligned} \frac{d(\sigma+\varphi)}{d\tau} &= \frac{a+b}{1+z}, \\ \frac{d(\sigma-\varphi)}{d\tau} &= \frac{a-b}{1-z}, \end{aligned} \tag{2.17}$$

which are integrated after z is known as a function of τ.

3. THE STUDY OF THE CUBIC POTENTIAL

The cubic polynomial in Eq. (2.16) is used but with a different sign in several textbooks of mechanics [3,4,11] to analyze the dynamics of the symmetric top.

In this section a similar analysis is made but by regardıng the polynomial to be considered as a cubic potential with a different sign, and Eq. (2.15) as the energy equation of a particle of unit mass moving in that cubic well with zero energy.

A periodic motion results in the variable z, which being a cosine should restrict the values of this variable to the interval $(-1,1)$. Actually motion can exist only for negative values of the potential energy in order to give a sum of zero to the total energy when added to the positive kinetic energy. That periodic motion is bound by two roots of the potential at which points the top stops and returns. See Fig. 1.

These two zeros of the potential are designated by sub indexes 2 and 3, while subindex 1 is reserved for the root outside the range of motion. In terms of its three real roots the potential is written as

$$V(z) = -\tfrac{1}{2}(z-z_1)(z-z_2)(z-z_3) \tag{3.1}$$

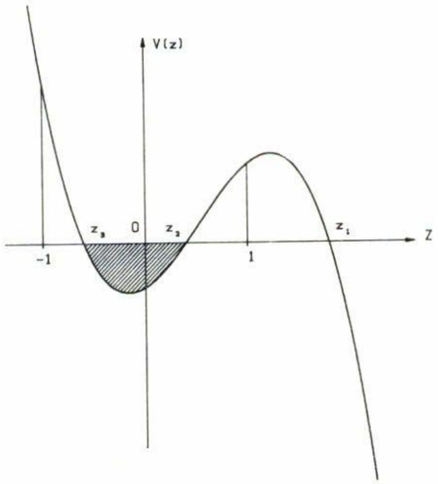

FIGURE 1. The cubic potential and the region of motion between z_2 and z_3.

comparing the foregoing with expression (2.16) one finds the physical constants in terms of the three roots as follows:

$$c = z_1 + z_2 + z_3, \tag{3.2}$$

$$2ab - 1 = z_1 z_2 + z_2 z_3 + z_3 z_1, \tag{3.3}$$

$$a^2 + b^2 - c = z_1 z_2 z_3. \tag{3.4}$$

Note that z values 1 and -1 are in general forbidden since at these values the potential is normally positive:

$$V(1) = \tfrac{1}{2}(a-b)^2, \tag{3.5}$$

$$V(-1) = \tfrac{1}{2}(a+b)^2. \tag{3.6}$$

As z grows the potential function diminishes limitless:

$$\lim_{z\to\infty} V(z) = -\infty; \tag{3.7}$$

and for higher negative values of z the potential tends to infinity:

$$\lim_{z\to-\infty} V(z) = \infty. \tag{3.8}$$

From these properties of the potential it follows that the third root z_1 should be larger than one, except in the case where the maximum of the potential falls at $z = 1$. In general one has

$$z_1 > 1 > z_2 > z > z_3 > -1. \tag{3.9}$$

In most computations the squares of $(a+b)$ and $(a-b)$ in terms of the roots of the potential should be expressed as

$$(a+b)^2 = (z_1+1)(z_2+1)(z_3+1), \tag{3.10}$$

$$(a-b)^2 = (z_1-1)(z_2-1)(z_3-1). \tag{3.11}$$

4. The Euler coordinates in terms of explicit functions of time

Integration of Eq. (2.15) can be performed by using Jacobi's elliptic functions. Transform to the new variable

$$z = z_3 + (z_2 - z_3)\,\mathrm{sn}^2(u,k), \tag{4.1}$$

where the value of parameter k is selected by

$$k^2 = \frac{z_2 - z_3}{z_1 - z_3}; \tag{4.2}$$

then the solution follows

$$z = z_3 + (z_2 - z_3)\,\mathrm{sn}^2(\alpha\tau,k), \tag{4.3}$$

$$\alpha = \frac{1}{2}\sqrt{z_1 - z_3}.$$

Eqs. (2.17) can now be integrated

$$\frac{d(\sigma+\varphi)}{d\tau} = \frac{a+b}{1+z_3+(z_2-z_3)\,\mathrm{sn}^2(\alpha\tau,k)}, \tag{4.4}$$

$$\frac{d(\sigma-\varphi)}{d\tau} = \frac{a-b}{1-z_3-(z_2-z_3)\,\mathrm{sn}^2(\alpha\tau,k)}. \tag{4.5}$$

