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Solution of the Helmholtz equation for spin-2 fields

G.F. TORRES DEL CASTILLO
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla
72000 Puebla, Pue., México

AND

J.E. ROJAS MARCIAL
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152, Puebla, Pue., México

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ABSTRACT. The Helmholtz equation for symmetric, traceless, second-rank tensor fields in three-dimensional flat space is solved in spherical and cylindrical coordinates by separation of variables making use of the corresponding spin-weighted harmonics. It is shown that any symmetric, traceless, divergenceless second-rank tensor field that satisfies the Helmholtz equation can be expressed in terms of two scalar potentials that satisfy the Helmholtz equation. Two such expressions are given, which are adapted to the spherical or cylindrical coordinates. The application to the linearized Einstein theory is discussed.

RESUMEN. La ecuación de Helmholtz para campos tensoriales simétricos, sin traza, de rango dos en espacio plano tridimensional se resuelve en coordenadas esféricas y cilíndricas por separación de variables usando los armónicos con peso de espín correspondientes. Se muestra que cualquier campo tensorial simétrico, sin traza y sin divergencia de rango dos que satisfaga la ecuación de Helmholtz puede expresarse en términos de dos potenciales escalares que satisfacen la ecuación de Helmholtz. Se dan dos de tales expresiones, las cuales están adaptadas a las coordenadas esféricas o cilíndricas. Se discute la aplicación a la teoría de Einstein linealizada.

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1. INTRODUCTION

The gravitational field in the linearized Einstein theory can be represented by two symmetric, traceless, second-rank tensor fields which, assuming that the fields vary harmonically in time, obey the Helmholtz equation outside the sources. As in the case of other nonscalar equations, the solution of the Helmholtz equation for second-rank tensor fields in noncartesian coordinates is a difficult problem owing to the coupling of the field components. However, when a nonscalar equation is written in spherical or cylindrical coordinates, a considerable simplification can be obtained by using spin-weighted quantities and the spin-weighted harmonics.

In this paper the Helmholtz equation for spin-2 fields (*i.e.*, symmetric, traceless, second-rank tensor fields) is solved by separation of variables in spherical and cylindrical coordinates making use of the spin-weighted harmonics. A similar treatment for the vector (spin-1)

Helmholtz equation is given in Refs. [1-3]. In Sect. 2 the Helmholtz equation for spin-2 fields is solved in spherical coordinates and it is shown that the divergenceless solutions of this equation can be expressed in terms of two scalar (Debye) potentials that satisfy the Helmholtz equation. The expressions for the divergenceless solutions in terms of potentials obtained here are equivalent to those found in Refs. [4,5] for the multipoles with $j > 1$. In Sect. 3 the Helmholtz equation for spin-2 fields is solved in cylindrical coordinates and an expression for the divergenceless solutions in terms of potentials adapted to the cylindrical coordinates is obtained. In Sect. 4 two alternative expressions for the solutions of the linearized Einstein vacuum field equations in terms of scalar potentials are given.

2. SEPARATION OF VARIABLES IN SPHERICAL COORDINATES

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis, a quantity η has spin-weight s if under the transformation

$$\mathbf{e}_1 + i\mathbf{e}_2 \to e^{i\alpha}(\mathbf{e}_1 + i\mathbf{e}_2) \tag{1}$$

it transforms according to

$$\eta \to e^{is\alpha}\eta. \tag{2}$$

The five independent components of a symmetric, traceless tensor, t_{ij}, can be combined to form the quantities

$$\begin{aligned} t_{\pm 2} &\equiv \tfrac{1}{2}(t_{11} - t_{22} \pm 2it_{12}) = \tfrac{1}{2}(t_{33} + 2t_{11} \pm 2it_{12}), \\ t_{\pm 1} &\equiv \mp\tfrac{1}{2}(t_{13} \pm it_{23}), \\ t_0 &\equiv \tfrac{1}{2}t_{33}, \end{aligned} \tag{3}$$

so that t_s has spin-weight s. If the components t_{ij} are real, then

$$\overline{t_s} = (-1)^s t_{-s}, \tag{4}$$

where the bar denotes complex conjugation. Similarly, in the case of a vector field F_i, the combinations

$$F_{\pm 1} \equiv \pm\frac{1}{\sqrt{2}}(F_1 \pm iF_2), \qquad F_0 \equiv -\frac{1}{\sqrt{2}}F_3, \tag{5}$$

have spin-weight ± 1 and 0.

By choosing the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the basis $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r\}$ induced by the spherical coordinates, one finds that the Helmholtz equation for a symmetric, traceless tensor field t,

$$\nabla^2 t + k^2 t = 0, \tag{6}$$

written in terms of the spin-weighted components (3) amounts to

$$\begin{aligned}
\frac{1}{r^2}\partial_r(r^2\partial_r t_{+2}) + \frac{1}{r^2}\eth\bar{\eth} t_{+2} + \frac{4}{r^2}\eth t_{+1} + k^2 t_{+2} &= 0,\\
\frac{1}{r^2}\partial_r(r^2\partial_r t_{+1}) - \frac{4}{r^2}t_{+1} + \frac{1}{r^2}\eth\bar{\eth} t_{+1} - \frac{1}{r^2}\bar{\eth} t_{+2} + \frac{3}{r^2}\eth t_0 + k^2 t_{+1} &= 0,\\
\frac{1}{r^2}\partial_r(r^2\partial_r t_0) - \frac{6}{r^2}t_0 + \frac{1}{r^2}\eth\bar{\eth} t_0 + \frac{2}{r^2}(\eth t_{-1} - \bar{\eth} t_{+1}) + k^2 t_0 &= 0,\\
\frac{1}{r^2}\partial_r(r^2\partial_r t_{-1}) - \frac{4}{r^2}t_{-1} + \frac{1}{r^2}\bar{\eth}\eth t_{-1} + \frac{1}{r^2}\eth t_{-2} - \frac{3}{r^2}\bar{\eth} t_0 + k^2 t_{-1} &= 0,\\
\frac{1}{r^2}\partial_r(r^2\partial_r t_{-2}) + \frac{1}{r^2}\bar{\eth}\eth t_{-2} - \frac{4}{r^2}\bar{\eth} t_{-1} + k^2 t_{-2} &= 0,
\end{aligned} \tag{7}$$

where, acting on a quantity η with spin-weight s [6,7],

$$\begin{aligned}
\eth\eta &\equiv -\sin^s\theta\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)(\eta\sin^{-s}\theta),\\
\bar{\eth}\eta &\equiv -\sin^{-s}\theta\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)(\eta\sin^{s}\theta).
\end{aligned} \tag{8}$$

(The expressions (7) can be readily obtained by using the spinor formalism of Ref. [8].)

We seek separable solutions of Eqs. (7) of the form

$$\begin{aligned}
t_{\pm 2} &= \left[\frac{j(j+1)}{(j-1)(j+2)}\right]^{1/2} g_{\pm 2}(r)\, {}_{\pm 2}Y_{jm}(\theta,\phi),\\
t_{\pm 1} &= \left[j(j+1)\right]^{1/2} g_{\pm 1}(r)\, {}_{\pm 1}Y_{jm}(\theta,\phi),\\
t_0 &= j(j+1) g_0(r) Y_{jm}(\theta,\phi),
\end{aligned} \tag{9}$$

where j is an integer greater than 1 and ${}_sY_{jm}$ are spin-weighted spherical harmonics [6,7]. The tensor field given by Eqs. (9) is an eigentensor of J^2 and J_3 with eigenvalues $j(j+1)$ and m, respectively (see, *e.g.*, Ref. [1]). Substituting Eqs. (9) into Eqs. (7) one obtains the set of ordinary differential equations

$$\begin{aligned}
\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{(j-1)(j+2)}{r^2} + k^2\right] g_{\pm 2} + \frac{4(j-1)(j+2)}{r^2} g_{\pm 1} &= 0,\\
\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)+4}{r^2} + k^2\right] g_{\pm 1} + \frac{1}{r^2} g_{\pm 2} + \frac{3j(j+1)}{r^2} g_0 &= 0,\\
\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)+6}{r^2} + k^2\right] g_0 + \frac{2}{r^2}(g_{-1} + g_{+1}) &= 0.
\end{aligned} \tag{10}$$

By combining Eqs. (10) we find decoupled equations for $g_{+2} - g_{-2} - 2(j+2)(g_{+1} - g_{-1})$, $g_{+2} - g_{-2} + 2(j-1)(g_{+1} - g_{-1})$, $g_{+2} + g_{-2} - 4(j+2)(g_{+1} + g_{-1}) + 6(j+1)(j+2)g_0$, $g_{+2} + g_{-2} - 2(g_{+1} + g_{-1}) - 2j(j+1)g_0$ and $g_{+2} + g_{-2} + 4(j-1)(g_{+1} + g_{-1}) + 6j(j-1)g_0$ [9], whose solutions are spherical Bessel functions provided $k \neq 0$. Thus, from Eqs. (9) we get

$$\begin{aligned}
t_{\pm 2} &= \frac{1}{2}\big[(j-1)j(j+1)(j+2)\big]^{1/2}\big\{a j_{j+2}(kr) + b n_{j+2}(kr) \\
&\quad - 2\big[c j_j(kr) + d n_j(kr)\big] + e j_{j-2}(kr) + f n_{j-2}(kr) \\
&\quad \pm 2\big[-A j_{j+1}(kr) - B n_{j+1}(kr) + C j_{j-1}(kr) + D n_{j-1}(kr)\big]\big\}\, {}_{\pm 2}Y_{jm}, \\
t_{\pm 1} &= \frac{1}{2}\big[j(j+1)\big]^{1/2}\big\{-(j+2)\big[a j_{j+2}(kr) + b n_{j+2}(kr)\big] \\
&\quad + c j_j(kr) + d n_j(kr) + (j-1)\big[e j_{j-2}(kr) + f n_{j-2}(kr)\big] \\
&\quad \pm (j+2)\big[A j_{j+1}(kr) + B n_{j+1}(kr)\big] \\
&\quad \pm (j-1)\big[C j_{j-1}(kr) + D n_{j-1}(kr)\big]\big\}\, {}_{\pm 1}Y_{jm}, \\
t_0 &= \left\{\frac{(j+1)(j+2)}{2}\big[a j_{j+2}(kr) + b n_{j+2}(kr)\big]\right. \\
&\quad \left. + \frac{j(j+1)}{3}\big[c j_j(kr) + d n_j(kr)\big] + \frac{j(j-1)}{2}\big[e j_{j-2}(kr) + f n_{j-2}(kr)\big]\right\} Y_{jm},
\end{aligned} \tag{11}$$

where $a, b, c, d, e, f, A, B, C$, and D are arbitrary constants.