It is convenient to introduce two constant parameters γ and λ associated to these two equations such that

$$\begin{aligned}
\mathrm{sn}(\gamma,k') &= \sqrt{\frac{1+z_3}{1+z_2}},\\
\mathrm{cn}(\gamma,k') &= \frac{\sqrt{(z_2-z_3)(1+z_1)(1+z_2)}}{a+b},\\
\mathrm{dn}(\gamma,k') &= \sqrt{\frac{(z_2-z_3)(1+z_1)}{(z_1-z_3)(1+z_2)}}
\end{aligned} \tag{4.6}$$

and

$$\mathrm{sn}(\lambda, k') = \sqrt{\frac{(z_1 - z_3)(1 - z_2)}{(z_1 - z_2)(1 - z_3)}},$$
$$\mathrm{cn}(\lambda, k') = \sqrt{\frac{(z_2 - z_3)(z_1 - 1)}{(z_1 - z_2)(1 - z_3)}}, \tag{4.7}$$
$$\mathrm{dn}(\lambda, k') = \frac{\sqrt{(z_2 - z_3)(z_1 - 1)(1 - z_2)}}{a - b}.$$

The integrals (4.4-5) become

$$\frac{\sigma + \varphi}{2} = \frac{\mathrm{sn}(\gamma, k')\,\mathrm{dn}(\gamma, k')}{\mathrm{cn}(\gamma, k')} \int_0^{\alpha\tau} \frac{du}{1 - \mathrm{cn}^2(\gamma, k')\,\mathrm{cn}^2(u, k)},$$
$$\frac{\sigma - \varphi}{2} = k'^2 \frac{\mathrm{sn}(\lambda, k')\,\mathrm{cn}(\lambda, k')}{\mathrm{dn}(\lambda, k')} \int_0^{\alpha\tau} \frac{du}{1 - \mathrm{dn}^2(\lambda, k')\,\mathrm{sn}^2(u, k)}, \tag{4.8}$$

which are the same elliptic integrals of third kind similar to those found in the Euler case of motion of the free asymmetric top. Compare with Eq. (AI.15) in Appendix I.

Two particular cases deserve singling out from many others for special attention, as they are not found in standard treatments on the dynamics of the symmetric top [2,3,4].

Consider first the case of coincidence of the roots z_3 and z_2 that occurs when the top precedes and rotates with no nutation. Angle θ remains constant at

$$\cos\theta = z_3. \tag{4.9}$$

This value, is the minimum of the potential, and constitutes a stable equilibrium for the nutation motion.

In such cases the right hand terms of the Eqs. (2.17) are constants. When squared these equations give us, according to (3.10) and (3.11),

$$\left(\frac{d(\sigma + \varphi)}{d\tau}\right)^2 = \left(\frac{a + b}{1 + z_3}\right)^2 = z_1 + 1 \tag{4.10}$$

and

$$\left(\frac{d(\sigma - \varphi)}{d\tau}\right)^2 = \left(\frac{a - b}{1 - z_3}\right)^2 = z_1 - 1, \tag{4.11}$$

and adding and subtracting these equations term to term the angular velocities of angles φ and σ are found in the intersection (Fig. 2) of the circle

$$\left(\frac{d\sigma}{d\tau}\right)^2 + \left(\frac{d\varphi}{d\tau}\right)^2 = z_1 \tag{4.12}$$

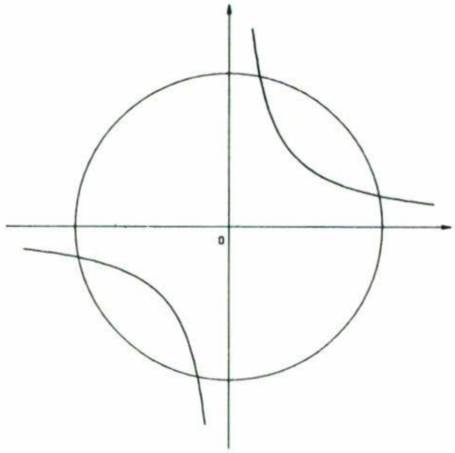

Figure 2. The values of the angular velocities in the intersection of a circle and of an equilateral hyperbola. No nutation case.

and of the equilateral hyperbola

$$\frac{d\sigma}{d\tau}\frac{d\varphi}{d\tau} = \frac{1}{2}. \tag{4.13}$$

Around the minimal value of z small vibrations give rise to a harmonic nutation. The trigonometric approximation of which is valid whenever

$$z_2 - z_3 \ll z_1. \tag{4.14}$$

The other case to be especially considered is one in which the maximum of the potential is itself a root of the selfsame potential, and

$$z_2 = z_1 = 1, \tag{4.15}$$

since in this case the motion ceases to be periodic, parameter k is equal to one, and the elliptic functions become hyperbolic functions. Instead of (4.3) one has

$$z = z_3 + (1 - z_3)\tanh^2(\alpha\tau), \tag{4.16}$$

with

$$\alpha = \tfrac{1}{2}\sqrt{1 - z_3}. \tag{4.17}$$

In this case integration of (2.17) is not obtained in terms of elementary functions although the integral can be decomposed into the sum of a linear function of time and of a variable but finite angle with a known rapidly convergent limit as time tends to infinity.