The cases $j = 1$ and $j = 0$ must be treated separately since ${}_sY_{jm} = 0$ for $|s| > j$. We find that, also in these cases, the separable solutions of Eqs. (7) are given by Eqs. (11).

As in the case of the vector Helmholtz equation, the fact that the radial equations can be decoupled is related with the existence of an operator that commutes with J^2, J_3 and $\nabla^2 + k^2$ [2]. Such an operator can be chosen as

$$K \equiv L_k S_k + 2, \tag{12}$$

where L_k and S_k are the operators corresponding to the cartesian components of the orbital and spin angular momentum, respectively, and the summation convention applies.

For a symmetric, traceless tensor field t, the spin-weighted components of Kt are given by

$$\begin{aligned}
(Kt)_{+2} &= 2\eth t_{+1}, \\
(Kt)_{+1} &= -\tfrac{1}{2}\bar{\eth} t_{+2} - 3t_{+1} + \tfrac{3}{2}\eth t_0, \\
(Kt)_0 &= -\bar{\eth} t_{+1} - 4t_0 + \eth t_{-1}, \\
(Kt)_{-1} &= \tfrac{1}{2}\eth t_{-2} - 3t_{-1} - \tfrac{3}{2}\bar{\eth} t_0, \\
(Kt)_{-2} &= -2\bar{\eth} t_{-1}.
\end{aligned} \tag{13}$$

Using Eqs. (9) (which give the eigentensors of J^2 and J_3) and (13) one can readily find the common eigentensors of J^2, J_3 and K; for these tensor fields only one of the five radial functions $g_s(r)$ is independent and, therefore, Eqs. (10) reduce to a single equation (*cf.* Ref. [2]).

Since $J^2 = L^2 + 2L_kS_k + S^2$ and, for spin-2 fields, $S^2t = 6t$, it follows that

$$K = \tfrac{1}{2}(J^2 - L^2 - 2), \tag{14}$$

which shows that the eigentensors of J^2, S^2, and K are also eigentensors of L^2 and using Eqs. (13–14) it is easy to see that the separable solution (11) is a superposition of five eigentensors of L^2 with eigenvalues $\ell(\ell+1)$, where ℓ coincides with the index of the spherical Bessel functions appearing in Eqs. (11). Assuming that under the parity transformation, $\mathbf{r} \to -\mathbf{r}$, $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are left unchanged and $\mathbf{e}_\theta$ changes sign and taking into account that ${}_sY_{jm}$ is transformed into $(-1)^j\ {}_{-s}Y_{jm}$, one finds that the five eigentensors of L^2 contained in Eqs. (11) are also eigentensors of the parity operator with eigenvalue $(-1)^\ell$.

The divergence of a second-rank, symmetric, traceless tensor field, t, is the vector field $\operatorname{div} t$ whose *cartesian* components are given by $(\operatorname{div} t)_i = \partial_j t_{ij}$, where $\partial_j \equiv \partial/\partial x_j$. The components of the divergence of t with respect to the basis $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r\}$ are determined by [see, *e.g.*, Ref. [8], Eq. (44)]

$$(\operatorname{div} t)_s = \frac{1}{\sqrt{2}}\left\{-\frac{1}{r}\bar{\eth} t_{s+1} - \frac{2}{r^3}\partial_r(r^3 t_s) + \frac{1}{r}\eth t_{s-1}\right\}, \tag{15}$$

with the spin-weighted components of div t defined as in Eq. (5). Substituting Eqs. (11) into Eqs. (15) and using the recurrence relations for the spin-weighted spherical harmonics and for the spherical Bessel functions, one finds that the separable solution of the Helmholtz equation given by Eqs. (11) has vanishing divergence if and only if

$$\begin{aligned}
a &= \frac{j(2j-1)c}{3(j+2)(2j+1)}, \qquad b = \frac{j(2j-1)d}{3(j+2)(2j+1)}, \qquad e = \frac{(j+1)(2j+3)c}{3(j-1)(2j+1)}, \\
f &= \frac{(j+1)(2j+3)d}{3(j-1)(2j+1)}, \qquad A = \frac{j-1}{j+2}C, \qquad B = \frac{j-1}{j+2}D.
\end{aligned} \tag{16}$$

Substituting Eqs. (16) into Eqs. (11) and making use of the recurrence relations for the Bessel functions one gets

$$\begin{aligned} t_{+2} &= -\frac{ik}{r^2}\partial_r r^2 \partial\partial\psi_1 + \frac{1}{2}\left(\frac{1}{r^2}\partial_r^2 r^2 - k^2\right)\partial\partial\psi_2, \\ t_{+1} &= \frac{ik}{2r}\bar{\partial}\partial\partial\psi_1 - \frac{1}{2r^2}\partial_r r\bar{\partial}\partial\partial\psi_2, \\ t_0 &= \frac{1}{2r^2}\bar{\partial}\bar{\partial}\partial\partial\psi_2, \\ t_{-1} &= \frac{ik}{2r}\partial\bar{\partial}\bar{\partial}\psi_1 + \frac{1}{2r^2}\partial_r r\partial\bar{\partial}\bar{\partial}\psi_2, \\ t_{-2} &= \frac{ik}{r^2}\partial_r r^2 \bar{\partial}\bar{\partial}\psi_1 + \frac{1}{2}\left(\frac{1}{r^2}\partial_r^2 r^2 - k^2\right)\bar{\partial}\bar{\partial}\psi_2, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \psi_1 &\equiv \frac{i(2j+1)}{k^2(j+2)}\big[Cj_j(kr) + Dn_j(kr)\big]Y_{jm}, \\ \psi_2 &\equiv \frac{(2j-1)(2j+3)}{3k^2(j-1)(j+2)}\big[cj_j(kr) + dn_j(kr)\big]Y_{jm}. \end{aligned} \tag{18}$$