5. Jacobi's theorem

According to Jacobi's theorem [12] the rotation matrix R of the Lagrange's symmetric top is equal to the product of two rotation matrices R_1 and R_2 of free asymmetric bodies

$$R = R_2 \tilde{R}_1, \tag{5.1}$$

where each rotation is represented by an orthogonal matrix in three dimensions, and where the tilde denotes the transpose matrix.

I will use the notation

$$\boldsymbol{\omega}\times = \tilde{R}\dot{R} \tag{5.2}$$

for an antisymmetric matrix having as vector $\boldsymbol{\omega}$ the components of the angular velocity in the body system, and where a dot above a letter denotes the time derivative. R with a dot above denotes the matrix having as entries the time derivatives of the entries of R. The notation $\boldsymbol{\omega}\times$ for an antisymmetric matrix and its relation to vector $\boldsymbol{\omega}$ is found in Piña [9].

In a similar way to (5.2) we define

$$\boldsymbol{\Omega}\times = \dot{R}\tilde{R}, \tag{5.3}$$

where $\boldsymbol{\Omega}$ is the angular velocity in the inertial system. Both vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are transformed by the rotation matrix

$$\boldsymbol{\Omega} = R\boldsymbol{\omega}. \tag{5.4}$$

Substitution of (5.1) in these equations give us

$$\begin{aligned}\boldsymbol{\omega}\times &= R_1\tilde{R}_2(\dot{R}_2\tilde{R}_1 + R_2\dot{\tilde{R}}_1)\\ &= R_1\boldsymbol{\omega}_2 \times \tilde{R}_1 - \dot{R}_1\tilde{R}_1,\end{aligned}$$

where $\dot{R}_1\tilde{R}_1 + R_1\dot{\tilde{R}}_1 = 0$, which comes from $R_1\tilde{R}_1 = 1$, has been used, and thus

$$\boldsymbol{\omega} = R_1\boldsymbol{\omega}_2 - \boldsymbol{\Omega}_1 \tag{5.5}$$

it follows also from (5.3)

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_2 - R_2\boldsymbol{\omega}_1. \tag{5.6}$$

It has been commonplace to select direction

$$\mathbf{k} = (0\ 0\ 1) \tag{5.7}$$

for the constant direction of the angular momentum vector in the inertial system of the free asymmetric top, and I will do the same for the two equivalent tops. I will use $\mathbf{l}_1$ and $\mathbf{l}_2$ for the angular momentum directions in the body system of the two free tops, namely

$$\mathbf{l}_1 = \tilde{R}_1 \mathbf{k}, \qquad \mathbf{l}_2 = \tilde{R}_2 \mathbf{k}. \tag{5.8}$$

Energy conservation for the two free tops is written in the form

$$\frac{2E_1}{L_1} = \mathbf{l}_1 \cdot \boldsymbol{\omega}_1 = \mathbf{k} \cdot \boldsymbol{\Omega}_1,$$

$$\frac{2E_2}{L_2} = \mathbf{l}_2 \cdot \boldsymbol{\omega}_2 = \mathbf{k} \cdot \boldsymbol{\Omega}_2, \tag{5.9}$$

where E_1, E_2 are the energies and L_1, L_2 the constant magnitudes of the angular momenta of the two free tops.

On the other hand for the symmetric Lagrange top one finds the constants of motion

$$a = \mathbf{k} \cdot \boldsymbol{\omega}, \qquad b = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{5.10}$$

These are the angular velocity components along the figure axis and the force direction, respectively, where we have used the dimensionless time unit of Sect. 2.

Substitution of (5.9-10) into (5.5-6) produces the results

$$\frac{2E_1}{L_1} + a = \mathbf{l}_1 \cdot \boldsymbol{\omega}_2, \qquad \frac{2E_2}{L_2} - b = \mathbf{l}_2 \cdot \boldsymbol{\omega}_1. \tag{5.11}$$

To satisfy Eqs. (5.11) one can assume that $\boldsymbol{\omega}_1$ is a linear combination with constant coefficients of $\mathbf{l}_2$ and $\boldsymbol{\omega}_2$, and also that $\boldsymbol{\omega}_2$ is a linear combination of $\mathbf{l}_1$ and $\boldsymbol{\omega}_1$. But to assume a too general relation could lead us to the unwanted conclusion that $\mathbf{l}_1 \cdot \mathbf{l}_2$ is a constant, which would be in contradiction to the fact that $z = \mathbf{l}_1 \cdot \mathbf{l}_2$ is in general a variable corresponding to $\cos\theta$ of the Lagrange top. From (5.1) one finds

$$z = \cos\theta = \mathbf{k} \cdot R \cdot \mathbf{k} = \mathbf{l}_1 \cdot \mathbf{l}_2. \tag{5.12}$$

Let us assume then that

$$\boldsymbol{\omega}_2 = p\boldsymbol{\omega}_1, \tag{5.13}$$

with p a constant to be determined shortly.