Clearly, the functions ψ_1 and ψ_2 are separable solutions of the scalar Helmholtz equation. It may be noticed that Eqs. (17) contain no reference to the value of j. On the other hand, for $j = 1, 0$, from Eqs. (11) and (15) one finds that if $k \neq 0$ and $\operatorname{div} t = 0$ then, necessarily, $t = 0$.

Thus, by virtue of the completeness of the spin-weighted spherical harmonics, any divergenceless solution of the spin-2 Helmholtz equation (6) can be expressed as a superposition of separable solutions of the form (17) where, now, ψ_1 and ψ_2 are two solutions of the scalar Helmholtz equation that are superpositions of solutions of the form (18). In view of Eq. (4), if ψ_1 and ψ_2 are real then t is real. (The factor i was included in Eqs. (17) in order to produce this relation.)

The components (17) can be written in terms of certain tensor operators U_{ij}, V_{ij} [4], whose *cartesian* components are defined by

$$U_{ij}(\psi) \equiv L_i X_j \psi + L_j X_i \psi, \qquad V_{ij}(\psi) \equiv \varepsilon_{imn}\partial_m U_{nj}(\psi), \tag{19}$$

where now

$$\mathbf{L} \equiv \mathbf{r} \times \nabla, \qquad \mathbf{X} \equiv \nabla \times \mathbf{L} - \nabla. \tag{20}$$

It is easy to see that for any well-behaved function ψ, $U_{ij}(\psi)$ and $V_{ij}(\psi)$ are symmetric, traceless, divergenceless tensor fields. By computing the spin-weighted components of

$U_{ij}(\psi)$ and $V_{ij}(\psi)$ with respect to the basis $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r\}$ [8] one finds that the expressions (17) are equivalent to

$$t_{ij} = kU_{ij}(\psi_1) + V_{ij}(\psi_2). \tag{21}$$

Equations (19) imply that

$$\varepsilon_{imn}\partial_m V_{nj}(\psi) = -U_{ij}(\nabla^2\psi), \tag{22}$$

and therefore the tensor field (21) satisfies

$$(\mathrm{curl}\, t)_{ij} \equiv \varepsilon_{imn}\partial_m t_{nj} = k^2 U_{ij}(\psi_2) + kV_{ij}(\psi_1). \tag{23}$$

According to Eqs. (17), the scalar potentials generating a divergenceless solution of the spin-2 Helmholtz equation are determined by

$$\bar{\partial}\bar{\partial}\partial\partial\psi_2 = 2r^2 t_0, \tag{24a}$$

and, by comparing Eqs. (21) and (23),

$$\bar{\partial}\bar{\partial}\partial\partial\psi_1 = \frac{2r^2}{k}(\mathrm{curl}\, t)_0. \tag{24b}$$

The usefulness of Eqs. (24) can be illustrated by obtaining the expansion of a circularly polarized plane wave in spherical waves. The cartesian components of a spin-2 field corresponding to a circularly polarized plane wave with helicity $\pm$ propagating in the z-direction are given by

$$(t_{ij}) = A \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{ikz}, \tag{25}$$

where A is a constant. From Eqs. (3) and (25) one finds that $2r^2 t_0 = x_i x_j t_{ij} = A(r\sin\theta \times e^{\pm i\phi})^2 e^{ikz}$; therefore, by using the expansion of e^{ikz} in terms of spherical harmonics and the recurrence relations for the associated Legendre functions (see, *e.g.*, Ref. [10]), Eq. (24a) yields

$$\bar{\partial}\bar{\partial}\partial\partial\psi_2 = -\frac{1}{k^2}\sum\left[4\pi(2j+1)(j-1)j(j+1)(j+2)\right]^{1/2} i^j j_j(kr) Y_{j,\pm 2},$$

hence, we can choose

$$\psi_2 = -\frac{1}{k^2}\sum_{j=2}^{\infty}\left[\frac{4\pi(2j+1)}{(j-1)j(j+1)(j+2)}\right]^{1/2} i^j j_j(kr) Y_{j,\pm 2}. \tag{26a}$$

Since the tensor field (25) is such that curl $t = \pm kt$, from Eqs. (24) we see that

$$\psi_1 = \pm\psi_2. \tag{26b}$$

By substituting Eqs. (26) into Eqs. (17) or (21) one gets the desired expansion (*cf.* Ref. [11]).