Sustitution of this restriction in (5.5-6) allows us to find

$$\boldsymbol{\omega} = (p-1)\boldsymbol{\Omega}_1, \tag{5.14}$$

and

$$\boldsymbol{\Omega} = (1 - 1/p)\boldsymbol{\Omega}_2 \tag{5.15}$$

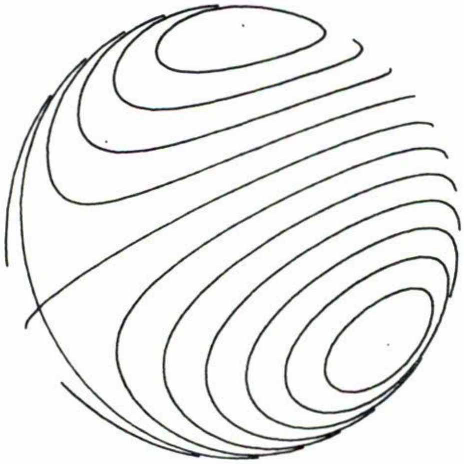

FIGURE 3. Curves described by the angular momentum vector at the intersection of elliptic cones and the sphere of constant angular momentum magnitude. Euler case of motion of an asymmetric top without torques. Using the addition theorem, the elliptic functions were computed by a rational iteration.

obliges one to admit that constant p cannot be equal to one, were such the case there would be no motion taking place since the angular velocity of the symmetric body would be zero. One is thus led to conclude

$$p \neq 1. \tag{5.16}$$

In Appendix II while accepting the proportionality (5.13) for two free tops, we come to the conclusion that $p^2 = 1$ and therefore

$$p = -1, \tag{5.17}$$

which means

$$\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2. \tag{5.18}$$

This important result in (5.11) and (5.9) gives us

$$\frac{2E_1}{L_1} = -\frac{a}{2} \tag{5.19}$$

and

$$\frac{2E_2}{L_2} = \frac{b}{2}. \tag{5.20}$$

Whilst Eq. (5.20) is identical to that of Yamada and Shieh [10], Eq. (5.19) is different in sign from that of these authors.

Eqs. (5.14-15) then become

$$\boldsymbol{\Omega} = 2\boldsymbol{\Omega}_2 \tag{5.21}$$

double amplification of the herpolode of body 2 become the curve that traces the angular velocity in the inertial system of the Lagrange top when projected on a plane orthogonal to the force direction; and

$$\boldsymbol{\omega} = -2\boldsymbol{\Omega}_1 \tag{5.22}$$

double amplification and inversion of the herpolode of body 1 becomes the curve traced by the angular velocity of the Lagrange top in the body system when projected on the plane orthogonal to the figure axis.

These are two important and useful properties.

6. THE TWO FREE TOPS OF THE JACOBI'S THEOREM

In order to write the expressions of the quantities of the two free tops I use the fundamental Eq. (5.18) that imposes a large number of constraints on the relative motion of the two free tops. As in the previous section subindexes 1 and 2 will be used to distinguish one from the other. In Eqs. (AI.5) and (AI.10) of Appendix I, we predict that the variable components along the principal axis of the three vectors $\mathbf{l}_1$, $\mathbf{l}_2$ and $\boldsymbol{\omega}_1$ (or $\boldsymbol{\omega}_2$) will be respectively proportional to the same three elliptic functions $\mathrm{sn}(\alpha\tau, k)$, $\mathrm{dn}(\alpha\tau, k)$, $\mathrm{cn}(\alpha\tau, k)$.

This property and the unit character of vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ according to the properties of the motion in the Euler case, implies the necessity of determining only one μ parameter (see Eq. (AI.12) in Appendix I) for each top in order to know these unit vectors.

However Eqs. (4.3) and (5.12) impose new conditions on the unit vectors that loses all freedom but that of the simultaneous sign of each of its components. Jacobi discovered a simple relation between parameters γ, λ and parameters μ of these vectors which become the sum of and difference between those parameters.

Considering the distinct possibilities for choosing the sign for parameters a and b, one has the following expression for the unit vectors in explicit form

$$\mathbf{l}_1 = \begin{pmatrix} \sqrt{\dfrac{(1-z_3^2)(1-z_2^2)}{z_1-z_2}}\,\dfrac{az_1-b}{a^2-b^2}\,\mathrm{dn}(u,k) \\ \sqrt{\dfrac{(z_1^2-1)(1-z_2^2)}{z_1-z_3}}\,\dfrac{b-az_3}{a^2-b^2}\,\mathrm{sn}(u,k) \\ \sqrt{\dfrac{(z_1^2-1)(1-z_3^2)}{z_1-z_2}}\,\dfrac{b-az_2}{a^2-b^2}\,\mathrm{cn}(u,k) \end{pmatrix} \tag{6.1}$$

and

$$\mathbf{l}_2 = \begin{pmatrix} \sqrt{\dfrac{(1-z_3^2)(1-z_2^2)}{z_1 - z_2}}\, \dfrac{bz_1 - a}{a^2 - b^2}\, \mathrm{dn}(u,k) \\ \sqrt{\dfrac{(z_1^2-1)(1-z_2^2)}{z_1 - z_3}}\, \dfrac{a - bz_3}{a^2 - b^2}\, \mathrm{sn}(u,k) \\ \sqrt{\dfrac{(z_1^2-1)(1-z_3^2)}{z_1 - z_2}}\, \dfrac{a - bz_2}{a^2 - b^2}\, \mathrm{cn}(u,k) \end{pmatrix} \tag{6.2}$$

These expressions correspond to two unit vectors with a scalar product equal to the z variable in (4.1). They could be simplified if a and b were expressed in terms of the three z-roots. However that simplification is possible only after the sign of a and b are known.

The angular velocity of a torque free body is restricted by the condition of having a constant scalar product with the corresponding unit vector $\mathbf{l}$. But in this case the angular velocity should have a constant scalar product with both vectors, the value of that product is predetermined by Eqs. (5.19) and (5.20). At first sight it seems that these are too many conditions to be satisfied. Actually the solution is not overdetermined and one finds

$$\boldsymbol{\omega}_2 = \frac{1}{2} \begin{pmatrix} \dfrac{(a^2-b^2)\,\mathrm{dn}(u,k)}{\sqrt{(z_1-z_2)(1-z_3^2)(1-z_2^2)}} \\ \dfrac{(a^2-b^2)\,\mathrm{sn}(u,k)}{\sqrt{(z_1-z_3)(z_1^2-1)(1-z_2^2)}} \\ \dfrac{(a^2-b^2)\,\mathrm{cn}(u,k)}{\sqrt{(z_1-z_2)(z_1^2-1)(1-z_3^2)}} \end{pmatrix} = -\boldsymbol{\omega}_1. \tag{6.3}$$

The three vectors (6.1-3) are unique up to a simultaneous change of sign of any component of the three vectors.

The inertia moments of the two tops are then determined by the proportionality between the components of $\boldsymbol{\omega}$ and $\mathbf{l}$. One finds

$$\begin{aligned} \frac{L_1}{A_1} &= -\frac{b - az_1}{2(c - a^2 - z_1)}, & \frac{L_2}{A_2} &= \frac{a - bz_1}{2(c - b^2 - z_1)}, \\ \frac{L_1}{B_1} &= -\frac{b - az_3}{2(c - a^2 - z_3)}, & \frac{L_2}{B_2} &= \frac{a - bz_3}{2(c - b^2 - z_3)}, \\ \frac{L_1}{C_1} &= -\frac{b - az_2}{2(c - a^2 - z_2)}, & \frac{L_2}{C_2} &= \frac{a - bz_2}{2(c - b^2 - z_2)}, \\ \frac{2E_1}{L_1} &= -\frac{a}{2}, & \frac{2E_2}{L_2} &= \frac{b}{2}. \end{aligned} \tag{6.4}$$

These quantities were computed previously by Yamada and Shieh [10] with a different sign in the four quantities of the first column.

The present selection of signs is of paramount importance to satisfy the Jacobi theorem and avoid the absurdity of being devoid of motion.

On the other hand the main arguments in Yamada and Shieh remain valid and are very important for the Jacobi theorem. In particular the argument of these authors about the physical possible motion of the two auxiliary tops is still valid after our change of sign in quantities of the first column of (6.4).

Appendix I. The asymmetric free top without torques

In this Appendix I will mention some properties of the dynamics of motion of the asymmetric top. (See for example Piña [9]).

The Euler equation of motion for an asymmetric top is

$$\dot{\mathbf{l}} = L\mathbf{l} \times (I^{-1}\mathbf{l}), \tag{AI.1}$$

where $\mathbf{l}$ is the unit direction along the angular momentum in the body frame, L the angular momentum magnitude and I the inertia matrix in the principal inertia moment frame,

$$I^{-1} = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & 1/B & 0 \\ 0 & 0 & 1/C \end{pmatrix}, \tag{AI.2}$$

where A, B, C are the principal inertia moments.