In the case where $k = 0$, the Helmholtz equation reduces to the Laplace equation and by assuming a separable solution of the form (9) we obtain [*cf.* Eqs. (11)]

$$\begin{aligned}
t_{\pm 2} &= \frac{1}{2}\big[(j-1)j(j+1)(j+2)\big]^{1/2}\big\{ar^{j+2} + br^{-j-3} - 2[cr^j + dr^{-j-1}] \\
&\quad + er^{j-2} + fr^{-j+1} \pm 2\big[-Ar^{j+1} - Br^{-j-2} + Cr^{j-1} + Dr^{-j}\big]\big\}\ {}_{\pm 2}Y_{jm}, \\
t_{\pm 1} &= \frac{1}{2}\big[j(j+1)\big]^{1/2}\big\{-(j+2)\big[ar^{j+2} + br^{-j-3}\big] + cr^j + dr^{-j-1} \\
&\quad + (j-1)\big[er^{j-2} + fr^{-j+1}\big] \pm (j+2)\big[Ar^{j+1} + Br^{-j-2}\big] \\
&\quad \pm (j-1)\big[Cr^{j-1} + Dr^{-j}\big]\big\}\ {}_{\pm 1}Y_{jm}, \\
t_0 &= \left\{\frac{(j+1)(j+2)}{2}\big[ar^{j+2} + br^{-j-3}\big] + \frac{j(j+1)}{3}\big[cr^j + dr^{-j-1}\big]\right. \\
&\quad \left. + \frac{j(j-1)}{2}\big[er^{j-2} + fr^{-j-1}\big]\right\} Y_{jm},
\end{aligned} \tag{27}$$

for $j = 0, 1, 2, \ldots$

Substituting expressions (27) into Eqs. (15) one finds that the solution of the Laplace equation given by Eqs. (27) has vanishing divergence if and only if

$$a = c = d = f = A = D = 0. \tag{28}$$

When these relations are inserted into Eqs. (27), for $j > 1$, they can be written as [*cf.* Eqs. (17)]

$$\begin{aligned}
t_{+2} &= -\frac{i}{r^2}\partial_r r^2 \eth\eth\psi_1 + \frac{1}{2r^2}\partial_r^2 r^2 \eth\eth\psi_2, \\
t_{+1} &= \frac{i}{2r}\bar{\eth}\eth\eth\psi_1 - \frac{1}{2r^2}\partial_r r \bar{\eth}\eth\eth\psi_2, \\
t_0 &= \frac{1}{2r^2}\bar{\eth}\bar{\eth}\eth\eth\psi_2, \\
t_{-1} &= \frac{i}{2r}\eth\bar{\eth}\bar{\eth}\psi_1 + \frac{1}{2r^2}\partial_r r \eth\bar{\eth}\bar{\eth}\psi_2, \\
t_{-2} &= \frac{i}{r^2}\partial_r r^2 \bar{\eth}\bar{\eth}\psi_1 + \frac{1}{2r^2}\partial_r^2 r^2 \bar{\eth}\bar{\eth}\psi_2,
\end{aligned} \tag{29}$$

where

$$\begin{aligned}\psi_1 &= \frac{i}{(j-1)(j+2)}\left[(j-1)Cr^j + (j+2)Br^{-j-1}\right]Y_{jm},\\ \psi_2 &= \frac{1}{(j-1)(j+2)}\left[\frac{j-1}{j+1}er^j + \frac{j+2}{j}br^{-j-1}\right]Y_{jm}.\end{aligned} \tag{30}$$

In this case, ψ_1 and ψ_2 are separable solutions of the scalar Laplace equation.

From Eqs. (27-28) we see that the divergenceless solutions of the Laplace equation with $j=1$ are given by

$$j=1:\begin{cases} t_{\pm 2} = 0,\\ t_{\pm 1} = \dfrac{3}{\sqrt{2}}(\pm Br^{-3} - br^{-4})\,{}_{\pm 1}Y_{1m},\\ t_0 = 3br^{-4}Y_{1m},\end{cases} \tag{31}$$

which can be written in the form (29) with

$$\begin{aligned}\psi_1 &= -\frac{iB}{r^2}(Y_{1m}\ln r + 3h_m),\\ \psi_2 &= -\frac{b}{r^2}(Y_{1m}\ln r + 3h_m),\end{aligned} \tag{32}$$

where $h_m(\theta,\phi)$ is a solution of

$$\bar{\eth}\eth h_m + 2h_m = Y_{1m}, \quad (m = \pm 1, 0). \tag{33}$$

Owing to Eq. (33), the scalar potentials (32) satisfy the Laplace equation. Finally, in the case where $j=0$, Eqs. (27-28) yield

$$j=0:\begin{cases} t_{\pm 2} = 0,\\ t_{\pm 1} = 0,\\ t_0 = br^{-3}Y_{00}.\end{cases} \tag{34}$$

This solution is of the form (29) with

$$\psi_1 = 0, \qquad \psi_2 = \frac{b}{r}Y_{00}\ln(r\csc\theta), \tag{35}$$

which are solutions of the Laplace equation. It may be noticed that the scalar potentials (32) and (35) diverge at $\theta = 0,\pi$, and are not separable.

Thus, any divergenceless solution of the spin-2 Laplace equation can be expressed in the form (29), where ψ_1 and ψ_2 are solutions of the scalar Laplace equation. Equations (29) are equivalent to

$$t_{ij} = U_{ij}(\psi_1) + V_{ij}(\psi_2). \tag{36}$$

3. Separation of variables in cylindrical coordinates

Taking now the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the orthonormal basis $\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$ induced by the circular cylindrical coordinates, the Helmholtz equation for a symmetric, traceless, second-rank tensor field is equivalent to the set of uncoupled equations

$$\partial_z^2 t_s + \bar{\eth}\eth t_s + k^2 t_s = 0, \quad (s = \pm 2, \pm 1, 0), \tag{37}$$

where, acting on a quantity η with spin-weight s [3],

$$\eth\eta \equiv -\rho^s \left(\partial_\rho + \frac{i}{\rho}\partial_\phi\right)(\rho^{-s}\eta),$$

$$\bar{\eth}\eta \equiv -\rho^{-s} \left(\partial_\rho - \frac{i}{\rho}\partial_\phi\right)(\rho^{s}\eta). \tag{38}$$

We seek solutions of Eqs. (37) of the form

$$t_s = g_s(z)\,{}_sF_{\alpha m}(\rho, \phi), \qquad (s = \pm 2, \pm 1, 0), \tag{39}$$

where the ${}_sF_{\alpha m}$ are spin-weighted cylindrical harmonics [3]. A tensor field of the form (39) is eigentensor of J_3 and of the square of the linear momentum perpendicular to the z-axis, $P_1^2 + P_2^2$, with eigenvalues m and α^2, respectively [3]. Substituting Eqs. (39) into Eqs. (37) one finds

$$\left(\frac{d^2}{dz^2} - \gamma^2\right) g_s = 0, \qquad (s = \pm 2, \pm 1, 0), \tag{40}$$

where

$$\gamma^2 \equiv \alpha^2 - k^2, \tag{41}$$

therefore, if $\gamma \neq 0$, $g_s(z) = A_s e^{\gamma z} + B_s e^{-\gamma z}$ and if $\gamma = 0$, $g_s(z) = A_s + B_s z$, where A_s and B_s are arbitrary constants. Thus, assuming that α is different from zero, Eqs. (37) admit separable solutions of the form

$$t_s = \left(A_s e^{\gamma z} + B_s e^{-\gamma z}\right)\left[C_s({}_sJ_{\alpha m}) + D_s({}_sN_{\alpha m})\right], \qquad (\gamma \neq 0), \tag{42a}$$

and

$$t_s = \big(A_s + B_s z\big)\big[C_s({}_sJ_{\alpha m}) + D_s({}_sN_{\alpha m})\big], \qquad (\gamma = 0), \tag{42b}$$

where A_s, B_s, C_s and D_s are arbitrary constants and [3]

$$ {}_sJ_{\alpha m}(\rho,\phi) \equiv J_{m+s}(\alpha\rho)e^{im\phi}, \qquad {}_sN_{\alpha m}(\rho,\phi) \equiv N_{m+s}(\alpha\rho)e^{im\phi}, \tag{43}$$

with J_ν and N_ν being Bessel functions. For $\alpha = 0$, the functions ${}_sF_{\alpha m}$ diverge when ρ goes to zero or to infinity, or they do not vanish when ρ goes to infinity.

The components of the divergence of a symmetric, traceless, second-rank tensor field with respect to the basis $\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$ are given by [see, *e.g.*, Ref. [8], Eq. (44)]

$$(\operatorname{div} t)_s = \frac{1}{\sqrt{2}}\big\{-\bar{\eth} t_{s+1} - 2\partial_z t_s + \eth t_{s-1}\big\}, \tag{44}$$

therefore, the tensor field (42a) has vanishing divergence if and only if

$$\begin{aligned}
&\frac{\alpha}{2}\big(A_0C_0 + A_{\pm2}C_{\pm2}\big) = \gamma A_{\pm1}C_{\pm1}, &&\frac{\alpha}{2}\big(A_0D_0 + A_{\pm2}D_{\pm2}\big) = \gamma A_{\pm1}D_{\pm1},\\
&\frac{\alpha}{2}\big(B_0C_0 + B_{\pm2}C_{\pm2}\big) = -\gamma B_{\pm1}C_{\pm1}, &&\frac{\alpha}{2}\big(B_0D_0 + B_{\pm2}D_{\pm2}\big) = -\gamma B_{\pm1}D_{\pm1},\\
&\frac{\alpha}{2}\big(A_{+1}C_{+1} + A_{-1}C_{-1}\big) = \gamma A_0C_0, &&\frac{\alpha}{2}\big(A_{+1}D_{+1} + A_{-1}D_{-1}\big) = \gamma A_0D_0,\\
&\frac{\alpha}{2}\big(B_{+1}C_{+1} + B_{-1}C_{-1}\big) = -\gamma B_0C_0, &&\frac{\alpha}{2}\big(B_{+1}D_{+1} + B_{-1}D_{-1}\big) = -\gamma B_0D_0.
\end{aligned} \tag{45}$$

Introducing the combinations

$$\begin{aligned}
&a_1 \equiv \frac{1}{2\alpha}\big(A_{+1}C_{+1} - A_{-1}C_{-1}\big), &&a_2 \equiv \frac{1}{2\alpha}\big(A_{+1}D_{+1} - A_{-1}D_{-1}\big),\\
&b_1 \equiv \frac{1}{2\alpha}\big(B_{+1}C_{+1} - B_{-1}C_{-1}\big), &&b_2 \equiv \frac{1}{2\alpha}\big(B_{+1}D_{+1} - B_{-1}D_{-1}\big),\\
&a_3 \equiv \frac{1}{2\alpha\gamma}\big(A_{+1}C_{+1} + A_{-1}C_{-1}\big), &&a_4 \equiv \frac{1}{2\alpha\gamma}\big(A_{+1}D_{+1} + A_{-1}D_{-1}\big),\\
&b_3 \equiv -\frac{1}{2\alpha\gamma}\big(B_{+1}C_{+1} + B_{-1}C_{-1}\big), &&b_4 \equiv -\frac{1}{2\alpha\gamma}\big(B_{+1}D_{+1} + B_{-1}D_{-1}\big),
\end{aligned} \tag{46}$$