To be concrete let us assume

$$L/A > 2E/L > L/B > L/C, \tag{AI.3}$$

where E is the total energy (purely kinetic) of the free top.

$$\frac{2E}{L} = L\mathbf{l} \cdot I^{-1} \cdot \mathbf{l}. \tag{AI.4}$$

One has the solution

$$l_x = \sqrt{\frac{\dfrac{2E}{L} - \dfrac{L}{C}}{\dfrac{L}{A} - \dfrac{L}{C}}}\, \mathrm{dn}(\alpha t, k),$$

$$l_y = \sqrt{\frac{\dfrac{L}{A} - \dfrac{2E}{L}}{\dfrac{L}{A} - \dfrac{L}{B}}}\, \mathrm{sn}(\alpha t, k), \tag{AI.5}$$

$$l_z = \sqrt{\frac{\dfrac{L}{A} - \dfrac{2E}{L}}{\dfrac{L}{A} - \dfrac{L}{C}}}\,\mathrm{cn}(\alpha t, k),$$

where l_x, l_y, l_z are the components of $\mathbf{l}$ and sn, cn, dn are Jacobi elliptic functions, α and k are two constants defined by

$$\alpha^2 = \left(\frac{2E}{L} - \frac{L}{C}\right)\left(\frac{L}{A} - \frac{L}{B}\right) \tag{AI.6}$$

and

$$k^2 = \frac{\left(\dfrac{L}{B} - \dfrac{L}{C}\right)\left(\dfrac{L}{A} - \dfrac{2E}{L}\right)}{\left(\dfrac{2E}{L} - \dfrac{L}{C}\right)\left(\dfrac{L}{A} - \dfrac{L}{B}\right)}. \tag{AI.7}$$

Computation of functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ is performed [14] using the arithmetic-geometric mean associated to Landen and Gauss' names.

Vector $\mathbf{l}$ determines two Euler angles according to

$$\mathbf{l} = \begin{pmatrix} \sin\theta \sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix} = R\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = R\mathbf{k}, \tag{AI.8}$$

where $\mathbf{k}$ is the unit vector (5.7). Note that the vector $\mathbf{l}$ is a function only of products or ratios of differences of the four quantities $L/A, L/B, L/C$, and $2E/L$. Therefore a change of sign in these four quantities, were it physical, would not affect that vector $\mathbf{l}$. The third Euler angle satisfies the differential equation

$$\dot{\varphi} = \frac{\mathbf{l}\cdot\boldsymbol{\omega} - \mathbf{k}\cdot\mathbf{l}\,\mathbf{k}\cdot\boldsymbol{\omega}}{1 - (\mathbf{k}\cdot\mathbf{l})^2}, \tag{AI.9}$$

where $\boldsymbol{\omega}$ is the angular velocity vector in the body reference system:

$$\boldsymbol{\omega} = LI^{-1}\mathbf{l}. \tag{AI.10}$$

One finds

$$\frac{1}{\alpha}\dot{\varphi} = \frac{\dfrac{2E}{L} - \dfrac{L}{C}}{\sqrt{\left(\dfrac{2E}{L} - \dfrac{L}{C}\right)\left(\dfrac{L}{A} - \dfrac{L}{B}\right)\left(1 - \dfrac{\frac{L}{A} - \frac{2E}{L}}{\frac{L}{A} - \frac{L}{C}}\,\mathrm{cn}^2(\alpha t, k)\right)}} + \frac{L}{\alpha C}. \tag{AI.11}$$

It is convenient to write conical coordinates [13] for vector $\mathbf{l}$:

$$\mathbf{l} = \begin{pmatrix} \mathrm{sn}(\mu, k')\,\mathrm{dn}(\alpha t, k) \\ \mathrm{dn}(\mu, k')\,\mathrm{sn}(\alpha t, k) \\ \mathrm{cn}(\mu, k')\,\mathrm{cn}(\alpha t, k) \end{pmatrix}; \tag{AI.12}$$

then $\mathbf{l}$ moves on the unit sphere intersected by the elliptic cone $\mu =$ constant. Parameter μ plays an essential role in the complete integration of the remaining third Euler angle φ. Comparing the previous expressions one obtains the three equations

$$\begin{aligned} \mathrm{sn}^2(\mu, k') &= \frac{\dfrac{2E}{L} - \dfrac{L}{C}}{\dfrac{L}{A} - \dfrac{L}{C}}, \\ \mathrm{dn}^2(\mu, k') &= \frac{\dfrac{L}{A} - \dfrac{2E}{L}}{\dfrac{L}{A} - \dfrac{L}{B}}, \\ \mathrm{cn}^2(\mu, k') &= \frac{\dfrac{L}{A} - \dfrac{2E}{L}}{\dfrac{L}{A} - \dfrac{L}{C}}, \end{aligned} \tag{AI.13}$$

with

$$(k')^2 = 1 - k^2 = \frac{\left(\dfrac{L}{A} - \dfrac{L}{C}\right)\left(\dfrac{2E}{L} - \dfrac{L}{B}\right)}{\left(\dfrac{L}{A} - \dfrac{L}{B}\right)\left(\dfrac{2E}{L} - \dfrac{L}{C}\right)}. \tag{AI.14}$$

Eq. (AI.11) is then written as

$$\frac{1}{\alpha}\dot{\varphi} = \frac{\mathrm{sn}(\mu, k')\,\mathrm{dn}(\mu, k')}{\mathrm{cn}(\mu, k')} \frac{1}{1 - \mathrm{cn}^2(\mu, k')\,\mathrm{cn}^2(\alpha t, k)} + \frac{L}{\alpha C}, \tag{AI.15}$$

which is integrated in terms of Jacobi's Θ and H functions [11]

$$\varphi = \frac{1}{i}\left[\frac{1}{2}\ln\frac{\Theta(u - i\mu + iK')}{\Theta(u + i\mu - iK')} + \alpha t \frac{H'(i\mu - iK')}{H(i\mu - iK')}\right] + \frac{Lt}{C}, \tag{AI.16}$$

where K' denotes the complete elliptic integral of first kind of argument k', $K' = K(k')$.