and assuming that the conditions (45) hold, one finds that the components (42a) can be written as [*cf.* Eqs. (17) and (29)]

$$
\begin{aligned}
t_{+2} &= -i\partial_z \partial\partial\psi_1 + \frac{1}{2}\big(\partial_z^2 - k^2\big)\partial\partial\psi_2, \\
t_{+1} &= \frac{i}{2}\bar{\partial}\partial\partial\psi_1 - \frac{1}{2}\partial_z\bar{\partial}\partial\partial\psi_2, \\
t_0 &= \frac{1}{2}\bar{\partial}\bar{\partial}\partial\partial\psi_2, \\
t_{-1} &= \frac{i}{2}\partial\bar{\partial}\bar{\partial}\psi_1 + \frac{1}{2}\partial_z\partial\bar{\partial}\bar{\partial}\psi_2, \\
t_{-2} &= i\partial_z\bar{\partial}\bar{\partial}\psi_1 + \frac{1}{2}(\partial_z^2 - k^2)\bar{\partial}\bar{\partial}\psi_2,
\end{aligned}
\tag{47}
$$

where

$$
\begin{aligned}
\psi_1 &= \frac{2i}{\alpha^2}\big[(a_1 e^{\gamma z} + b_1 e^{-\gamma z})_0 J_{\alpha m} + (a_2 e^{\gamma z} + b_2 e^{-\gamma z})_0 N_{\alpha m}\big], \\
\psi_2 &= \frac{2}{\alpha^2}\big[(a_3 e^{\gamma z} + b_3 e^{-\gamma z})_0 J_{\alpha m} + (a_4 e^{\gamma z} + b_4 e^{-\gamma z})_0 N_{\alpha m}\big],
\end{aligned}
\tag{48}
$$

which are solutions of the scalar Helmholtz equation.

Similarly, one finds that if the field given by Eqs. (42*b*) has vanishing divergence then its components can be written in the form (47), where

$$
\begin{aligned}
\psi_1 &= \frac{2i}{\alpha^2}\big[(a_1 + b_1 z)_0 J_{\alpha m} + (a_2 + b_2 z)_0 N_{\alpha m}\big], \\
\psi_2 &= \frac{2}{\alpha^2}\big[(a_3 + b_3 z)_0 J_{\alpha m} + (a_4 + b_4 z)_0 N_{\alpha m}\big],
\end{aligned}
\tag{49}
$$

which satisfy the scalar Helmholtz equation, and

$$
\begin{aligned}
a_1 &\equiv \frac{1}{2\alpha}\big(A_{+1}C_{+1} - A_{-1}C_{-1}\big), & a_2 &\equiv \frac{1}{2\alpha}\big(A_{+1}D_{+1} - A_{-1}D_{-1}\big), \\
b_1 &\equiv \frac{1}{2\alpha}\big(B_{+1}C_{+1} - B_{-1}C_{-1}\big), & b_2 &\equiv \frac{1}{2\alpha}\big(B_{+1}D_{+1} - B_{-1}D_{-1}\big), \\
a_3 &\equiv -\frac{1}{2\alpha^2}\big(A_{+2}C_{+2} + A_{-2}C_{-2}\big), & a_4 &\equiv -\frac{1}{2\alpha^2}\big(A_{+2}D_{+2} + A_{-2}D_{-2}\big), \\
b_3 &\equiv \frac{1}{2\alpha}\big(A_{+1}C_{+1} + A_{-1}C_{-1}\big), & b_4 &\equiv \frac{1}{2\alpha}\big(A_{+1}D_{+1} + A_{-1}D_{-1}\big).
\end{aligned}
$$

It can be shown that Eqs. (47) are equivalent to

$$
t_{ij} = W_{ij}(\psi_1) + Z_{ij}(\psi_2),
\tag{50}
$$

where the tensor operators W_{ij} and Z_{ij} are given in cartesian coordinates by

$$W_{ij}(\psi) \equiv M_i N_j \psi + M_j N_i \psi, \qquad Z_{ij}(\psi) \equiv \varepsilon_{imn} \partial_m W_{nj}(\psi), \tag{51}$$

with

$$\mathbf{M} \equiv \mathbf{e}_z \times \nabla, \qquad \mathbf{N} \equiv \nabla \times \mathbf{M}, \tag{52}$$

[*cf.* Eqs. (19-21)]. It is easy to see that for any well-behaved function ψ, $W_{ij}(\psi)$ and $Z_{ij}(\psi)$ are symmetric, traceless, divergenceless tensor fields, and that

$$\varepsilon_{imn} \partial_m Z_{nj}(\psi) = -W_{ij}(\nabla^2 \psi). \tag{53}$$

The solutions (39) with $\alpha = 0$ can also be written in the form (47), in terms of two scalar potentials ψ_1 and ψ_2 that satisfy the Helmholtz equation but they are not separable. Owing to the completeness of the spin-weighted cylindrical harmonics, any divergenceless solution of the spin-2 Helmholtz equation can be written in the form (47), where ψ_1 and ψ_2 are solutions of the scalar Helmholtz equation. If ψ_1 and ψ_2 are real, then t_{ij} is real.