The Jacobi's theta functions (Θ and H) are easily computed numerically using its Fourier series which are rapidly convergent functions of the number of harmonics [11].

The imaginary exponential of (AI.16) can be written by using properties of elliptic functions [11] as the product of two terms

$$\begin{aligned}
\exp(i\varphi) &= \exp(i\varphi_1)\exp(i\varphi_2), \\
\exp(i\varphi_1) &= \exp\left(iL\frac{t}{C} + \alpha t\frac{H'(i\mu - iK')}{H(i\mu - iK')}\right), \\
\exp(i\varphi_2) &= \frac{\Theta(0)\Theta(u - i\mu - iK')}{\Theta(i\mu - iK')\Theta(u)\sqrt{1 - k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(i\mu - iK')}}.
\end{aligned} \tag{AI.17}$$

The rotation matrix is then determined by the three Euler angles, however it can be written in an explicit form. The angular momentum direction in (AI.5) provides three known components that form the third row of the rotation matrix. The other two rows of this matrix form the real and imaginary parts of vector

$$\mathbf{V} = \exp(i\varphi)\begin{pmatrix} \cos\psi + i\cos\theta\sin\psi \\ -\sin\psi + i\cos\theta\cos\psi \\ -i\sin\theta \end{pmatrix}, \tag{AI.18}$$

which is written in terms of the unit vector $\mathbf{l}$ as

$$\mathbf{V} = \exp(i\varphi)\frac{1}{\sqrt{(1 - \mathbf{k}\cdot\mathbf{l})^2}}\big[\mathbf{l}\times\mathbf{k} - i(\mathbf{k} - \mathbf{k}\cdot\mathbf{l}\,\mathbf{l})\big]. \tag{AI.19}$$

Substitution of expressions for $\mathbf{l}$ and $\exp(i\varphi)$ in these equations allows one to express that complex vector in the form

$$\begin{aligned}
\mathbf{V} &= \frac{\exp(i\varphi_1)}{\mathrm{dn}(i\mu - iK')}\frac{\Theta(0)\Theta(u - i\mu + iK')}{\Theta(u)\Theta(i\mu - iK')} \\
&\quad\times\big[\mathbf{i}k\,\mathrm{sn}(u - i\mu + iK') + \mathbf{j}\,\mathrm{dn}(u - i\mu + iK') - i\mathbf{k}\big]
\end{aligned} \tag{AI.20}$$

and from this vector comes the explicit expression for the herpolhode

$$\begin{aligned}
\Omega_x + i\Omega_y = \mathbf{V}\cdot\omega &= \delta\frac{\exp(i\varphi_1)}{\mathrm{dn}(i\mu - iK')}\frac{\Theta(0)\Theta(u - i\mu + iK')}{\Theta(u)\Theta(i\mu - iK')} \\
&\quad\times\left[-i\,\mathrm{cn}(u - i\mu + iK')\right],
\end{aligned} \tag{AI.21}$$

where δ is the constant

$$\delta = \sqrt{\left(\frac{L}{B} - \frac{L}{C}\right)\left(\frac{L}{A} - \frac{2E}{L}\right)}. \tag{AI.22}$$

Appendix II. Proof that proportional angular velocities of the Euler asymmetric free bodies actually have the same magnitude

In this appendix, the expressions of Appendix I are used to prove $p^2 = 1$ when the angular velocities of two free tops, like those described in the previous appendix, are proportional

$$\boldsymbol{\omega}_2 = p\boldsymbol{\omega}_1. \tag{5.13}$$

The α and k parameters of both angular velocities should be the same then

$$\left(\frac{2E_1}{L_1}-\frac{L_1}{C_1}\right)\left(\frac{L_1}{A_1}-\frac{L_1}{B_1}\right)=\left(\frac{2E_2}{L_2}-\frac{L_2}{C_2}\right)\left(\frac{L_2}{A_2}-\frac{L_2}{B_2}\right) \tag{AII.1}$$

and

$$\frac{\left(\frac{L_1}{A_1}-\frac{2E_1}{L_1}\right)\left(\frac{L_1}{B_1}-\frac{L_1}{C_1}\right)}{\left(\frac{2E_1}{L_1}-\frac{L_1}{C_1}\right)\left(\frac{L_1}{A_1}-\frac{L_1}{B_1}\right)}=\frac{\left(\frac{L_2}{A_2}-\frac{2E_2}{L_2}\right)\left(\frac{L_2}{B_2}-\frac{L_2}{C_2}\right)}{\left(\frac{2E_2}{L_2}-\frac{L_2}{C_2}\right)\left(\frac{L_2}{A_2}-\frac{L_2}{B_2}\right)}. \tag{AII.2}$$

Dividing (AII.2) by (AII.1)

$$\left(\frac{L_1}{A_1}-\frac{2E_1}{L_1}\right)\left(\frac{L_1}{B_1}-\frac{L_1}{C_1}\right)=\left(\frac{L_2}{A_2}-\frac{2E_2}{L_2}\right)\left(\frac{L_2}{B_2}-\frac{L_2}{C_2}\right) \tag{AII.3}$$

and subtracting Eqs. (AII.3) from (AII.1) one finds

$$\left(\frac{2E_1}{L_1}-\frac{L_1}{B_1}\right)\left(\frac{L_1}{A_1}-\frac{L_1}{C_1}\right)=\left(\frac{2E_2}{L_2}-\frac{L_2}{B_2}\right)\left(\frac{L_2}{A_2}-\frac{L_2}{C_2}\right). \tag{AII.4}$$

The proportionality between the two angular velocities gives three other equations for the constant coefficients that multiply the elliptic functions

$$\left(\frac{L_2}{A_2}\right)^2\frac{\frac{2E_2}{L_2}-\frac{L_2}{C_2}}{\frac{L_2}{A_2}-\frac{L_2}{C_2}}=p^2\left(\frac{L_1}{A_1}\right)^2\frac{\frac{2E_1}{L_1}-\frac{L_1}{C_1}}{\frac{L_1}{A_1}-\frac{L_1}{C_1}}, \tag{AII.5}$$

$$\left(\frac{L_2}{B_2}\right)^2\frac{\frac{L_2}{A_2}-\frac{2E_2}{L_2}}{\frac{L_2}{A_2}-\frac{L_2}{B_2}}=p^2\left(\frac{L_1}{B_1}\right)^2\frac{\frac{L_1}{A_1}-\frac{2E_1}{L_1}}{\frac{L_1}{A_1}-\frac{L_1}{B_1}}, \tag{AII.6}$$

$$\left(\frac{L_2}{C_2}\right)^2\frac{\frac{L_2}{A_2}-\frac{2E_2}{L_2}}{\frac{L_2}{A_2}-\frac{L_2}{C_2}}=p^2\left(\frac{L_1}{C_1}\right)^2\frac{\frac{L_1}{A_1}-\frac{2E_1}{L_1}}{\frac{L_1}{A_1}-\frac{L_1}{C_1}}. \tag{AII.7}$$

Adding term to term Eqs. (AII.5) and (AII.7) gives

$$\frac{2E_2}{L_2}\left(\frac{L_2}{A_2}+\frac{L_2}{C_2}\right)-\frac{L_2^2}{A_2C_2}=p^2\left[\frac{2E_1}{L_1}\left(\frac{L_1}{A_1}+\frac{L_1}{C_1}\right)-\frac{L_1^2}{A_1C_1}\right]. \tag{AII.8}$$

Multiplying term to term Eqs. (AII.4) and (AII.5) and dividing by (AII.1)

$$\left(\frac{L_2}{A_2}\right)^2\frac{\dfrac{2E_2}{L_2}-\dfrac{L_2}{B_2}}{\dfrac{L_2}{A_2}-\dfrac{L_2}{B_2}}=p^2\left(\frac{L_1}{A_1}\right)^2\frac{\dfrac{2E_1}{L_1}-\dfrac{L_1}{B_1}}{\dfrac{L_1}{A_1}-\dfrac{L_1}{B_1}}. \tag{AII.9}$$

Adding term to term Eqs. (AII.9) to (AII.6) gives

$$\frac{2E_2}{L_2}\left(\frac{L_2}{A_2}+\frac{L_2}{B_2}\right)-\frac{L_2^2}{A_2B_2}=p^2\left[\frac{2E_1}{L_1}\left(\frac{L_1}{A_1}+\frac{L_1}{B_1}\right)-\frac{L_1^2}{A_1B_1}\right]. \tag{AII.10}$$

And subtracting term to term (AII.10) from (AII.8) gives

$$\left(\frac{L_2}{A_2}-\frac{2E_2}{L_2}\right)\left(\frac{L_2}{B_2}-\frac{L_2}{C_2}\right)=p^2\left(\frac{L_1}{A_1}-\frac{2E_1}{L_1}\right)\left(\frac{L_1}{B_1}-\frac{L_1}{C_1}\right), \tag{AII.11}$$

which comparing with (AII.3) proofs

$$p^2=1 \qquad \text{and} \qquad p=-1 \tag{AII.12}$$

taking in account (5.16).

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