4. Application to the linearized Einstein equations

The Einstein vacuum field equations linearized about the Minkowski metric can be written in cartesian coordinates in the gauge-invariant form

$$\partial_i E_{ij} = 0, \qquad \partial_i B_{ij} = 0, \tag{54a}$$

$$\frac{1}{c}\frac{\partial}{\partial t} E_{ij} = \varepsilon_{imn} \partial_m B_{nj}, \qquad \frac{1}{c}\frac{\partial}{\partial t} B_{ij} = -\varepsilon_{imn} \partial_m E_{nj}, \tag{54b}$$

where E_{ij} and B_{ij} are symmetric traceless tensor fields defined by

$$E_{ij} \equiv K_{0i0j}, \qquad B_{ij} \equiv -\frac{1}{2} K_{0imn} \varepsilon_{jmn}, \tag{55}$$

and $K_{\alpha\beta\gamma\delta}$ is the curvature tensor to first order in the metric perturbation (see *e.g.*, Refs. [4,12]). From Eqs. (54) it follows that the fields E_{ij} and B_{ij} obey the wave equation; therefore, assuming that E_{ij} and B_{ij} have a time dependence of the form $e^{-i\omega t}$, the fields E_{ij} and B_{ij} satisfy the Helmholtz equation with $k = \omega/c$. According to the results of Sect. 2, if $\omega \neq 0$, E_{ij} can be expressed in the form

$$E_{ij} = kU_{ij}(\psi_1) + V_{ij}(\psi_2), \tag{56}$$

[*cf.* Eq. (21)] where ψ_1 and ψ_2 are regular solutions of the scalar Helmholtz equation. From Eqs. (54*b*) and (23) we see that the field B_{ij} corresponding to (56) is given by

$$B_{ij} = -i\big[kU_{ij}(\psi_2) + V_{ij}(\psi_1)\big]. \tag{57}$$

In the static case ($\omega = 0$), E_{ij} and B_{ij} must satisfy the Laplace equation; hence, E_{ij} can be expressed in the form $E_{ij} = U_{ij}(\psi_1) + V_{ij}(\psi_2)$, where ψ_1 and ψ_2 are solutions of the scalar Laplace equation [Eq. (36)]. Equations (54*b*) and (22) give $0 = \varepsilon_{imn}\partial_m E_{nj} = V_{ij}(\psi_1)$, which implies that $U_{ij}(\psi_1) = 0$ [see Eqs. (29-35)]. Thus

$$E_{ij} = V_{ij}(\psi_2). \tag{58}$$

In a similar manner, it follows that

$$B_{ij} = V_{ij}(\psi_4), \tag{59}$$

where ψ_4 is a solution of the scalar Laplace equation. (It may be noticed that Eqs. (58-59) can be obtained from Eqs. (56-57) by simply setting $k = 0$.)

Alternatively, the components E_{ij} and B_{ij} can be expressed in the form (50). Assuming again that the time dependence of the fields is given by a factor $e^{-i\omega t}$, in the case where $\omega \neq 0$,

$$E_{ij} = kW_{ij}(\psi_1) + Z_{ij}(\psi_2), \tag{60}$$

where ψ_1 and ψ_2 are solutions of the scalar Helmholtz equation and the factor k has been introduced for convenience. Then, Eqs. (54*b*) and (53) imply that

$$B_{ij} = -i\big[kW_{ij}(\psi_2) + Z_{ij}(\psi_1)\big]. \tag{61}$$

On the other hand, when $\omega = 0$,

$$E_{ij} = Z_{ij}(\psi_1), \qquad B_{ij} = Z_{ij}(\psi_2), \tag{62}$$

where ψ_1 and ψ_2 are solutions of the scalar Laplace equation.

In the standard approach, the Einstein vacuum field equations linearized about the Minkowski metric are written in terms of the metric perturbations, which are affected by the gauge transformations induced by the infinitesimal coordinate changes. By contrast, the curvature perturbations $K_{\alpha\beta\gamma\delta}$, which are equivalent to the fields E_{ij} and B_{ij}, provide a gauge-invariant description of the gravitational field. Eqs. (56-57), which are adapted to the spherical coordinates, yield a multipole expansion of the gravitational field. The gravitational potentials appearing in Eqs. (56-62) for fields generated by localized sources can be expressed in terms of the energy-momentum tensor of the sources as in Refs. [4,5].

5. CONCLUDING REMARKS

It is known that in flat space-time a massless field of arbitrary spin can be expressed in terms of two real potentials or of a single complex scalar potential (see, *e.g.*, Ref. [13]); the results presented above and in Refs. [1,3] show specifically that there exist operators such that when applied to a solution of the scalar Helmholtz equation which is eigenfunction of J^2 $(= L^2)$ and J_3 $(= L_3)$, they yield solutions of the spin-1 and spin-2 massless field equations that are eigenfunctions of J^2 and J_3 with the eigenvalues of the potential and that, similarly, there exist operators that map a solution of the scalar Helmholtz equation which is eigenfunction of $P_1^2 + P_2^2$ and J_3 into solutions of the spin-1 and spin-2 massless field equations that are eigenfunctions of $P_1^2 + P_2^2$ and J_3 with the same respective eigenvalues.

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